INVESTMENTS IN EDUCATION DEVELOPMENT

Topological phases in quantum lattice models

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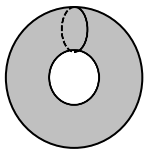
Essential properties of systems described by TQFT

TQFT	Condensed matter	Quantum information
absence of metric $S_{\mathrm{CS}}[A] = \frac{k}{4\pi} \sum_{\ell,m,n} \int_{X} \mathrm{d}^3x\, \epsilon_{\ell mn} \mathrm{tr} \left(A_\ell \cdot \partial_m A_n + \frac{2i}{3} A_\ell \cdot A_m \cdot A_n \right)$ - no clocks, no rulers	**topological phases**	**no local errors**
$\mathcal{H} = 0$	**ground state**	**fault-tolerance**
finite-dimensional Hilbert space on compact manifolds without boundary  $Z(\Sigma \times S^1) = N_\Sigma$ e.g. k^{genus}	**finite** ground state degeneracy (on torus) and **gap** Δ to the first excited state	**qubits** **exponential suppression of non-local errors** $\exp(\Delta/k_B T)$
bulk TQFT/ edge CFT correspondence on manifolds with boundary	**edge modes,** e.g. Majorana bound states **transport/braiding**	**quantum information processing**

Outline

Topological phases and lattice models

- Toric code
- Kitaev honeycomb lattice model

Novel exact solution of the Kitaev model and topological phase transition

- hard-core boson - effective spin representation and fermionization
- topological degeneracy on torus

Edge and vortex modes and non-Abelian fractional statistics

Phys. Rev. B 80, 125415 (2009)
Phys. Rev. Lett. 101, 240404 (2008)

Exact solution of the Yao-Kivelson model

- ground state degeneracy on torus
- phase diagram

Phys. Rev. B 82, 125122 (2010),
JSTAT – Th. Exp. P06020 (2011),
New J. Phys. 14, 045007 (2012)

Square-octagon model

- kaleidoscope of topological phases

Phys. Rev. B 81, 104429 (2010)
New J. Phys 13, 095014 (2011)

Toric code

$$H = - \Sigma_{plaquette}\ Q_p - \Sigma_{stars}\ Q_s$$

$$Q_p = \Pi_{plaq.} \sigma^z = \Pi_{plaq.} \sigma^z{}_1 \sigma^z{}_2 \sigma^z{}_3 \sigma^z{}_4$$

Eigenvalues and eigenvectors of - Q_p :

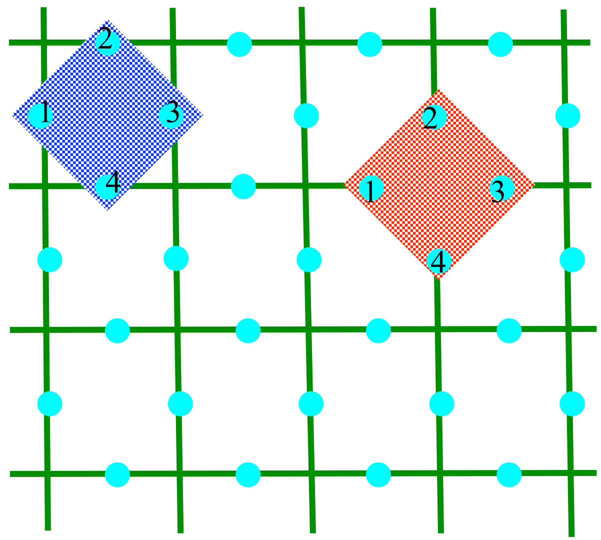
-1: {|0000>, |0011>, |0110>, |0101>,

|1111>, |1100>, |1001>, |1010>}
(even bit string parity states)

+1: {|0001>, |0010>, |0100>, |1000>,

|1110>, |1101>, |1011>, |0111>}
(odd bit string parity)

$$Q_s = \Pi_{star} \sigma^x = \Pi_{star} \sigma^x{}_1 \sigma^x{}_2 \sigma^x{}_3 \sigma^x{}_4$$



A.Y.Kitaev, Ann. Phys. 303, 2 (2003)

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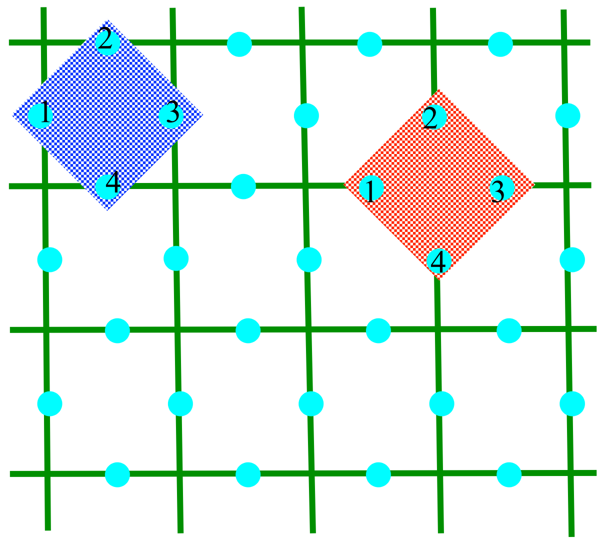

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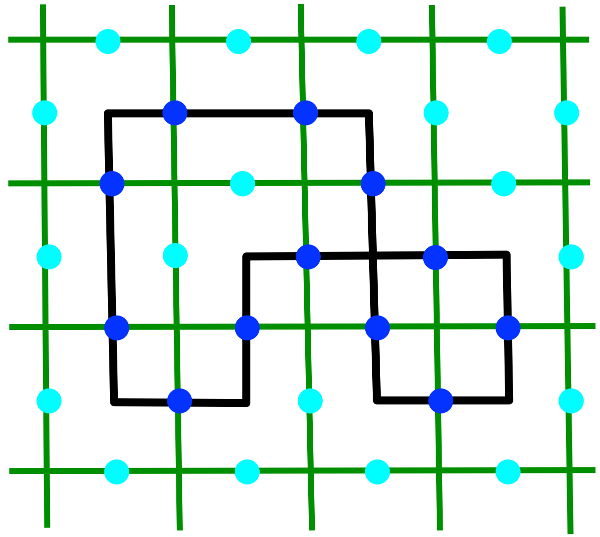

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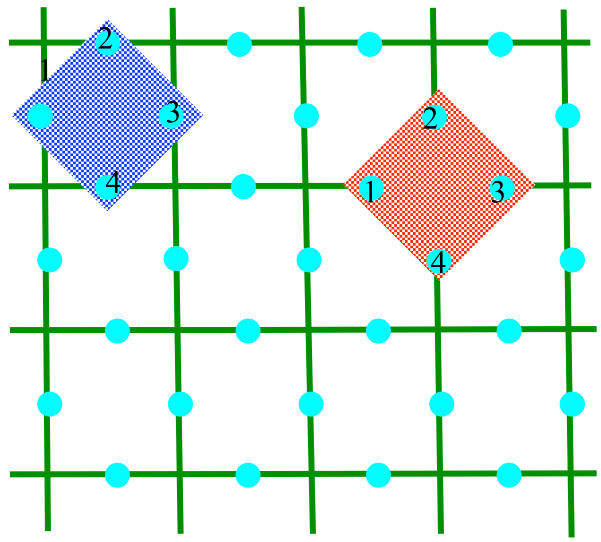

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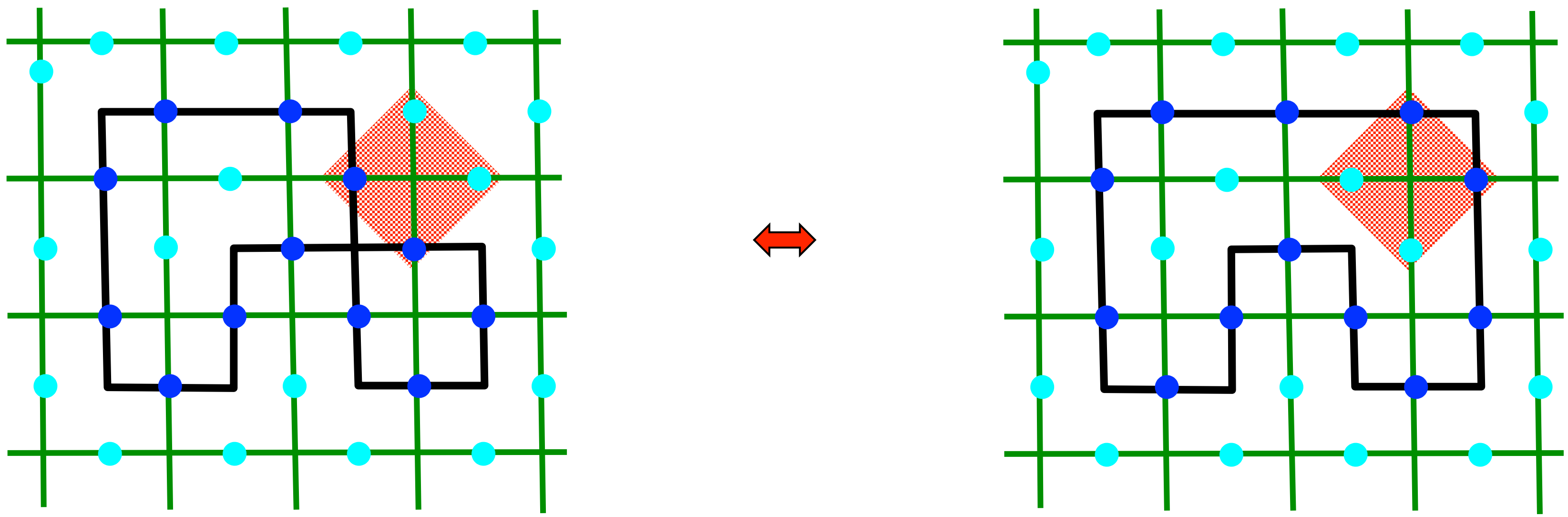
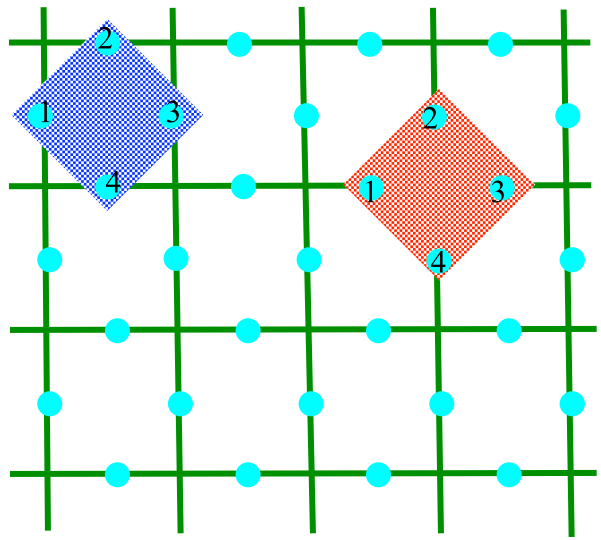


Toric code

$H = -\ \Sigma_{plaquette}\ Q_p\ -\ \Sigma_{stars}\ Q_s$



$Q_p = \Pi_{plaq.}\sigma^z = \Pi_{plaq.}\sigma^z{}_1\sigma^z{}_2\sigma^z{}_3\sigma^z{}_4$

$Q_s = \Pi_{star}\sigma^x = \Pi_{star}\sigma^x{}_1\sigma^x{}_2\sigma^x{}_3\sigma^x{}_4$

$[Q_s, Q_p] = 0$

$[Q_{s/p}, H] = 0$

A.Y.Kitaev, Ann. Phys. 303, 2 (2003)

In the ground state, all Q_p and Q_s operators must equal +1, we say

|g.s.> is **stabilized** by all Q_p and Q_s operators: i.e.

$$\boldsymbol{Q_{p/s}|\mathrm{g.s.}\rangle = +1\ |\mathrm{g.s.}\rangle}$$

for all plaquettes and stars.

