Mezinárodní centrum pro informaci a neurčitost

Registrační číslo: CZ.1.07/2.3.00/20.0060

evropský sociální fond v ČR
EVROPSKÁ UNIE
MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY
OP Vzdělávání pro konkurenceschopnost
INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Tento projekt je spolufinancován Evropským sociálním fondem a státním rozpočtem České republiky.

Majorization relations and entanglement generation in a b.s.

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Palacký University, Olomouc 14.02.2013

What this talk is about

- Beams splitter

$$\begin{pmatrix} \hat{a}' \\ \hat{b}' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}, \tag{1}$$

- Input: $|k, 0\rangle$ states.
- Which input can give more entanglement in the output for different values of k.
- For fixed k, comparison of the entanglement for different outputs that correspond to different θ's.
- Up to what extend the comparison is possible.
- The tool: majorization theory.

Basics for majorization theory

Definition 1
The theory of majorization gives the means to compare two probability distributions and to conclude which of the two is more "disordered".
For two d-dimensional real vectors $\mathbf{p}$ and $\mathbf{q}$.
We say that $\mathbf{p}$ is majorized by $\mathbf{q}$ ($\mathbf{p} \prec \mathbf{q}$) iff:

$$\sum_{i=1}^{k} p_i^{\downarrow} \leq \sum_{i=1}^{k} q_i^{\downarrow} \tag{2}$$

for $k = 1, \ldots, d-1$ and

$$\sum_{i=1}^{d} p_i^{\downarrow} = \sum_{i=1}^{d} q_i^{\downarrow}, \tag{3}$$

Definition 2

A more intuitive definition is to say that $\mathbf{p}$ is majorized by $\mathbf{q}$ iff there exists a set of $d-$dimensional permutation matrices Π_n and a probability distribution $\{t_n\}$ such that

$$\mathbf{p} = \sum_n t_n \Pi_n \cdot \mathbf{q}. \tag{4}$$

Roughly speaking: $\mathbf{p}$ is majorized by $\mathbf{q}$ iff we can obtain $\mathbf{p}$ by randomly permuting the components of vector $\mathbf{q}$ and afterwards taking the average over all permutations.

Basics for majorization theory

Majorization and doubly stochastic matrices

A real $d \times d$ matrix $\mathbf{D} = [D_{ij}]$ is doubly stochastic if all its entries are non-negative, and each row and each column sums to 1.

Theorem 1: $\mathbf{p} \prec \mathbf{q}$ iff $\mathbf{p} = \mathbf{D} \cdot \mathbf{q}$.

Theorem 2 (Birkhoff's theorem): The $d \times d$ doubly stochastic matrices form a convex set (Birkhoff's polytope) whose extreme points are all the $d \times d$ permutation matrices.

Birkhoff's polytope:

1. has $d!$ vertices (i.e., the number of $d \times d$ permutation matrices).
2. its dimension is $(d-1)^2$.
3. a point (a doubly stochastic matrix) belonging to this polytope can be expressed using $(d-1)^2 + 1$ extremal points at most (Caratheodory's theorem).

Majorization and measures of disorder

Theorem 3 (Hardy-Littlewood-Polya's theorem): $\mathbf{p} \prec \mathbf{q}$ iff $\sum_{i=1}^{d} h(p_i) \leq \sum_{i=1}^{d} h(q_i)$ for all convex functions h.

Consider, for example, the Shannon entropy:

$$S_1(\mathbf{p}) = -\sum_{i=1}^{d} p_i \ln p_i \tag{5}$$

or the Rényi entropy:

$$S_\alpha(\mathbf{p}) = \frac{1}{1-\alpha} \ln \left(\sum_{i=1}^{d} {p_i}^{\alpha} \right) \tag{6}$$

of order $\alpha \geq 0$, $\alpha \neq 1$.

