Optimal processing of reversible quantum channels

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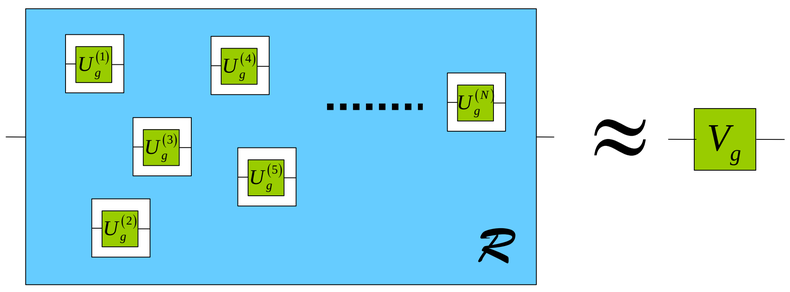
Abstract

We consider the general problem of the optimal transformation of N uses of (possibly different) unitary channels to a single use of another unitary channel in any finite dimension. We show how the optimal transformation can be fully parallelized, consisting in a preprocessing channel followed by a parallel action of all the N unitaries and a final postprocessing channel. Our techniques allow us to achieve an exponential reduction in the number of the free parameters of the optimization problem making it amenable to an efficient numerical treatment. Finally, we apply our general results to find the analytical solution for special cases of interest like the cloning of qubit phase gates.

Definition of the problem:

The goal is to use N unitary channels $\mathcal{U}_g^{(i)}$ (each of them once) to approximate as closely as possible a target unitary channel $\mathcal{V}_g$, which is described by a different unitary representation V_g of the same group G.



When $\mathcal{U}_g^{(i)} = \mathcal{U}_g$ the problem coincides with $N \to M$ cloning of unitary transformations.

Higher order quantum maps

A higher order map is for example a transformation in which the input and the output are quantum transformations themselves. These maps have to obey admissibility conditions (linearity & complete positivity) and can be characterized similarly as channels via Choi-Jamiolkowski isomorphism .

channel **Choi operator of the channel**

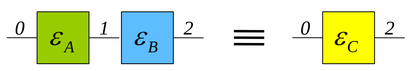
$$\mathcal{C} \leftrightarrow C = \mathcal{C} \otimes \mathcal{I}(|I\rangle\rangle\langle\langle I|) \qquad |A\rangle\rangle = \sum_{i,j} A_{ij}|i\rangle|j\rangle$$

higher order map

$$\mathcal{R} \leftrightarrow R \quad \textbf{Choi operator of the higher order map}$$

Star product – linking wires

Composition of 2 channels motivates star product of Choi matrixes



$$E_A * E_B \equiv E_C = Tr_1\left(E_A \otimes \mathbf{1}_2 . \mathbf{1}_0 \otimes E_B^{T_1}\right)$$

Insertion of a channel into a higher order map also corresponds to a star product.

Figure of merit

Closeness of two quantum channels $\mathcal{C}, \mathcal{D}$ can be quantified by quantum channel fidelity, which is defined via the corresponding Choi operators of the two channels:

$$f(C,D) := \frac{1}{d_0^2}\left(\mathrm{Tr}\left[\sqrt{\sqrt{C}D\sqrt{C}}\right]\right)^2$$

As a figure of merit for our task we use the channel fidelity between $R * \bigotimes_{i=1}^N |U_g^{(i)}\rangle\rangle\langle\langle U_g^{(i)}|$ and $|V_g\rangle\rangle\langle\langle V_g|$ uniformly averaged over the unknown unitaries, that is

$$F(R) = \int \mathrm{d}g\, f\left(R * \bigotimes_{i=1}^N |U_g^{(i)}\rangle\rangle\langle\langle U_g^{(i)}|, |V_g\rangle\rangle\langle\langle V_g|\right)$$

$$= \frac{1}{d_0^2}\int \mathrm{d}g\, \mathrm{Tr}\left[R\left(|U_g^*\rangle\rangle\langle\langle U_g^*| \otimes |V_g\rangle\rangle\langle\langle V_g|\right)\right]$$

