Optimal discrimination of measurements

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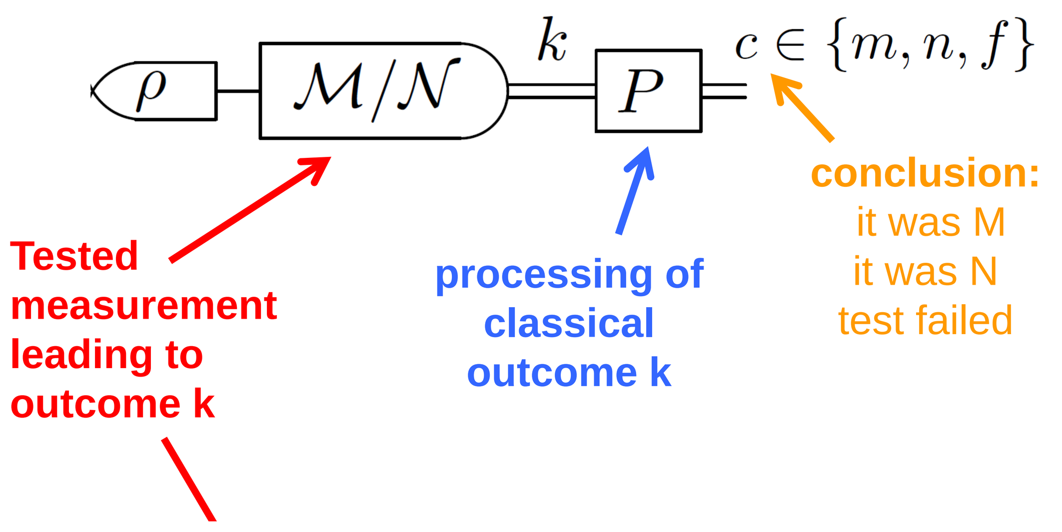
Abstract

We formulate discrimination problem for quantum measurements in the language of process positive operator valued measures (PPOVM). In this framework we present how a discrimination of a pair of Von Neumann qubit measurements can be converted into discrimination problem for a pair of pure states defining those possible measurements. In this way, the problem can be analytically solved using the results known for discrimination of states. This holds not only for minimum-error and unambiguous strategy, but also for strategy that interpolates between them, so called discrimination with fixed failure rate. We extend the result to the discrimination of imperfect measurements. We present a case study for unambiguous discrimination of two symmetric 3-outcome POVMs that demonstrates that optimal scheme is not always based on use of a maximally entangled state.

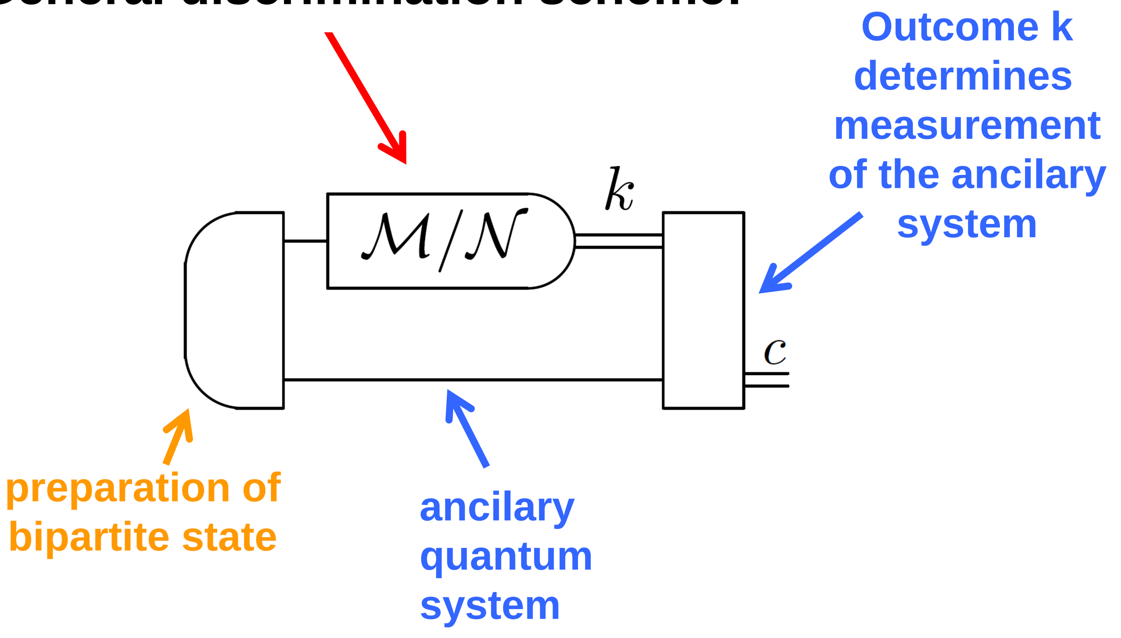
How to identify a measurement device?