Rényi entropies

$$S_\alpha(\mathbf{p}) = \frac{1}{1-\alpha} \ln \left(\sum_{i=1}^{d} {p_i}^\alpha \right), \ \ \alpha \geq 0, \ \ \alpha \neq 1$$

$$S_\alpha(\mathbf{p}) = \frac{\alpha}{1-\alpha} \ln \|\mathbf{p}\|_\alpha \ , \quad \mathbf{p} = (p_1, p_2, \ldots, p_d)$$

where $\|\mathbf{p}\|_\alpha = (|p_1|^\alpha + |p_2|^\alpha + \ldots + |p_d|^\alpha)^{\frac{1}{\alpha}}$

$$S_{\alpha\to 1}(\mathbf{p}) = -\sum_{i=1}^{d} p_i \ln p_i, \quad S_{\alpha\to\infty}(\mathbf{p}) = -\ln \ \max\{p_i\}$$

$$S_0 \geq S_1 \geq S_2 \geq \ldots \geq S_\infty$$

$$\begin{aligned} S_\alpha(\mathbf{p} \otimes \mathbf{q}) &= S_\alpha(\mathbf{p}) + S_\alpha(\mathbf{q}) & \forall \alpha \in \Re \\ S_\alpha(\mathbf{p}, \mathbf{q}) &\not\leq S_\alpha(\mathbf{p}) + S_\alpha(\mathbf{q}), & \alpha \neq 0, \ 1 \end{aligned}$$

Basics for majorization theory

Majorization provides only a partial ordering in the sense that if $\mathbf{p}$ is *not* majorized by $\mathbf{q}$ ($\mathbf{p} \nprec \mathbf{q}$) then this does not imply that $\mathbf{p} \succ \mathbf{q}$. When both $\mathbf{p} \nprec \mathbf{q}$ and $\mathbf{q} \nprec \mathbf{p}$ hold, we say that the two vectors are *incomparable*.

Basics for majorization theory

Majorization and quantum mechanics
Consider two $d-$level systems A and B,

$$|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i} |i\rangle_A |i\rangle_B \quad (7)$$

and
$\rho_A^\Psi \equiv \mathrm{tr}_B |\Psi\rangle\langle\Psi| = \sum_{i=1}^{d} \lambda_i |i\rangle\langle i|_A$, and analogously for B. We have the following theorem,
Theorem 4 (Nielsen's theorem): State $|\Psi\rangle$ can be converted deterministically into state $|\Phi\rangle$ by means of LOCC iff $\lambda_\Psi \prec \lambda_\Phi$, where λ_Ψ is the vector of eigenvalues of $\rho_A^\Psi \equiv \mathrm{tr}_B |\Psi\rangle\langle\Psi|$ and similarly for λ_Φ.

Basics for majorization theory

Catalysis

According to Theorem 4, if $\lambda(\Psi) \not\prec \lambda(\Phi)$ are incomparable, then there does not exist a strategy to convert one state into the other by LOCC with probability 1.

Nevertheless they can be *catalyzed*[1]:

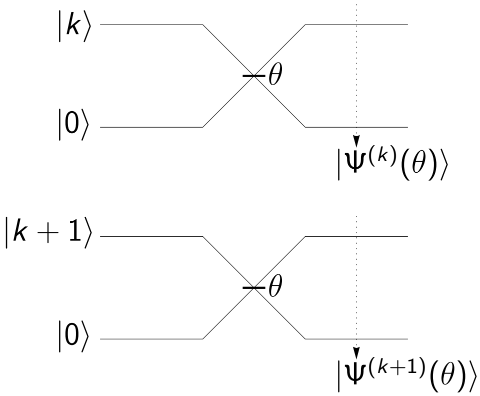
$$|\Psi\rangle \otimes |C\rangle \underset{\text{LOCC}}{\longrightarrow} |\Phi\rangle \otimes |C\rangle \quad (8)$$

Note that if catalysis is possible, then all *additive* measures of entanglement must satisfy $\mu(\Psi) > \mu(\Phi)$. In particular, we must have $S_\alpha(\lambda_\Psi) \geq S_\alpha(\lambda_\Phi)$ for all $\alpha \geq 0$.

[1] D. Jonathan and M. B. Plenio, Phys. Rev. Lett. **83**, 17 (1999).

Majorization with respect to photon number

We want to find if the output states $|\Psi^{(k)}(\theta)\rangle$ and $|\Psi^{(k+1)}(\theta)\rangle$ satisfy a majorization relation.