Toric code

$$H = -J\left(\Sigma_{stars}\ Q_s\ + \Sigma_{plaquette}\ Q_p\right)$$

$$Q_p = \Pi_{plaq.}\sigma^z = \Pi_{plaq.}\sigma^z{}_1\sigma^z{}_2\sigma^z{}_3\sigma^z{}_4$$

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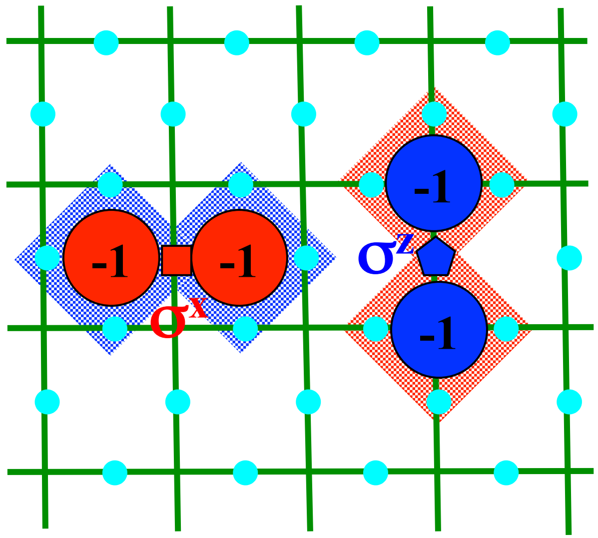


In the ground state, spins on all plaquettes and stars have to have even bit string parity (even number of spins points up).

Applying a single Pauli operator σ^x or σ^z on one spin changes the parity on neighboring plaquettes or stars respectively, thus exciting two quasiparticles.

<u>Four types of quasiparticles:</u>

Trivial (topological charge 0)

Magnetic charges m

- **σ^x** applied on g.s. flips $\langle Q_p \rangle$ to -1 on neighboring plaquettes and thus excites **2 magnetic charges;**
- the operators Q_s remain unaffected (commute with σ^x).

Electric charges e

- similarly **σ^z** flips $\langle Q_s \rangle$ to -1 on both connected stars;
- the operators Q_p remain unaffected (commute with σ^z).

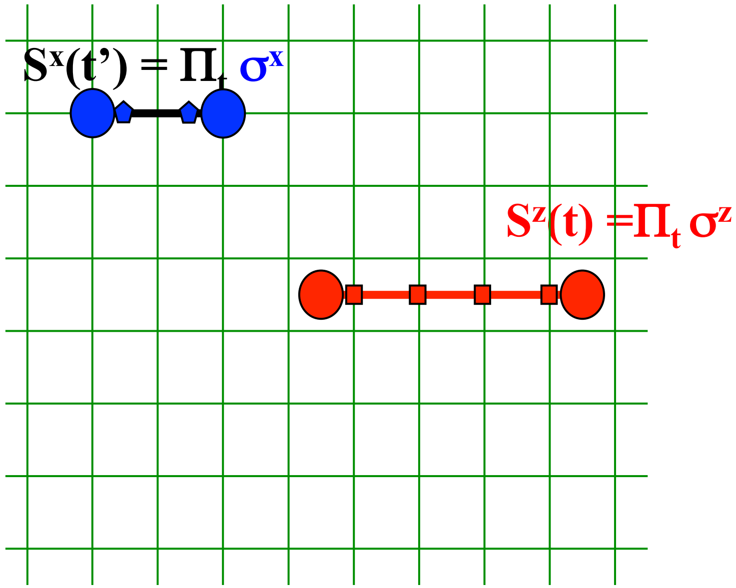
Composites em

Fractional statistics

Quasiparticles are moved and braided by applying an appropriate chain of Pauli operators:

Initial state with two magnetic and electric charges

$|\psi_{init}\rangle = S^z(t)\ S^x(t')\ |g.s.\rangle$



Fractional statistics

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Initial state with two magnetic and electric charges

$|\psi_{init}\rangle = \mathbf{S^z(t)}\ \mathbf{S^x(t')}\ |g.s.\rangle$

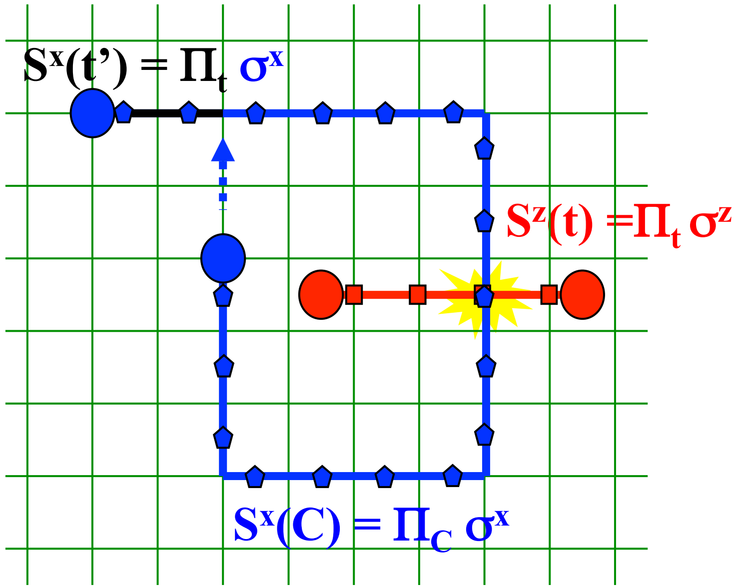
Final state after braiding one electric charge around magnetic particle

$|\psi_{final}\rangle = \mathbf{S^x(C)}\ \mathbf{S^z(t)}\ \mathbf{S^x(t')}\ |g.s.\rangle$

$|\psi_{final}\rangle = \mathbf{-}\ \mathbf{S^z(t)}\ \mathbf{S^x(C)}\ \mathbf{S^x(t')}\ |g.s.\rangle$

$S^x(t')\ |g.s.\rangle = S^x(C)\ S^x(t')\ |g.s.\rangle$

$|\psi_{final}\rangle = -\ \mathbf{S^z(t)}\ \mathbf{S^x(t')}\ |g.s.\rangle = -\ |\psi_{init}\rangle$



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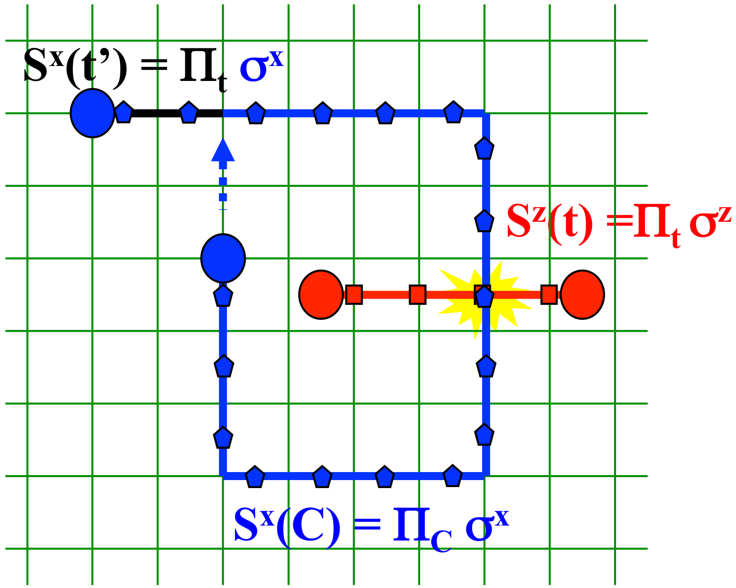
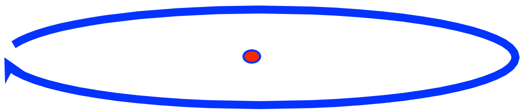


Thus encircling one magnetic charge by one electric charge

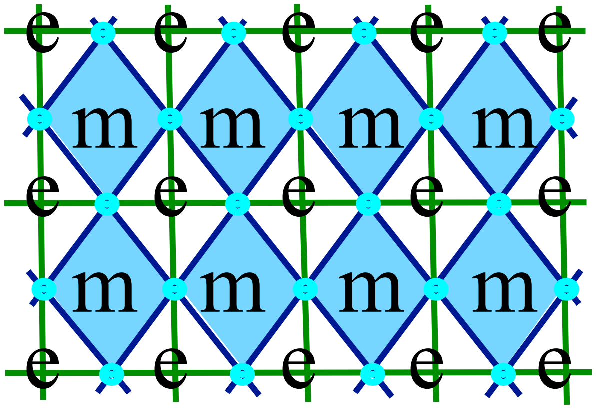


two exchanges: $|\psi\rangle \longrightarrow -1.|\psi\rangle$

reveals that teh particles behave as **anyons with Abelian fraction statistics**.