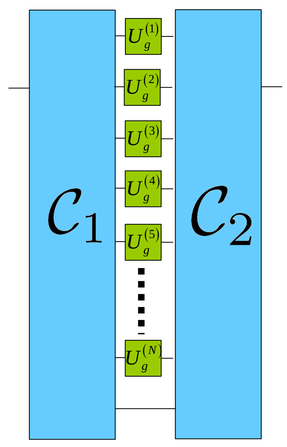
$$U_g := \bigotimes_{i=1}^N U_g^{(i)} .$$

Parallel strategy is optimal

Thanks to the symmetry of the figure of merit the optimal higher order map $\mathcal{R}$ can be chosen to be symmetric:

$$[R, W_h^* \otimes W_g] = 0 \qquad \forall g,h \in G, \qquad \begin{aligned} W_g &= U_g^* \otimes V_g \\ W_h &= U_h^* \otimes V_h \end{aligned}$$

where U_h^* acts on Hilbert space of inputs of all channels $\mathcal{U}_g^{(i)}$ and U_g^* acts on Hilbert space of all outputs.



As a consequence, we can prove that the optimal scheme does not require any non-circuital higher order map, but it can be realized by a three-step protocol:

i) application of a preprocessing channel $\mathcal{C}_1$,

ii) parallel application of the unitary channels

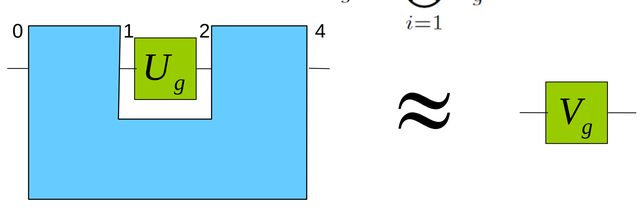
iii) final application of a post-processing channel $\mathcal{C}_2$.

Such strategies correspond to a subclass of higher order maps called deterministic 2-combs. They are characterized by the positive-semidefinitness of their Choi operator and by the following linear equations:

$$\mathrm{Tr}_3[R] = I_2 \otimes S_{10}$$
$$\mathrm{Tr}_1[S] = I_0$$

The N uses of unitary channels $\mathcal{U}_g^{(i)}$ can now be represented by a single quantum channel

$$U_g := \bigotimes_{i=1}^N U_g^{(i)}$$



In order to perform the optimization of the deterministic 2 comb $\mathcal{R}$ we proceed by decomposing the representations U_g, V_g of the group G into its irreducible representations (IRR):

$$U_h = \bigoplus_\beta U_h^{[\beta]} \otimes I_{m_\beta} \qquad V_h = \bigoplus_a V_h^{[a]} \otimes I_{m_a}$$

$$U_g = \bigoplus_\gamma U_g^{[\gamma]} \otimes I_{m_\gamma} \qquad V_g = \bigoplus_d V_g^{[d]} \otimes I_{m_d},$$

Similarly, we need to decompose the coupled representation:

$$V_g^{[d]} \otimes U_g^{[\gamma]*} = \bigoplus_L W_g^{[L]} \otimes I_{m_L^{d,\gamma}}$$

$$V_h^{[a]} \otimes U_h^{[\beta]*} = \bigoplus_K W_h^{[K]} \otimes I_{m_K^{a,\beta}}$$

The particular instance of the problem is then specified by the list of IRR into which U_g, V_g decompose and as we will see the optimal solution also depends on the list of IRR appearing in the coupled representation $U_g^* \otimes V_g$. Thus, the relevant numbers are:

d_a, d_β, d_K - dimensions of the IRR with the corresponding index

m_a - multiplicity of the IRR with index a in the decompositon of V_h

$m_K^{a\beta}$ - multiplicity of the IRR with index K in the decompositon of $V_h^{[a]} \otimes U_h^{[\beta]*}$