Majorization with respect to photon number

Denoting by $\mathcal{U}(\theta)$ the unitary transformation resulting from the beam splitter, we have:

$$|\Psi^{(k)}(\theta)\rangle = \mathcal{U}(\theta)\,|k,0\rangle = \sum_{n=0}^{k} \sqrt{P_n^{(k)}(\theta)}\,|n, k-n\rangle \qquad (9)$$

where

$$P_n^{(k)}(\theta) = \binom{k}{n} \cos^{2n}\theta \, \sin^{2(k-n)}\theta. \qquad (10)$$

The reduced density matrix corresponding to the first output mode is:

$$\rho^{(k)}(\theta) = \sum_{n=0}^{k} P_n^{(k)}(\theta)\,|n\rangle\langle n|. \qquad (11)$$

Majorization with respect to photon number

We wish to prove a majorization relation between $P_n^{(k)}(\theta)$ and $P_n^{(k+1)}(\theta)$, that is, we want to prove that there exists a doubly stochastic matrix $\mathbf{D}$ such that

$$\mathbf{P}^{(k+1)}(\theta) = \mathbf{D}^{(k+1)} \cdot \mathbf{P}^{(k)}(\theta) \qquad (12)$$

Using Pascal identity for the binomial coefficients, we obtain the recurrence equation:

$$\begin{aligned} P_n^{(k+1)}(\theta) &= \binom{k+1}{n} \cos^{2n}\theta \, \sin^{2(k+1-n)}\theta \\ &= \left(\binom{k}{n-1} + \binom{k}{n} \right) \cos^{2n}\theta \, \sin^{2(k+1-n)}\theta \\ &= P_{n-1}^{(k)}(\theta) \cos^2\theta + P_n^{(k)}(\theta) \sin^2\theta. \qquad (13) \end{aligned}$$

Majorization with respect to photon number

We can expand Eq. (13) as:

$$\begin{aligned} P_0^{(k+1)}(\theta) &= 0 + \sin^2\theta P_0^{(k)}(\theta), \\ P_1^{(k+1)}(\theta) &= \cos^2\theta P_0^{(k)}(\theta) + \sin^2\theta P_1^{(k)}(\theta), \\ &\vdots \\ P_{k+1}^{(k+1)}(\theta) &= \cos^2\theta P_k^{(k)}(\theta) + 0. \end{aligned} \tag{14}$$

We define:

$$\mathbf{P}^{(k+1)}(\theta) = \begin{pmatrix} P_0^{(k+1)}(\theta) \\ P_1^{(k+1)}(\theta) \\ P_2^{(k+1)}(\theta) \\ \vdots \\ P_k^{(k+1)}(\theta) \\ P_{k+1}^{(k+1)}(\theta) \end{pmatrix}, \quad \mathbf{P}^{(k)}(\theta) = \begin{pmatrix} P_0^{(k)}(\theta) \\ P_1^{(k)}(\theta) \\ P_2^{(k)}(\theta) \\ \vdots \\ P_k^{(k)}(\theta) \\ 0 \end{pmatrix}, \tag{15}$$

Majorization with respect to photon number

We can see:

$$\mathbf{P}^{(k+1)}(\theta) = \mathbf{D}^{(k+1)} \cdot \mathbf{P}^{(k)}(\theta) \tag{16}$$

with:

$$\mathbf{D}^{(k+1)} = \begin{pmatrix} \sin^2\theta & 0 & 0 & 0 & \cdots & \cos^2\theta \\ \cos^2\theta & \sin^2\theta & 0 & 0 & \cdots & 0 \\ 0 & \cos^2\theta & \sin^2\theta & 0 & \cdots & 0 \\ 0 & 0 & \cos^2\theta & \sin^2\theta & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \sin^2\theta \end{pmatrix}, \tag{17}$$

Thus, we have proven the majorization relation

$$\mathbf{P}^{(k+1)}(\theta) \prec \mathbf{P}^{(k)}(\theta), \qquad \forall\theta, \tag{18}$$

which implies that when increasing the number of the incident photons, the 2-mode output state can only be more entangled.