Unitarily equivalent toric code

We can redefine the toric code in such a way that the spins are on vertices of a square lattice



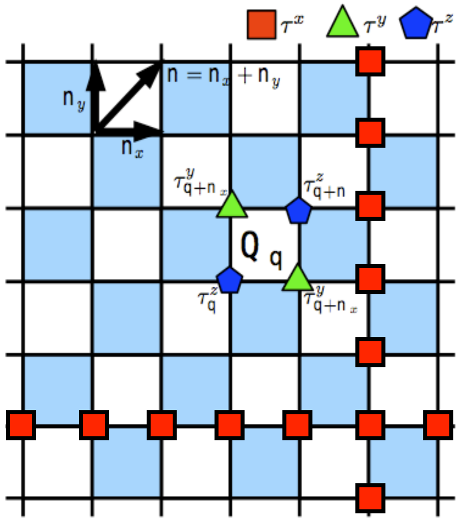
The transformed toric code Hamiltonian is then given only in the terms of new plaquette operators

$$H_{TC} = -J_{eff} \Sigma_p Q_p$$

$$Q_p = \boldsymbol{\tau}^z{}_p \boldsymbol{\tau}^y{}_{p+n_x} \boldsymbol{\tau}^y{}_{p+n_y} \boldsymbol{\tau}^z{}_{p+n_x+n_y}$$

This toric code connects naturally with the Kitaev honeycomb lattice model.

$\mathcal{D}(Z_2)$ topological phase "Toric code"



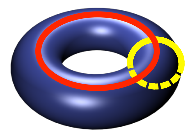
$$H_{TC} = -J\,\Sigma_p\, Q_p \qquad [Q_p, Q_q] = 0 \qquad [H_{TC}, Q_p] = 0$$

$\boldsymbol{Q}_p\, |\{Q_p\}\rangle = Q_p\, |\{Q_p\}\rangle$ where $Q_p = \pm 1$

$$Q_p = \tau^z_p \tau^y_{p+n_x} \tau^y_{p+n_y} \tau^z_{p+n_x+n_y}$$

Ground state: $\boldsymbol{Q}_p\, |\{Q_p\}\rangle = |\{Q_p\}\rangle$ on plane

on torus $\boldsymbol{Q}_p\, |\{Q_p\}, l_x, l_y\rangle = |\{Q_p\}, l_x, l_y\rangle$



$l_x, l_y = \pm 1$ => **Degeneracy(T^2) = 4**

Particle set

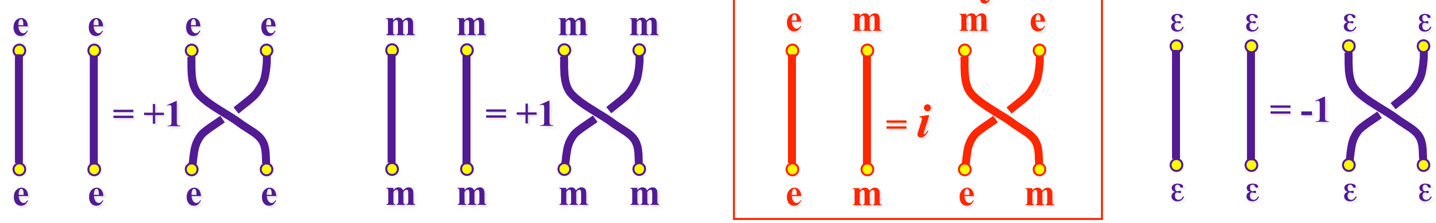
- trivial topological charge **1**
- electric charge **e**
- magnetic charge **m**
- fermion **ε**

Fusion rules

e x e = m x m = ε x ε = 1
e x m = ε
e x ε = m
m x ε = e

Braiding

bosons **bosons** **Abelian anyons** **fermions**



Ising topological phase

Particle set

- trivial topological charge **1**
- non-Abelian (Ising) anyon σ
- fermion ε

Degeneracy of torus

Deg(T^2) = 3

Fusion rules

$\varepsilon \times \varepsilon = 1$

$\varepsilon \times \sigma = \sigma$

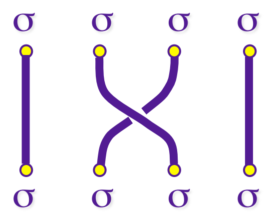
$\sigma \times \sigma = 1 + \varepsilon$

Braidings

non-Abelian anyons

$$|\psi_1^{\sigma\sigma}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\psi_\varepsilon^{\sigma\sigma}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

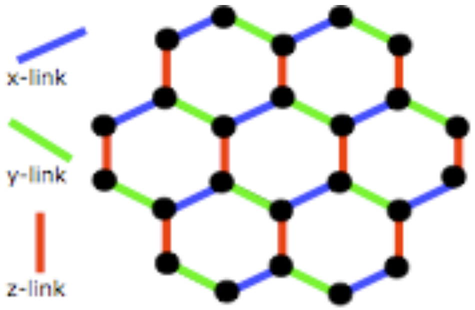
$$R = \begin{pmatrix} \exp\left\{i\frac{\pi}{8}\right\} & 0 \\ 0 & \exp\left\{-i\frac{3\pi}{8}\right\} \end{pmatrix}$$



Realizations of non-Abelian anyons σ

- **Ising lattice models - Kitaev honeycomb model, Yao-Kivelson model**
- e/4 charge quasiparticle in fractional quantum Hall state at the filling ν=5/2
- (half-)vortices in p-wave superconductors

Kitaev honeycomb lattice model



$$H_0 = J_x \sum_{\substack{i,j \\ x\text{-link}}} \sigma^x_i \sigma^x_j + J_y \sum_{\substack{i,j \\ y\text{-link}}} \sigma^y_i \sigma^y_j + J_z \sum_{\substack{i,j \\ z\text{-link}}} \sigma^z_i \sigma^z_j$$

$$= \Sigma_\alpha J_\alpha \Sigma_{i,j}\, \sigma^\alpha_i \sigma^\alpha_j = \Sigma_\alpha J_\alpha \Sigma_{i,j}\, K^\alpha_{ij} \qquad \alpha\text{ -link}$$

Kitaev, Ann. Phys. 321, 2 (2006)

Analytical insights into the model on a plane at the thermodynamic limit:

Phase diagram:

• phase A

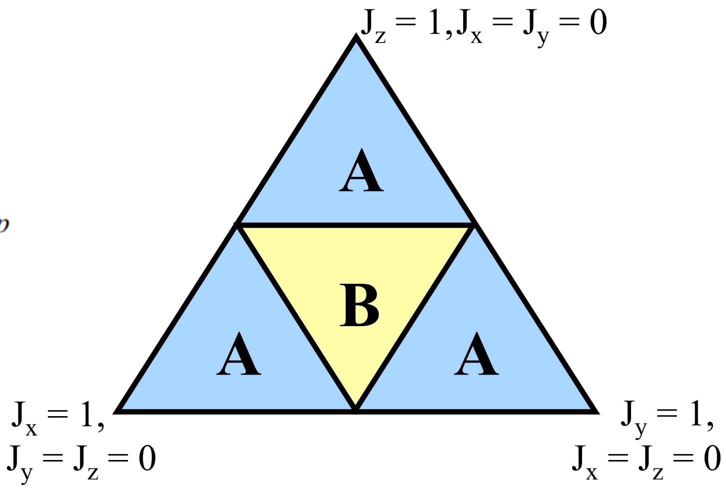
- can be mapped perturbatively onto **the toric code**:

$$|\uparrow>_{eff} = |\uparrow\uparrow>$$
$$|\downarrow>_{eff} = |\downarrow\downarrow>$$

$$H_{\text{eff}} = -\frac{J_x^2 J_y^2}{16|J_z|^3} \sum_p Q_p$$

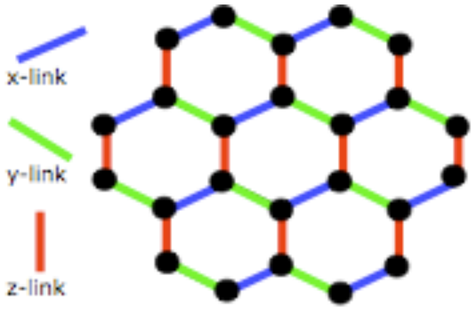
with the ground state stabilized by the plaquette operators:

$$\boldsymbol{Q}_{\mathbf{p}} |\{Q_p\}> = |\{Q_p\}>$$



• **phase B** - gapless.

Kitaev honeycomb lattice model



$$H_0 = J_x \sum_{\substack{i,j\\ \text{x-link}}} \sigma^x_i \sigma^x_j + J_y \sum_{\substack{i,j\\ \text{y-link}}} \sigma^y_i \sigma^y_j + J_z \sum_{\substack{i,j\\ \text{z-link}}} \sigma^z_i \sigma^z_j$$

$$= \Sigma_\alpha J_\alpha \Sigma_{i,j} \sigma^\alpha_i \sigma^\alpha_j = \Sigma_\alpha J_\alpha \Sigma_{i,j} K^\alpha_{ij} \qquad \alpha \text{ -link}$$

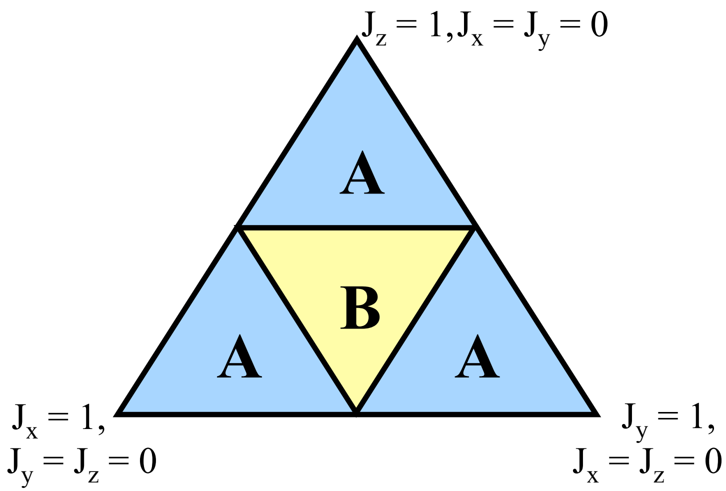
Kitaev, Ann. Phys. 321, 2 (2006)

Analytical insights into the model on a plane at the thermodynamic limit:

Including parity and time reversal breaking:

$$H = H_0 + H_1 = H_0 - \kappa \Sigma_q \underbrace{\Sigma^6_{l=1} P(q)^{(l)}_{,i}}$$