Bibliography:

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Quantum Comb applications:

(authors: A. Bisio, G. Chiribella, G. M. D'Ariano, S. Facchini, P. Perinotti)

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The resulting problem

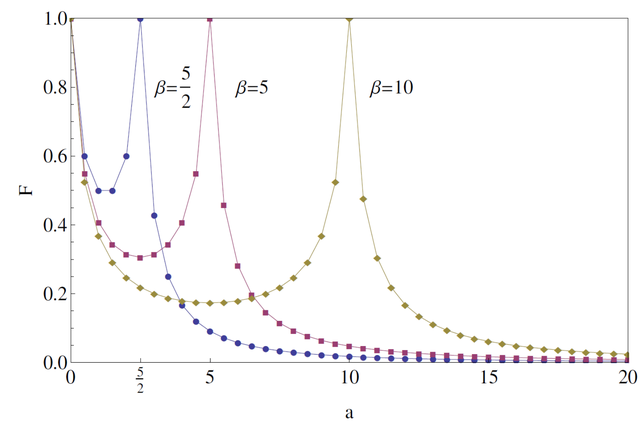
Using the symmetry structure of the problem we further simplified the general problem to an **exponentially smaller semidefinite programmig problem**:

$$\begin{aligned} \underset{p_K^a}{\text{maximize}} \quad & \Phi(p_K^a) = \sum_K \left(\sum_a \sqrt{q_K^a p_K^a}\right)^2 \\ \text{subject to} \quad & \sum_K p_K^a = 1 \quad \forall a \\ & p_K^a \geqslant 0 . \end{aligned}$$

where $q_K^a = \frac{m_a d_a}{d_K d_0^2} \sum_\beta m_K^{a\beta} d_\beta$.

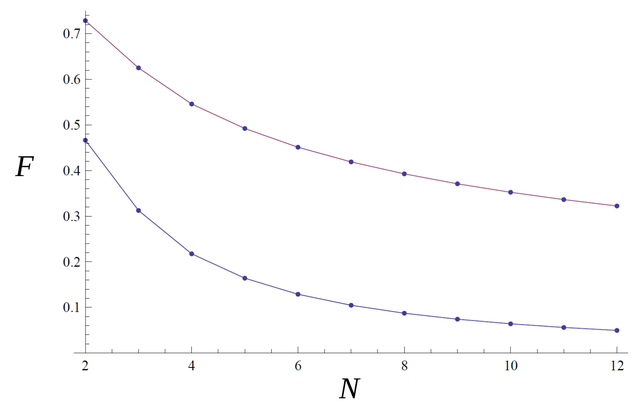
Examples

Transforming irreducible representations:



Average process fidelity F of an optimal transformation between irreducible representations β and a for the group *SU(2)*. The three lines correspond to different choice of the starting representation β, while the x axes represents the choice of the target irreducible representation a.

Cloning of a qubit unitary channels:



Average process fidelity of optimal 1 to N cloning of a qubit channel. The bottom line shows optimal process fidelity for cloning of all qubit unitary channels, whereas the top line corresponds to optimal cloning of only qubit phase gates.

Results

We considered a problem in which a single use of N unknown unitary channels $\mathcal{U}_g^{(i)}$ in an arbitrary sequential order and described by unitary representations U_g of a fixed compact group G are employed to create a target unitary channel V_g, which is described by a different unitary representation of G. As a figure of merit we chose the group average of the channel fidelity between the output channel and the ideal one.

We proved that in the optimal scheme the channels can be used in parallel and the optimization can be simplified to a semidefinite programming problem that significantly (in some cases even exponentially) reduces the number of variables involved in the optimization. As an application we provided the optimal solution for the tasks of transforming an *SU(2)* irreducible representation into a different one, and for the 1 to N cloning of unitary from *SU(2)* and *SU(1)*.

The last two cases illustrate how a stronger prior knowledge about the unknown unitaries enables a higher fidelity in the same way as it happens for phase covariant versus universal state cloning.

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