Suppose we are given a measurement device, which we can use only once to measure a single copy of a quantum state. We know that the device is either a measurement **M** or a measurement **N** with a priori probability η_M, η_N, respectively. Our goal is to design a test that would (optimally) identify the unknown measurement device. The only restriction on our test procedure is that it can use the unknown measurement only once.

Simple discrimination scheme:



General discrimination scheme:



In general, one can prepare a bipartite state, measure one part of it with the unknown measurement and the second part with the ancillary measurement we choose. The choice of the ancillary measurement can be either made independently of the outcome of the unknown measurement **k** or it can be influenced by **k** in any sophisticated way. Consequently, the guess about the identity of the unknown measurement has to be made.

We define:

$$p_c = \eta_{\mathcal{M}}\ p(m|\mathcal{M},\mathcal{T}) + \eta_{\mathcal{N}}\ p(n|\mathcal{N},\mathcal{T})$$
$$p_e = \eta_{\mathcal{M}}\ p(n|\mathcal{M},\mathcal{T}) + \eta_{\mathcal{N}}\ p(m|\mathcal{N},\mathcal{T})$$
$$p_f = \eta_{\mathcal{M}}\ p(f|\mathcal{M},\mathcal{T}) + \eta_{\mathcal{N}}\ p(f|\mathcal{N},\mathcal{T}).$$

where $p(c|\mathcal{M},\mathcal{T})$ denotes the probability of getting a conclusion c, when measurement $\mathcal{M}$ is tested by a test $\mathcal{T}$.

Discrimination strategies

Minimum error approach

$p_f = 0$ goal: $minimize\ p_e$

Unambiguous discrimination

$p_e = 0$ goal: $minimize\ p_f$

Fixed failure rate discrimination

$p_f = q$ goal: $maximize\ p_c$

Mathematical tool: Process POVM

The possible measurements $\mathcal{M},\mathcal{N}$ are defined by POVM elements $\{M_i\},\{N_i\}$ respectively. We can describe these measurements also as measure and prepare channels and we use Choi-Jamiolkowski isomorphism to associate with them positive operators

$$M = \sum_i |i\rangle\langle i|_O \otimes (M_i)_I^T \qquad N = \sum_i |i\rangle\langle i|_O \otimes (N_i)_I^T$$

An arbitrary test $\mathcal{T}$ that discriminates between $\mathcal{M},\mathcal{N}$ and is allowed by quantum mechanics, is described by

Process POVM [1] $\{T_m, T_n, T_f\} \in \mathcal{L}(\mathcal{H}_{in}\otimes\mathcal{H}_{out})$

$$T_m, T_n, T_f \geq 0 \qquad T_m + T_n + T_f = \rho\otimes I \qquad Tr\rho = 1$$

A conclusion c of the test appear with the probability given by:

$$p(c|\mathcal{M},\mathcal{T}) = Tr(MT_c) \quad c\in\{m,n,f\}$$

Thanks to block diagonal structure of *M, N* it suffice to consider $T_c = \sum_{i=1}^n H_i^{(c)}\otimes|i\rangle\langle i|$ and the optimization can be reformulated as:
i) fix $p_f = q$ and ii)

$$maximize\ p_c = \sum_i Tr(\eta_{\mathcal{M}} H_i^{(m)} M_i + \eta_{\mathcal{N}} H_i^{(n)} N_i),$$

under the constraints

$$\forall i \quad H_i^{(m)} + H_i^{(n)} + H_i^{(f)} = \rho$$

The problem resembles N discriminations of states, which are interlinked by the normalization constraints.

Discrimination of projective qubit measurements

The simplest case of the general problem is defined in a two dimensional Hilbert space by POVM elements

$$M_1 = |\varphi\rangle\langle\varphi| \qquad M_2 = I - M_1 = |\varphi^\perp\rangle\langle\varphi^\perp|$$
$$N_1 = |\psi\rangle\langle\psi| \qquad N_2 = I - N_1 = |\psi^\perp\rangle\langle\psi^\perp|.$$