From Nielsen's theorem: $|\Psi^{(k+1)}(\theta)\rangle \xrightarrow{LOCC} |\Psi^{(k)}(\theta)\rangle$.
Alice can perform a POVM measurement:

$$\mathcal{F}_1^{(k)} = \sum_{n=0}^{k} \sqrt{\frac{k+1-n}{k+1}} |n\rangle\langle n| \tag{19}$$

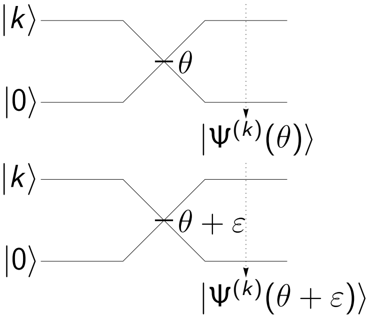
$$\mathcal{F}_2^{(k)} = \sum_{n=0}^{k} \sqrt{\frac{n+1}{k+1}} |n\rangle\langle n+1|. \tag{20}$$

Bob applies proper local unitaries:

$$\mathcal{U}_1^{(k)} = \sum_{n=0}^{k} |n\rangle\langle n+1| + |k+1\rangle\langle 0| \tag{21}$$

$$\mathcal{U}_2^{(k)} = \mathcal{I}. \tag{22}$$

Infinitesimal majorization



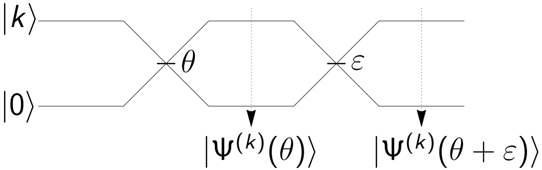
The input state is fixed to $|k, 0\rangle$, but we change the angle θ parameterizing the transmittance by an infinitesimal amount ε. Note that we take $\theta \geq 0$, $\varepsilon \geq 0$, and $\theta + \varepsilon \leq \frac{\pi}{4}$.

Parametric majorization relations

Infinitesimal majorization

$\mathbf{P}^{(k)}(\theta+\varepsilon) \overset{?}{\prec} \mathbf{P}^{(k)}(\theta)$

An equivalent way to see this scenario is depicted in:



Our goal is to probe whether the intermediate state majorizes or not the final output state. To this end we find it easier to use the first **definition 1** of majorization, involving the accumulations of the ordered vectors of eigenvalues of the reduced density matrix (we will refer to this vectors as OSC).

Parametric majorization relations

Infinitesimal majorization

This OSC vector will not have the same ordering as the parameter θ changes.

We adopt the notation $\mathbf{P}^{\downarrow r}(\theta)$, where $r = 1, 2, \ldots$ labels the regions of parameter θ in which the ordering of the OSC vector remains the same.

Every time two eigenvalues $P_n(\theta)$ and $P_m(\theta)$ are equal we have a change of ordering:

$$\theta = \arctan\left(\frac{(k-n)!n!}{(k-m)!m!}\right)^{\frac{1}{2(n-m)}}. \tag{23}$$

Parametric majorization relations

Infinitesimal majorization

Our goal now is to check if:

$$\mathbf{P}^{\downarrow r}(\theta+\varepsilon) \prec \mathbf{P}^{\downarrow r}(\theta) \tag{24}$$

holds or not within region r.

Equivalently, using the **definition 1**, we have to prove:

$$\sum_{n=0}^{j} P_n^{\downarrow r}(\theta+\varepsilon) \leq \sum_{n=0}^{j} P_n^{\downarrow r}(\theta) \Leftrightarrow \sum_{n=0}^{j} \frac{P_n^{\downarrow r}(\theta)}{d\theta} \leq 0,\ j=0,\ldots k. \tag{25}$$

By defining:

$$a_j^{\downarrow r}(\theta) = \sum_{n=0}^{j} \frac{dP_n^{\downarrow r}(\theta)}{d\theta}, \tag{26}$$

majorization relations become:

$$a_j^{\downarrow r}(\theta) \leq 0,\ j=0,\ldots k. \tag{27}$$

Infinitesimal majorization

Violation of at least one relation in

$$a_j^{\downarrow r}(\theta) \leq 0,\ j = 0, \ldots k. \tag{28}$$

is enough to disproof majorization in region r.