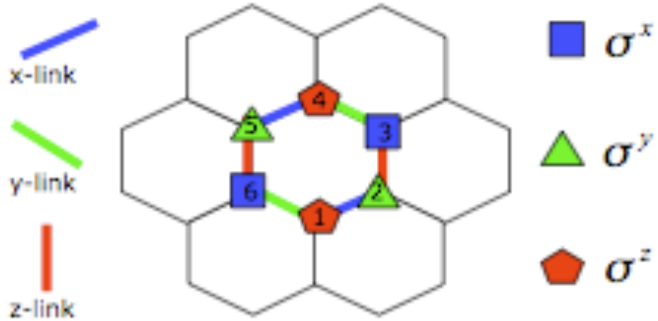
$$\sigma^x_1 \sigma^y_6 \sigma^z_5 + \sigma^z_2 \sigma^y_3 \sigma^x_4 + \sigma^y_1 \sigma^x_2 \sigma^z_3$$
$$+ \sigma^y_4 \sigma^x_5 \sigma^z_6 + \sigma^x_3 \sigma^z_4 \sigma^y_5 + \sigma^y_2 \sigma^z_1 \sigma^x_6$$



- corresponds to perturbative effect of magnetic field

- **phase B** acquires a gap and becomes
  non-abelian topological phase of the Ising type

Vortex operators



$$W_p = K^z_{1,2}K^x_{2,3}K^y_{3,4}K^z_{4,5}K^x_{5,6}K^y_{6,1} = \sigma^x_1\sigma^y_2\,\sigma^z_3\sigma^x_4\,\sigma^y_5\sigma^z_6$$

The vortex operators commute with H_0

$$[H_0, W_p] = 0$$

$$H_0 = -\Sigma_\alpha J_\alpha \Sigma_{i,j} K^\alpha_{ij}$$

$$K^\beta_{k+1,k+2}K^\alpha_{k,k+1} = -K^\alpha_{k,k+1}K^\beta_{k+1,k+2} \qquad (K^\alpha_{k,k+1})^2 = 1$$

and they also commute with the perturbative magnetic field term in the Hamiltonian:

$$H_1 = -\kappa\sum_{\boldsymbol{q}}\sum_{l=1}^{6} P(\boldsymbol{q})^{(l)} \qquad \sum_{l=1}^{6} P(\boldsymbol{q})^{(l)} = \sigma_1^x\sigma_6^y\sigma_5^z + \sigma_2^z\sigma_3^y\sigma_4^x + \sigma_1^y\sigma_2^x\sigma_3^z + \sigma_4^y\sigma_5^x\sigma_6^z + \sigma_3^x\sigma_4^z\sigma_5^y + \sigma_2^y\sigma_1^z\sigma_6^x$$

so for each energy eigenstate, we either have no vortex (+1) or a vortex (-1) at a plaquette p

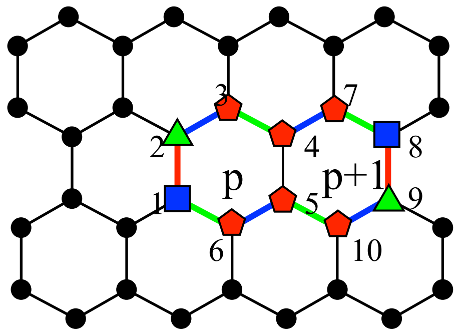
$$\boldsymbol{W_p}\,|E_n> \; = \pm 1\,|E_n>$$

The Hilbert space splits into vortex sectors, i.e. subspaces with particular vortex configurations

$$L = \bigoplus_{w_1,\ldots\ldots,w_m} L_{w_1,\ldots.,w_m}$$

Loop symmetries

Products of the vortex operators are loops:



$$K_{i,j}^{\alpha(1)} K_{j,k}^{\alpha(2)} \ldots K_{p,q}^{\alpha(M-1)} K_{q,i}^{\alpha(M)}$$

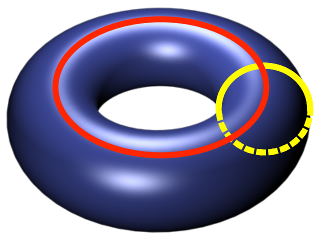
$(K^{\alpha}_{k,k+1})^2 = 1$

<u>Loops on the torus</u>

For a system of N spins on a torus (i.e. a system with N/2 plaquettes),
$\Pi_p W_p = 1$ which implies that there are N/2-1 independent vortex quantum numbers $\{w_1, \ldots , w_{N/2-1}\}$,

- all homologically trivial loops are generated by plaquette operators

- in addition, two distinct homologically nontrivial loops are needed to generate the full loop symmetry group (the third nontrivial loop is a product of these two).



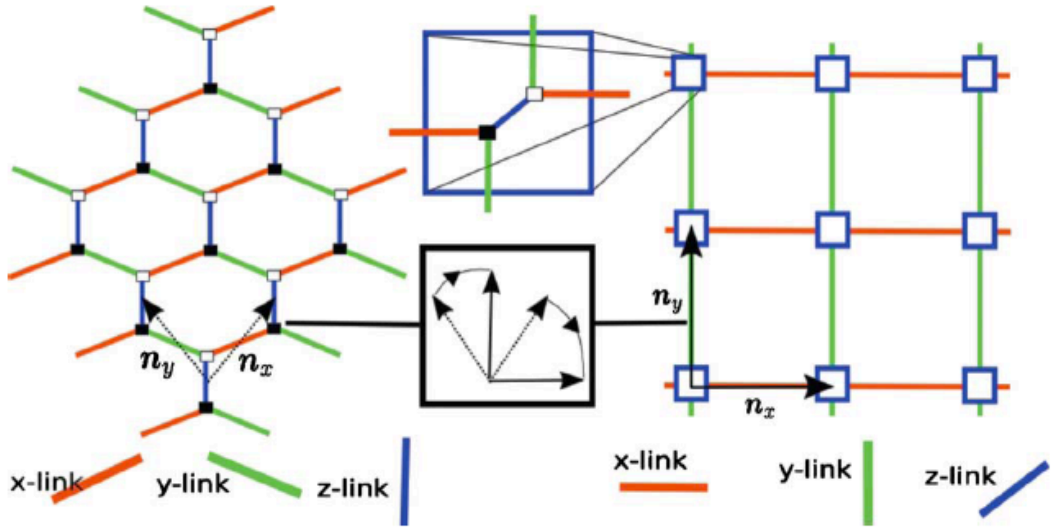
Effective spin – hardcore boson representation

Phys. Rev. B 80, 125415 (2009)
Phys. Rev. Lett. 101, 240404 (2008)

Kitaev model **exactly** maps onto
spin-hardcore boson representation

$$|\uparrow_{\blacksquare}\uparrow_{\square}\rangle = |\Uparrow,0\rangle, \quad |\downarrow_{\blacksquare}\downarrow_{\square}\rangle = |\Downarrow,0\rangle,$$
$$|\uparrow_{\blacksquare}\downarrow_{\square}\rangle = |\Uparrow,1\rangle, \quad |\downarrow_{\blacksquare}\uparrow_{\square}\rangle = |\Downarrow,1\rangle$$

Vidal, Schmidt, Dusuel



Pauli operators:

$$\sigma^x_{q,\blacksquare} = \tau^x_q(b^\dagger_q + b_q), \quad \sigma^x_{q,\square} = b^\dagger_q + b_q,$$
$$\sigma^y_{q,\blacksquare} = \tau^y_q(b^\dagger_q + b_q), \quad \sigma^y_{q,\square} = i\tau^z_q(b^\dagger_q - b_q),$$
$$\sigma^z_{q,\blacksquare} = \tau^z_q, \quad \sigma^z_{q,\square} = \tau^z_q(I - 2b^\dagger_q b_q),$$

Vortex and plaquette operators:

$$\boxed{W_q = (I - 2N_q)(I - 2N_{q+n_y})Q_q}$$

$$N_q = b^\dagger_q b_q \qquad Q_q = \tau^z_q \tau^y_{q+n_x} \tau^y_{q+n_y} \tau^z_{q+n}$$

This allows to write down an orthonormal basis of the full system in terms of the **toric code operators**:

$$|\{Q_q\}, \{N_q\}>$$

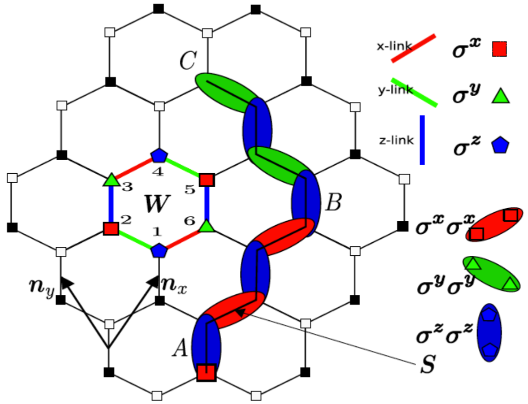
where $\{Q_q\}$ lists all honeycomb plaquette operators and $\{N_q\}$ lists the position vectors of any occupied bosonic modes. **On a torus,** the homologically nontrivial symmetries must be added

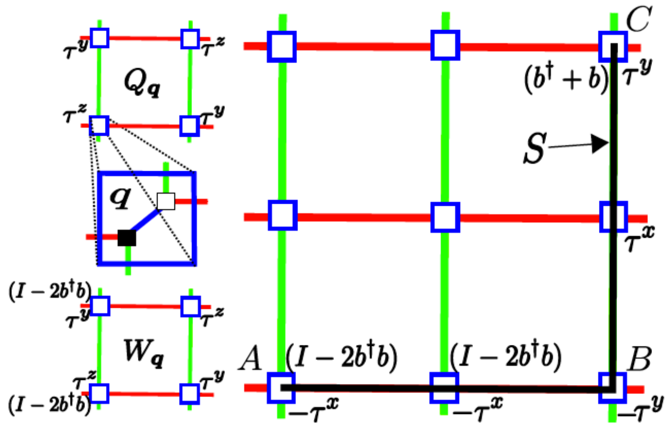
$$|\{Q_q\}, l_x, l_y, \{N_q\}>$$

Jordan-Wigner fermionization

$$S_q = c_q^\dagger + c_q = (b_q^\dagger + b_q) S'_q$$

Phys. Rev. B 80, 125415 (2009)