Thanks to existence of a positive trace preserving map Γ that acts as

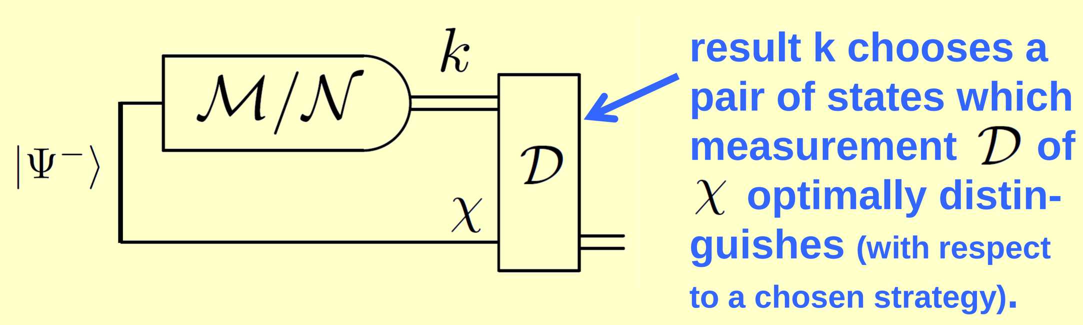
$$\Gamma(M_1) = M_2 \quad \Gamma(N_1) = N_2$$

the test can be made symmetric, i.e.

$$H_2^{(c)} = \Gamma(H_1^{(c)}) \quad \text{and} \quad \rho = \tfrac{1}{2}I$$

without loss of generality and the problem can be recast as discrimination of pure states $|\varphi\rangle,|\psi\rangle$. This implies that the following scheme will be always optimal:

Optimal scheme for discrimination of projective qubit measurements :



for k=1 the state χ is either $|\varphi^\perp\rangle$ or $|\psi^\perp\rangle$
for k=2 the state χ is either $|\varphi\rangle$ or $|\psi\rangle$

Unambiguous discrimination

In this case $\mathcal{D}$ is the Ivanovic-Dieks-Peres measurement [2] and the optimal performance cannot be achieved by a simple scheme.

Minimum error approach

Form of measurement $\mathcal{D}$ follows from the work of Helstrom [3]. Minimal probability of error can be achieved also by a simple scheme, where post-processing P is really necessary for very unbalanced prior probabilities.

Fixed failure rate discrimination

Form of optimal measurement for a discrimination of a pair of pure sates with a fixed failure rate was recently found by Sugimoto, et.al.[4], and Bagan, et.al. [5].

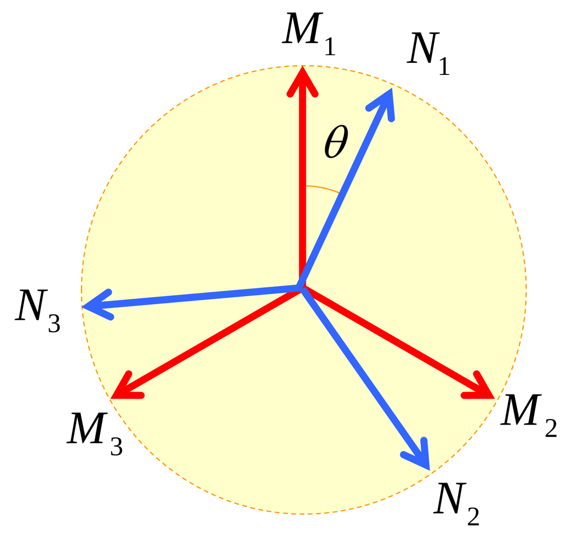
Noisy qubit measurements

$$M_1 = \delta|\varphi\rangle\langle\varphi| + \frac{1-\delta}{2}I \qquad M_2 = \delta|\varphi^\perp\rangle\langle\varphi^\perp| + \frac{1-\delta}{2}I$$
$$N_1 = \delta|\psi\rangle\langle\psi| + \frac{1-\delta}{2}I \qquad N_2 = \delta|\psi^\perp\rangle\langle\psi^\perp| + \frac{1-\delta}{2}I.$$

In this case the problem is equivalent to a discrimination of a pair of states defined by operators M_1, N_1 appearing with probability η_M, η_N .

Discrimination of 3-outcome qubit POVMs

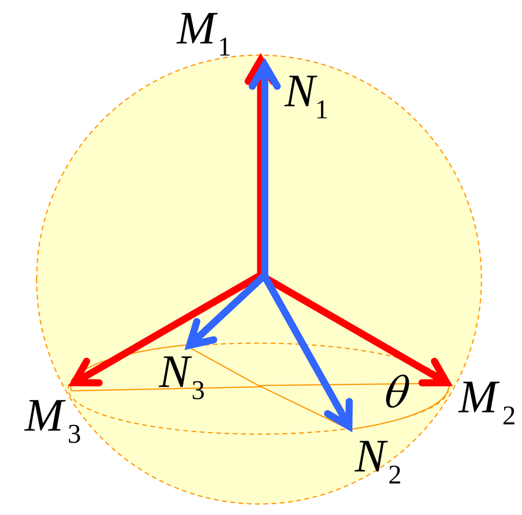
Let us consider two symmetric 3-outcome qubit POVMs that are mutually rotated by angle θ in the plane, where all the POVM elements lie (in the Bloch representation).



$$M_1 = \frac{2}{3}|0\rangle\langle 0|_Z$$
$$M_2 = R_X[\frac{2\pi}{3}](M_1)$$
$$M_3 = R_X[\frac{4\pi}{3}](M_1)$$
$$N_i = R_X[\theta](M_i) \quad \forall i$$

One can show that for unambiguous discrimination of this two measurements the optimal strategy is also based on maximally entangled state and an IDP measurement discriminating one of the 3 possible pairs.