A priori, if the above majorization relations do not hold, there may nevertheless be a majorization in the opposite direction if all relations are satisfied with $\geq$ instead of $\leq$. However, the $(k-1)$-th accumulation appears in all regions no matter what the ordering is, and its derivative

$$a_{k-1}(\theta) = -2k \sin^{2k-1}\theta \cos\theta \tag{29}$$

respects Eqs. (**??**) with a strict inequality, so majorization in the opposite direction is not possible.

Parametric majorization relations

Majorization always holds in the first region

The components of the OSC vector are:

$$P_n^{\downarrow 1}(\theta) = \binom{k}{n} \sin^{2n}\theta \cos^{2(k-n)}\theta. \tag{30}$$

We have:

$$a_j(\theta)^{\downarrow 1} = P_0^{\downarrow 1}(\theta) \sum_{n=0}^{j} \left[2n - 2(k-n)\tan^2\theta\right] \binom{k}{n} \tan^{2n-1}\theta \tag{31}$$

which can be expressed in a closed form as:

$$a_j(\theta)^{\downarrow 1} = -P_0^{\downarrow 1}(\theta)\, 2(k-j) \binom{k}{j} \tan^{2j+1}\theta \tag{32}$$

which is **non-positive** for $j = 0, \ldots k$.

Parametric majorization relations

Majorization always holds in the first region

Thus, within region r=1: $\mathbf{P}^{(k)}(\theta+\varepsilon) \prec \mathbf{P}^{(k)}(\theta),\ \forall k \geq 0$.

Region r=1: $\left(0, \arctan\frac{1}{\sqrt{k}}\right]$

For $k = 1$ the first region expands to the whole interval $\left(0, \frac{\pi}{4}\right]$

Passing to the second region

The first two components of the OSC vector $\mathbf{P}^{(k)}(\theta)$ switch places.

The derivative of the first accumulation becomes:

$$\alpha_0^{\downarrow 2} = 2k\left[1-(k-1)\tan^2\theta\right]\tan\theta\cos^{2k}\theta \tag{33}$$

which is positive until the value $\theta_1^+ = \arctan\frac{1}{\sqrt{k-1}}$.
Therefore majorization will be violated from the beginning of the region $r=2$ at least until that derivative remains positive:

$$\left(\arctan\frac{1}{\sqrt{k}}, \arctan\frac{1}{\sqrt{k-1}}\right] \tag{34}$$

Parametric majorization relations

Majorization relations violated

In general, in every region that begins with a positive derivative of the first accumulation: $\frac{dP_n(\theta)}{d\theta}$ for some n, majorization will be violated at least until this derivative remains positive.

We find that the derivative: $\frac{dP_n(\theta)}{d\theta}$ of a given element remains positive up to the value $\theta_n^+ \leq \arctan\left(\frac{n}{k-n}\right)^{\frac{1}{2n-1}}$.

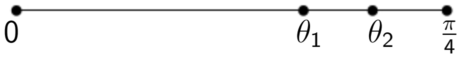
Example

We consider the case of three photons ($k = 3$). We have two cross-over angles:

$$\theta_1 = \arctan\frac{1}{\sqrt{3}},\ \theta_2 = \arctan\frac{1}{\sqrt[4]{3}}, \tag{35}$$

which define three regions of different orderings in $\left[0, \frac{\pi}{4}\right]$:

$$r = 1 : \quad [0, \theta_1) \tag{36}$$
$$r = 2 : \quad [\theta_1, \theta_2) \tag{37}$$
$$r = 3 : \quad [\theta_2, \tfrac{\pi}{4}] \tag{38}$$



Parametric majorization relations

Example

In region $r = 1$, it is easy to confirm that majorization holds.
In region $r = 2$ the accumulation derivatives are:

$$\begin{aligned} a_0^{\downarrow 2}(\theta) &= 3\cos^3\theta(-1+3\cos 2\theta)\sin\theta \\ a_1^{\downarrow 2}(\theta) &= -\frac{3}{2}\sin^3 2\theta \\ a_2^{\downarrow 2}(\theta) &= -6\cos\theta\sin^5\theta \\ a_3^{\downarrow 2}(\theta) &= 0 \end{aligned} \tag{39}$$

Where $a_0^{\downarrow 2}(\theta) > 0$ in the interval $[0, \arctan\frac{1}{\sqrt{2}})$.
That means that in the interval $(\arctan\frac{1}{\sqrt{3}}, \arctan\frac{1}{\sqrt{2}})$, i.e., from the beginning of region $r = 2$ up to where $a_0^{\downarrow 2}(\theta)$ remains positive, we are sure that there are no majorization relations.