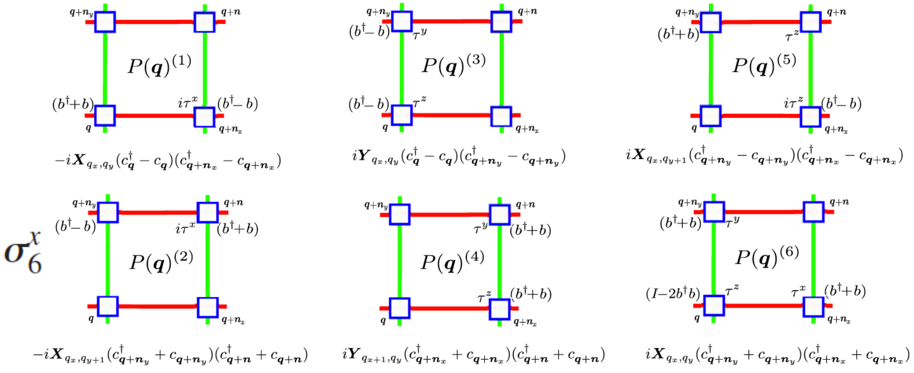




- including "magnetic field"

$$H_1 = -\kappa \sum \sum^{6} P(q)^{(l)}$$

$$\sigma_1^x\sigma_6^y\sigma_5^z + \sigma_2^z\sigma_3^y\sigma_4^x + \sigma_1^y\sigma_2^x\sigma_3^z + \sigma_4^y\sigma_5^x\sigma_6^z + \sigma_3^x\sigma_4^z\sigma_5^y + \sigma_2^y\sigma_1^z\sigma_6^x$$



- including torus topology and general vortex configurations

Quadratic fermionic Hamiltonian

$$H = \frac{1}{2}\sum_{qq'} \begin{bmatrix} c_q^\dagger & c_q \end{bmatrix} \begin{bmatrix} \xi_{qq'} & \Delta_{qq'} \\ \Delta_{qq'}^\dagger & -\xi_{qq'}^T \end{bmatrix} \begin{bmatrix} c_{q'} \\ c_{q'}^\dagger \end{bmatrix}$$

$$\xi_k = \varepsilon_k - \mu$$
$$\Delta_k = \alpha_k + i\beta_k$$
$$\mu = -2J_z$$
$$\varepsilon_k = 2J_x\cos(k_x) + 2J_y\cos(k_y)$$
$$\alpha_k = 4\kappa(\sin(k_x) - \sin(k_y) - \sin(k_x - k_y))$$
$$\beta_k = 2J_x\sin(k_x) + 2J_y\sin(k_y)$$

Fermionization on torus

The general Hamiltonian for an arbitrary vortex configuration

$$H = \frac{1}{2}\sum_{qq'} [c_q^\dagger \quad c_q] \begin{bmatrix} \xi_{qq'} & \Delta_{qq'} \\ \Delta_{qq'}^\dagger & -\xi_{qq'}^T \end{bmatrix} \begin{bmatrix} c_{q'} \\ c_{q'}^\dagger \end{bmatrix}$$

presents the Bogoliubov-de Gennes eigenvalue problem

$$\begin{bmatrix} \xi & \Delta \\ \Delta^\dagger & -\xi^T \end{bmatrix} = \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix} \begin{bmatrix} E & \mathbf{0} \\ \mathbf{0} & -E \end{bmatrix} \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix}^\dagger$$

with quasiparticle excitations

$$[\gamma_1^\dagger, \ldots, \gamma_M^\dagger, \quad \gamma_1, \ldots, \gamma_M] = [c_1^\dagger, \ldots, c_M^\dagger, \quad c_1, \ldots, c_M] \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix}$$

and the system thus reduces to free fermion Hamiltonian

$$H = \sum_{n=1}^{M} E_n \left(\gamma_n^\dagger \gamma_n - \frac{1}{2} \right)$$

Valid for
- all vortex sectors
- all sectors on torus

with the ground state in vortex free sector (in momentum representation) given as BCS states:

Plane:

$$\boxed{|\mathrm{gs}\rangle = \prod_k (u_k + v_k c_k^\dagger c_{-k}^\dagger) \boxed{|\{W_q\}, \{\varnothing\}\rangle}} \rightarrow \boxed{|\{1,1,\ldots,1\},\{0\}>}$$

Torus:

$$\boxed{|gs\rangle_{HC} = \prod_{\mathsf{k}} \left(u_{\mathsf{k}} + v_{\mathsf{k}} c_{\mathsf{k}}^\dagger c_{-\mathsf{k}}^\dagger \right) \boxed{|\{Q_{\mathsf{q}}\}, l_x \;, l_y \;, \{\emptyset\}\rangle}} \rightarrow \boxed{|\{1,1,\ldots,1\},l_x,l_y,\{0\}>}$$

Topological phase transition

The allowed values of momentum k_α in the various homology sectors on torus are given as

$$k_\alpha = \theta_\alpha + 2\pi\, n_\alpha / N_\alpha \qquad \boldsymbol{\theta_\alpha = (l_\alpha + 1)/2N_\alpha} \qquad n_\alpha = 0, 1, \ldots, N_\alpha - 1$$

So thus only one of the four configurations, (l_x= -1, l_y= -1), permits the momentum to be exactly (π, π):

$$\Delta_{\pi,\pi} = \alpha_k + i\,\beta_k = 0 \qquad \text{and thus} \qquad E_{\pi,\pi} = (\xi_{\pi,\pi}{}^2 + |\,\Delta_{\pi,\pi}\,|^2)^{1/2} = |\xi_{\pi,\pi}\,|$$

$$\alpha_k = 4\kappa(\sin(k_x)\text{-}\sin(k_y)\text{-}\sin(k_x\text{-}k_y)) = 0 \text{ and } \beta_k = 2J_x(\sin(k_x) + 2J_y(\sin(k_y) = 0$$

At the phase transition, where $\boldsymbol{J_z = J_x + J_y}$,

$\xi_{\pi,\pi} = \varepsilon_k - \mu = 2J_x\cos(k_x) + 2\,J_y\cos(k_y) - (-2\,J_z\,) = [-2(J_x + J_y\,)] + 2\,J_z$ **changes the sign**, giving

$$\xi_{\pi,\pi}\,/\,E_{\pi,\pi} = \xi_{\pi,\pi}/|\,\xi_{\pi,\pi}\,| = -1 \quad \text{and thus} \quad \boxed{u_{\pi,\pi} = [1/2\,(1 + \xi_{\pi,\pi}\,/\,E_{\pi,\pi}\,)]^{1/2} = 0}$$

$$\boxed{c^+_{\pi,\pi}\, c^+_{-\pi,-\pi} = (c^+_{\pi,\pi}\,)^2 = 0}$$

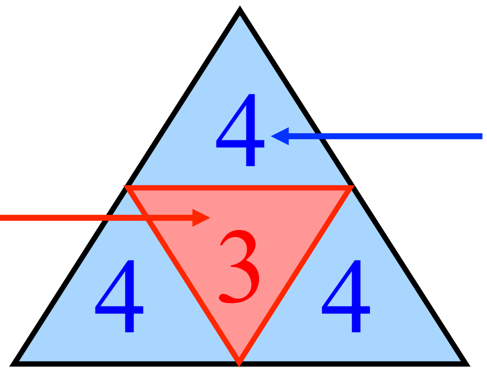
$$|g.s.\rangle = \Pi_k\,(\cancel{u_k} + v_k\, \cancel{c^+_k}\, c^+_{-k})\,|\{Q_q\},\, l_x^{(0)}, l_y^{(0)},\, \{0\}\rangle$$

One of the four BCS states on torus vanishes at transition to Ising phase

Non-Abelian phase

3-fold degenerate

ground state on torus



Abelian phases

4-fold degenerate

ground state on torus

G. Kells, J.K. Slingerland, J. Vala, Phys. Rev. B **80**, 125415 (2009)

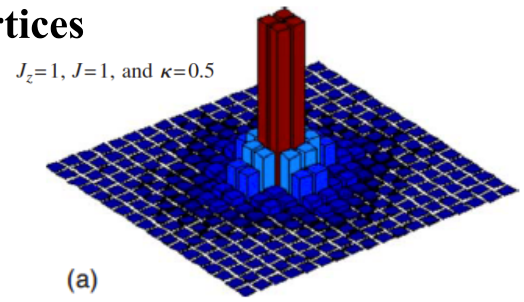
Other results on the Kitaev model

Edge modes and vortex modes

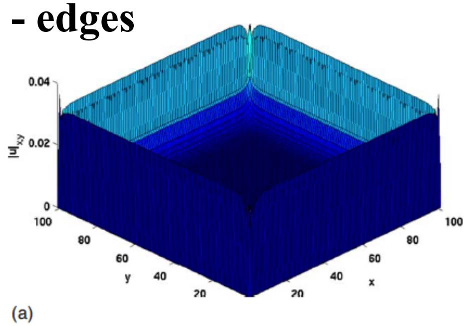
Phys. Rev. B 82, 125122 (2010),
JSTAT – Th. Exp. P06020 (2011)

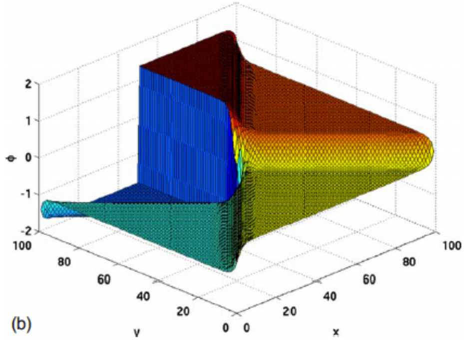
Numerical diagonalization of the Hamiltonian in any vortex sector for systems with 100x100 plaquettes: zero energy modes in the non-Abelian phase around:

- vortices



- edges



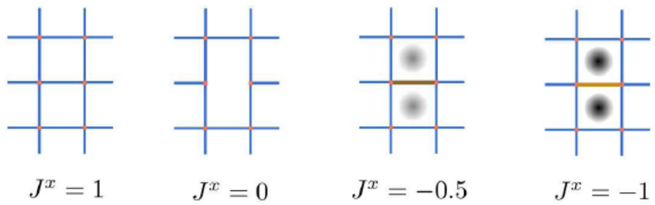


Simulation of non-Abelian anyon braiding – non-Abelian Berry matrix

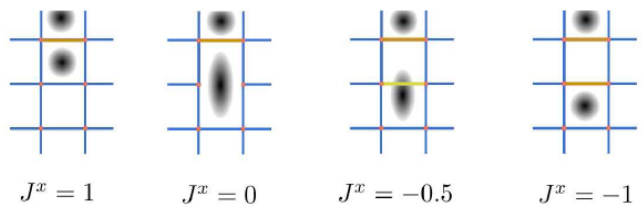
New. J. Phys.
14, 045007 (2012)