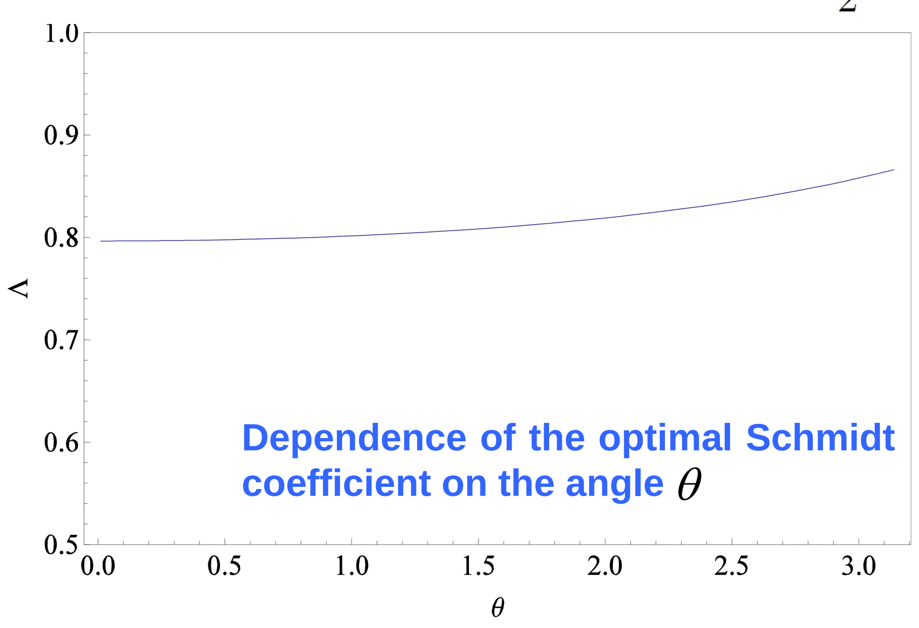
On the other hand, if the two measurements are related by a rotation along a measurement direction Z

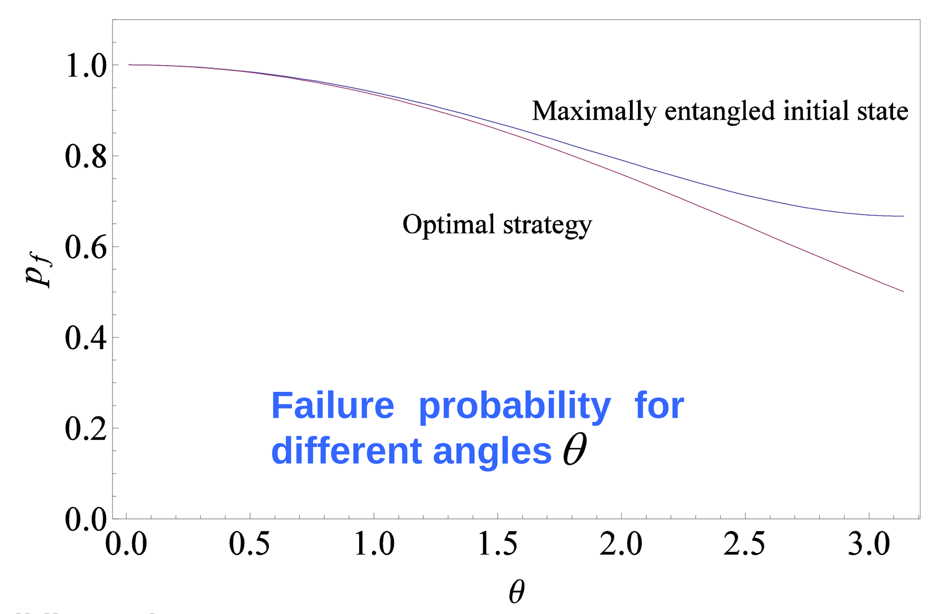


$$N_i = R_Z[\theta](M_i) \quad \forall i$$

then the optimal discrimination strategy needs to be based on less then maximally entangled state. For example, for $\theta = \pi$ the initial entangled state should be

$$|\xi\rangle = \lambda|0\rangle|0\rangle + \sqrt{1-\lambda^2}|1\rangle|1\rangle \quad with\ \lambda = \frac{\sqrt{3}}{2}$$





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