Changing the coupling between lattice sites results in

- vortex (anyon) creation

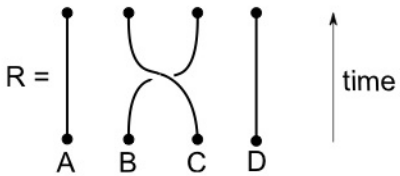


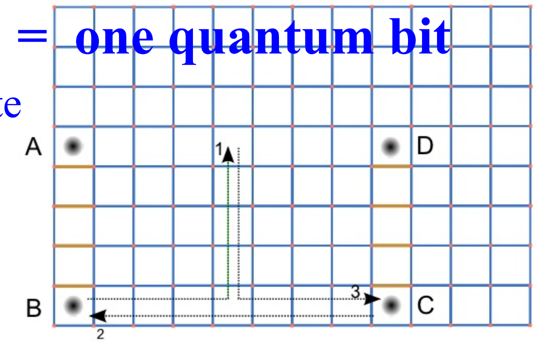
- vortex motion



System of four vortices = one quantum bit

has two-fold degenerate gr. state





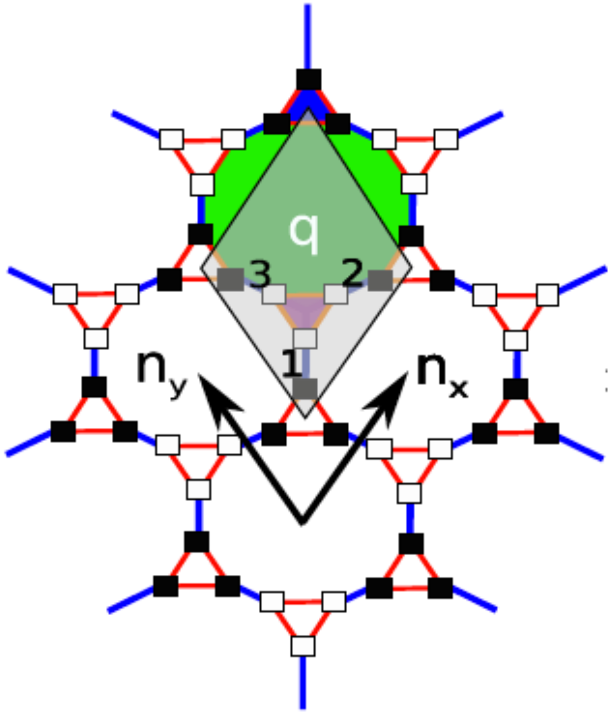
Error of our simulation

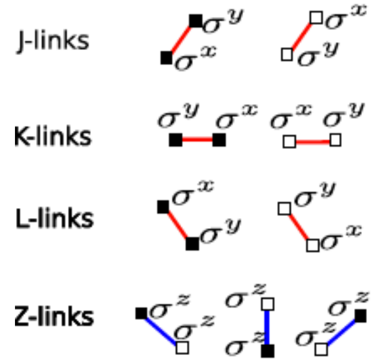
compared to effective field theory

$$\| \Delta \| = \| \mathcal{B}_E - \mathcal{B}_N \|$$
$$= 4.7970 \times 10^{-5}$$

Yao-Kivelson model

$$H = H_Z + H_J + H_K + H_L = -Z \sum_{Z\ \text{links}} \sigma^z \sigma^z - J \sum_{J\ \text{links}} \sigma^x \sigma^y - K \sum_{K\ \text{links}} \sigma^x \sigma^y - L \sum_{L\ \text{links}} \sigma^x \sigma^y$$





$$H_Z = -Z \sum_q \sum_{n=1}^{3} \sigma^z_{q,n,\blacksquare} \sigma^z_{q,n,\square},$$

6N spin system with N unit cells

$$H_J = -J \sum_q \sigma^y_{q,1,\square} \sigma^x_{q,2,\square} + \sigma^y_{q\rightarrow,1,\blacksquare} \sigma^x_{q,2,\blacksquare},$$

$$H_K = -K \sum_q \sigma^y_{q,2,\square} \sigma^x_{q,3,\square} + \sigma^y_{q\nwarrow,2,\blacksquare} \sigma^x_{q,3,\blacksquare},$$

Each spin site is specified by

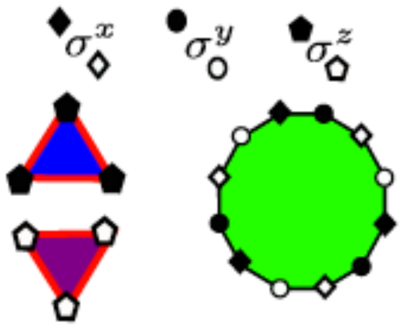
- the position vector q
- the index n
- the black/white

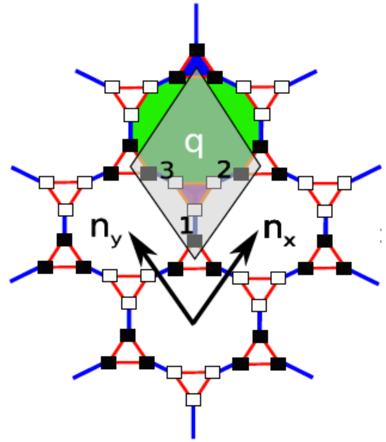
$$H_L = -L \sum_q \sigma^y_{q,3,\square} \sigma^x_{q,1,\square} + \sigma^y_{q\downarrow,3,\blacksquare} \sigma^x_{q,1,\blacksquare},$$

$$q\nwarrow = q - n_x + n_y \qquad q\downarrow = q - n_y$$

Symmetries

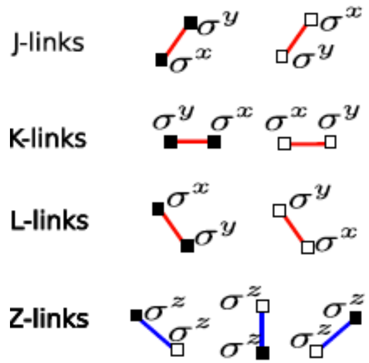
Loop symmetries





Homologically trivial loops $$W_q = \langle n | W_q | n \rangle = \pm 1$$

vortex



Homologically **non**trivial loops on torus

$$x = N_x n_x \qquad L^{(x)}_{q_y} = \prod_{q_x} \sigma^x_{q,1,\square} \sigma^x_{q,1,\blacksquare} \sigma^y_{q,2,\square} \sigma^y_{q,2,\blacksquare}$$

$$y = N_y n_y \qquad L^{(y)}_{q_x} = \prod_{q_y} \sigma^y_{q,1,\square} \sigma^y_{q,1,\blacksquare} \sigma^x_{q,3,\square} \sigma^x_{q,3,\blacksquare}$$

Time-reversal symmetry is spontaneously broken by triangular plaquettes

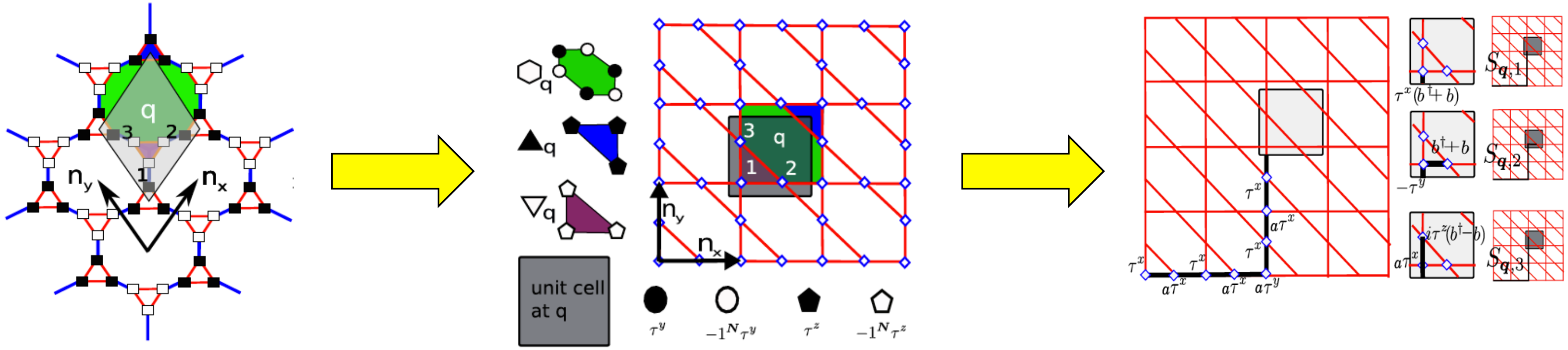
- time-reversal operator, $T\sigma^a = -\sigma^a$, changes the eigenvalues of all triangular plaquettes
- each state is at least two-fold degenerate as $[T, H] = 0$

Effective spin – hardcore boson representation and fermionization

Mapping Z-”dimers” to effective spins and hardcore bosons

$$|\uparrow_{\blacksquare}\uparrow_{\square}\rangle = |\Uparrow,0\rangle, \quad |\downarrow_{\blacksquare}\downarrow_{\square}\rangle = |\Downarrow,0\rangle$$

$$|\uparrow_{\blacksquare}\downarrow_{\square}\rangle = |\Uparrow,1\rangle, \quad |\downarrow_{\blacksquare}\uparrow_{\square}\rangle = |\Downarrow,1\rangle$$



$$H(k) = \begin{bmatrix} \xi(k) & \Delta(k) \\ \Delta(k)^\dagger & -[\xi(-k)]^T \end{bmatrix}$$

Phase diagram: gapped phases

$$H(\boldsymbol{k}) = \begin{bmatrix} \xi(\boldsymbol{k}) & \Delta(\boldsymbol{k}) \\ \Delta(\boldsymbol{k})^{\dagger} & -[\xi(-\boldsymbol{k})]^{T} \end{bmatrix} \qquad \xi = \begin{bmatrix} 2Z & J(1+\theta_x) & L(1+\theta_y) \\ J(1+\theta_x^*) & 2Z & iK(1+\theta_y\theta_x^*) \\ L(1+\theta_y^*) & -iK(1+\theta_x\theta_y^*) & 2Z \end{bmatrix}$$

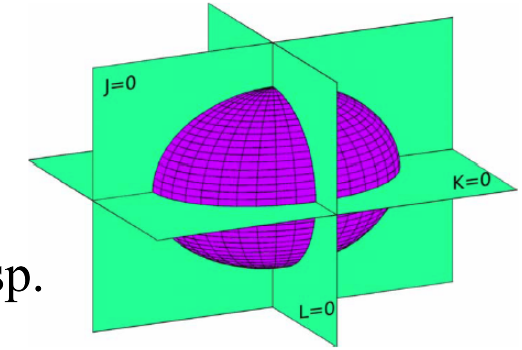
Abelian phase: inside sphere

- The minimum gap always occurs at $\boldsymbol{k} = 0$, coinciding with $\Delta = 0$, so the eigenvalues can be calculated by diagonalizing the matrix ξ

$$E = \pm 2(Z + a\sqrt{J^2+K^2+L^2})$$

with the minimal gap and the phase transition occuring at

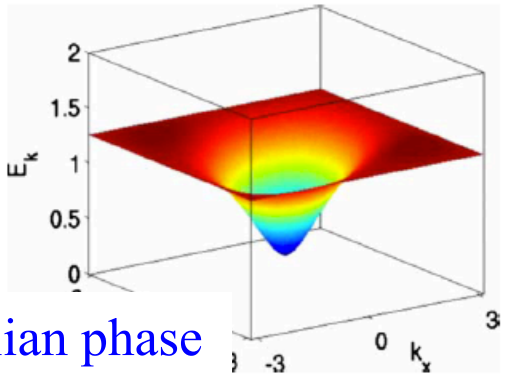
$$|2Z - 2\sqrt{J^2+K^2+L^2}| \qquad \text{and} \qquad Z = \sqrt{J^2+K^2+L^2} \quad \text{resp.}$$



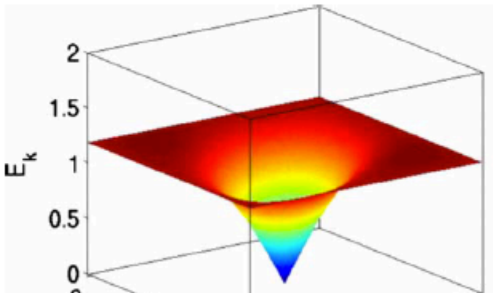
Non-Abelian phase: outside sphere and off planes

Dispersion relations

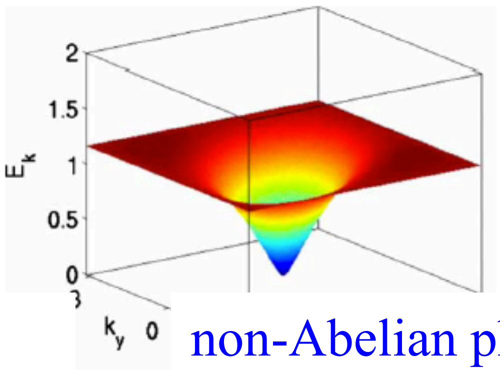
- the minimum gap at k=0 is true also for the J=K=L



Abelian phase



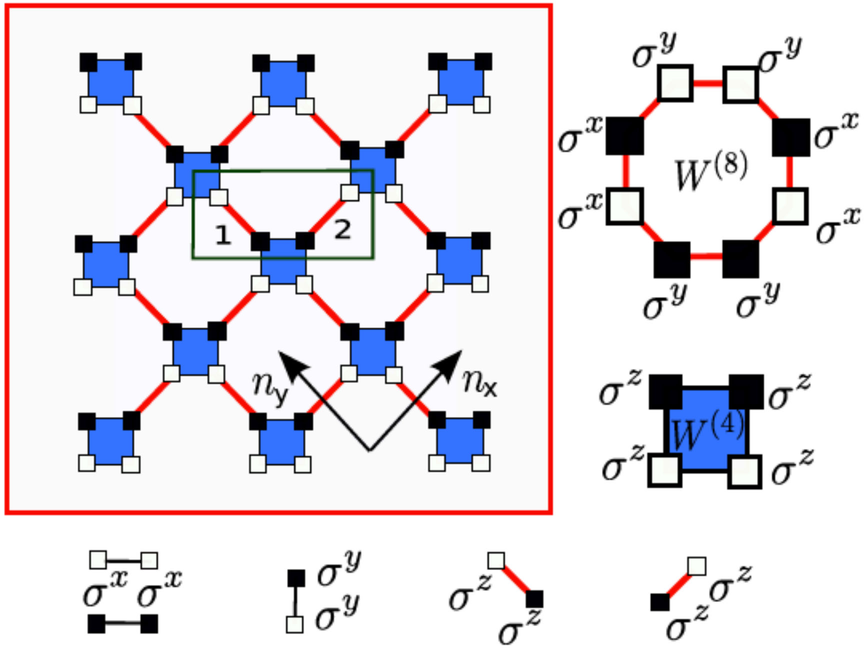
phase transition



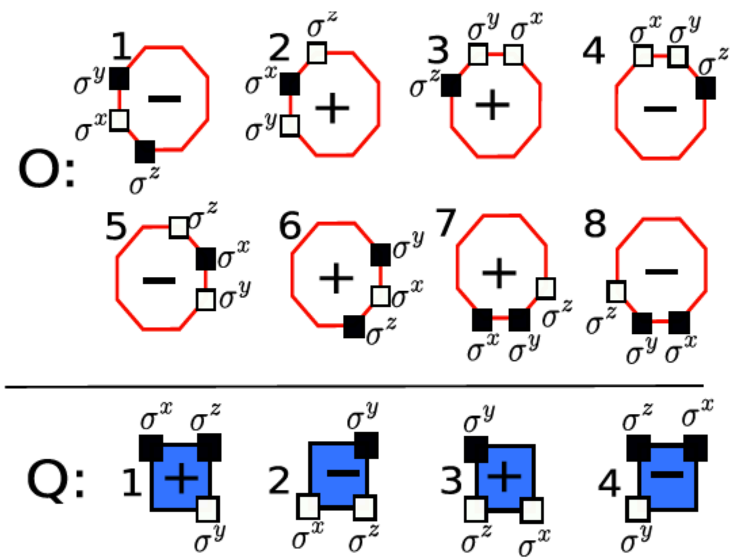
non-Abelian phase

Square-octagon model

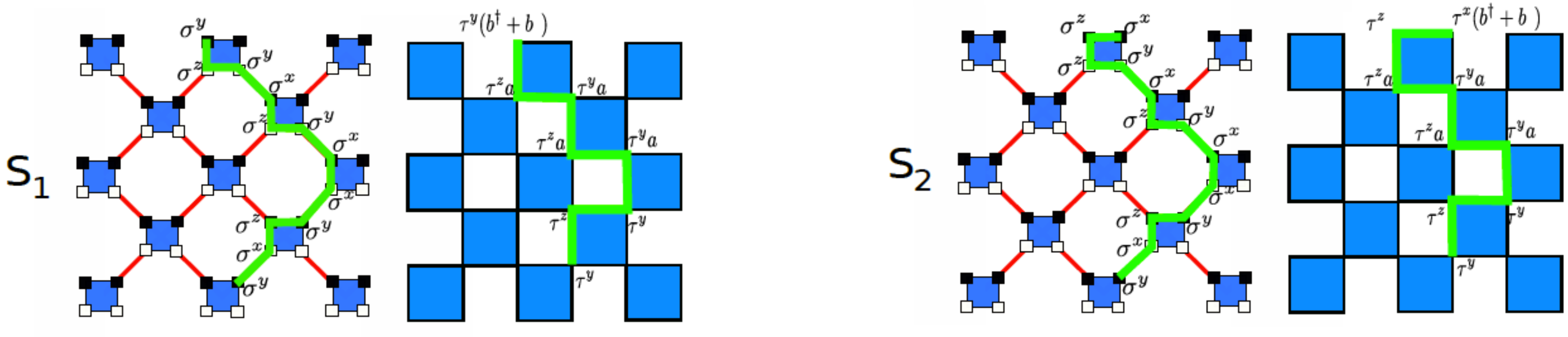
$$H = -J_z \sum_{\text{z-links}} \sigma^z\sigma^z - J_x \sum_{\text{x-links}} \sigma^x\sigma^x - J_y \sum_{\text{y-links}} \sigma^y\sigma^y$$



Time-reversal symmetry breaking terms



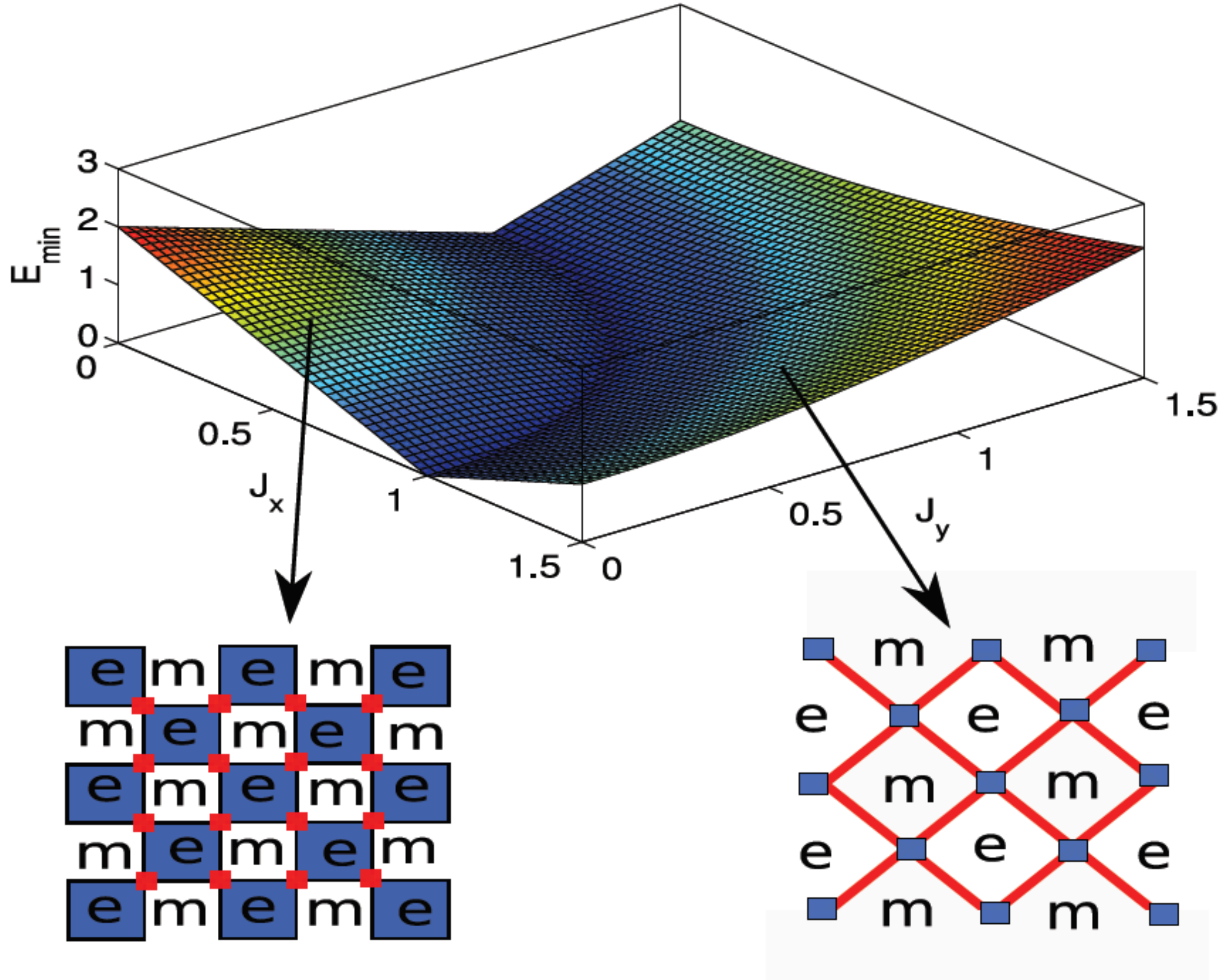
Jordan-Wigner fermionization



Solution and Abelian phases

$$H = \frac{1}{2}\sum_{\boldsymbol{k},nm} \left[\, c^\dagger_{\boldsymbol{k}n} \quad c_{-\boldsymbol{k}n} \,\right] H(\boldsymbol{k}) \begin{bmatrix} c_{\boldsymbol{k}m} \\ c^\dagger_{-\boldsymbol{k}m} \end{bmatrix} \qquad H(\boldsymbol{k}) = \begin{bmatrix} \xi_J(\boldsymbol{k}) + \xi_\kappa(\boldsymbol{k}) & \Delta_J(\boldsymbol{k}) + \Delta_\kappa(\boldsymbol{k}) \\ (\Delta_J(\boldsymbol{k}) + \Delta_\kappa(\boldsymbol{k}))^\dagger & -(\xi_J(-\boldsymbol{k}) + \xi_\kappa(-\boldsymbol{k}))^T \end{bmatrix}$$

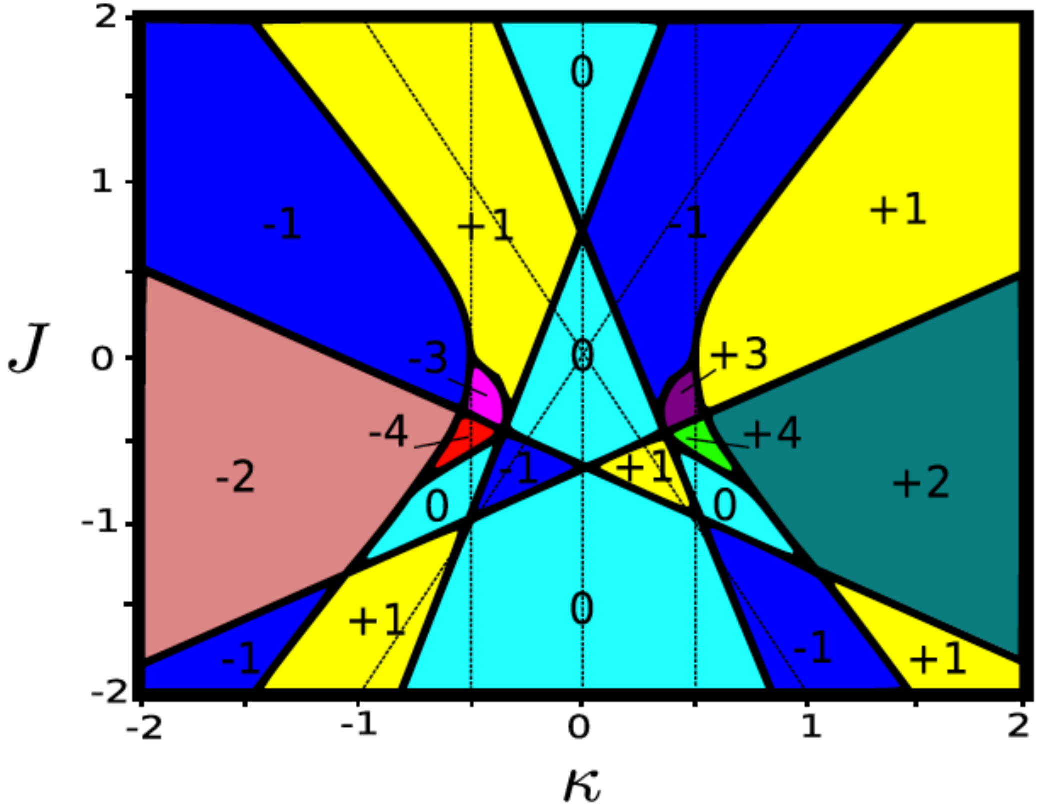
Example: calculation of the gap (abelian phases, no time reversal breaking terms)



Kaleidoscope of topological phases with Majorana species

<u>Chern number</u> $$\nu = \frac{1}{2\pi i}\int dk_x dk_y Tr\left(P(k_x,k_y)\left[\frac{\partial P}{\partial k_x}\frac{\partial P}{\partial k_y} - \frac{\partial P}{\partial k_y}\frac{\partial P}{\partial k_x}\right]\right)$$

where $P = \Sigma\,|n\rangle\langle n|$ projects onto the negative energy eigenspace of M



New J. Phys. 14, 095014 (2011)

Two distinct non-Abelian phases in one system:

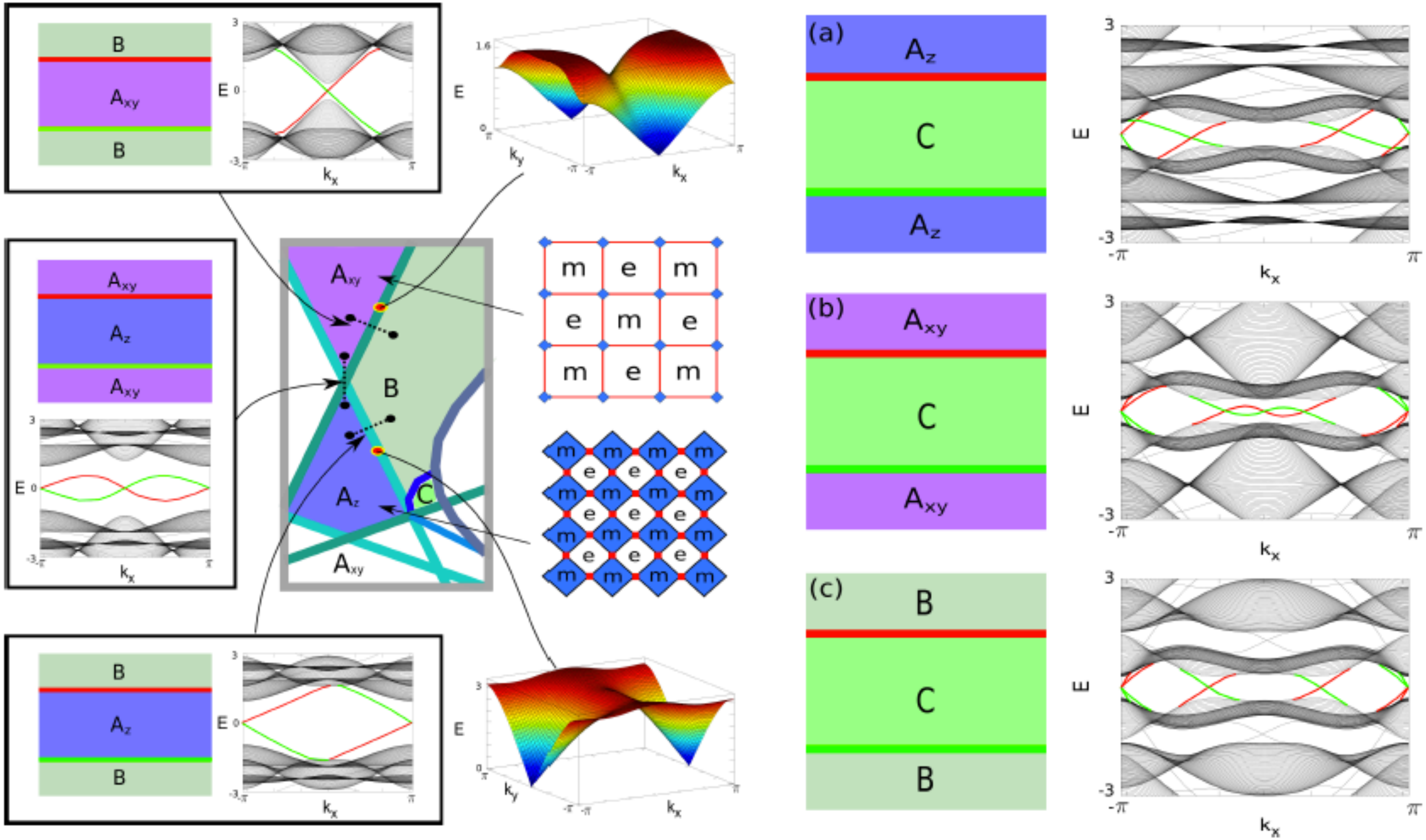
Ising phase characterized by $\nu = \pm 1$

$SU(2)_2$ phase with $\nu = \pm 3$

Topological phase transition and bulk-edge correspondence

$$\#\ \text{left-movers} - \#\ \text{right-movers} = \Delta_{\text{phase boundary}}\ \nu$$

New J. Phys. 14, 095014 (2011)



Conclusions

Novel solution of the Kitaev honeycomb lattice model was presented which combines powerful wavefunction descriptions:

- BCS product
- stabilizer formalism

The novel solution represents a microscopic model of non-Abelian topological phase which

- provides closed expression for the ground state with the vacuum state explicitely given

$$\boxed{|gs\rangle_{HC} = \prod \left(u_{\mathsf{k}} + v_{\mathsf{k}} c_{\mathsf{k}}^{\dagger} c_{-\mathsf{k}}^{\dagger}\right) \boxed{|\{Q_{\mathsf{q}}\},\{\emptyset\}\rangle}}$$

$$\boxed{|gs\rangle_{HC} = \prod_{\mathsf{k}} \left(u_{\mathsf{k}} + v_{\mathsf{k}} c_{\mathsf{k}}^{\dagger} c_{-\mathsf{k}}^{\dagger}\right) \boxed{|\{Q_{\mathsf{q}}\}, l_x^{(0)}, l_y^{(0)}, \{\emptyset\}\rangle}}$$

- yields important insights into the relatins between the toric code and the Ising non-Abelian phase
- allows calculation of the vortex states and edge states
- allows direct calculation of the non-Abelian fractional statistics
- generalizes to other models: Yao-Kivelson model and square octagon model
- square-octagon model exhibits five topological phase including two non-Abelian

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sfi science foundation ireland

ICHEC Irish Centre for High-End Computing

IRCSET