Parametric majorization relations

Example

In region $r = 3$ the accumulation derivatives are:

$$\begin{aligned} a_0^{\downarrow 3}(\theta) &= 3\cos^3\theta(-1+3\cos 2\theta)\sin\theta \\ a_1^{\downarrow 3}(\theta) &= \frac{3}{2}\sin 4\theta \\ a_2^{\downarrow 3}(\theta) &= -6\cos\theta\sin^5\theta \\ a_3^{\downarrow 3}(\theta) &= 0. \end{aligned} \tag{40}$$

Where $a_2^{\downarrow 3}(\theta) > 0$ in region $r = 3:\ [\theta_2, \frac{\pi}{4}]$. while the other accumulation derivatives are negative within this region. Hence, for $r = 3$, the states are always incomparable.

Parametric majorization relations

Example

So:

One wonders if some measure of entaglement can depict all this.

The answer is yes and no...

Parametric majorization relations

Example

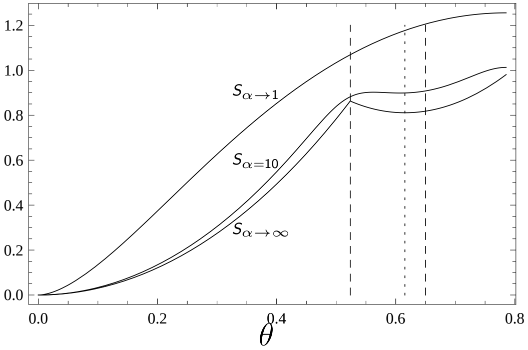


Figure: Evolution of Rényi entropies across the three regions.

Majorization violation in $r = 2$: up to the local minimum of the min-entropy (equivelently up to where $a_0^{\downarrow 2}(\theta) > 0$).
Majorization violation in $r = 3$ is not manifested by the S_α.

Parametric majorization relations

Catalysis

$$|\Psi^{(k)}(\phi)\rangle \nprec |\Psi^{(k)}(\theta)\rangle \xrightarrow{catalysis} |\Psi^{(k)}(\phi)\rangle \otimes |C\rangle \prec |\Psi^{(k)}(\theta)\rangle \otimes |C\rangle \quad (41)$$

for $\phi > \theta$.

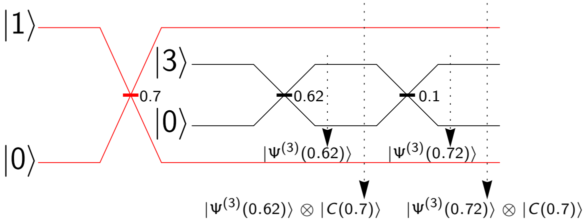
$$\begin{aligned} S_\alpha(|\Psi^{(k)}(\phi)\rangle \otimes |C\rangle) &> S_\alpha(|\Psi^{(k)}(\theta)\rangle \otimes |C\rangle) \Leftrightarrow \\ S_\alpha(|\Psi^{(k)}(\phi)\rangle) + \cancel{S_\alpha(|C\rangle)} &> S_\alpha(|\Psi^{(k)}(\theta)\rangle) + \cancel{S_\alpha(|C\rangle)} \quad (42) \end{aligned}$$

So it makes sense to look for catalyzable incomparable states in regions of the parameter where all of the Réniy entropies increase.

Another constraint is that the dimension of the catalyzable state should be $d \geq 4$.

Parametric majorization relations

Catalysis-example



$$\mathbf{P}^{(3)}(0.72) \not\prec \mathbf{P}^{(3)}(0.62) \tag{43}$$

$$\mathbf{P}^{(3)}(0.72) \otimes \mathbf{P}^{(1)}(0.7) \prec \mathbf{P}^{(3)}(0.62) \otimes \mathbf{P}^{(1)}(0.7) \tag{44}$$

Several other numerical examples can be found and some of them, like the ones provided above, are experimentally feasible.

Conclusion and further research

- $\mathbf{P}^{(k+1)}(\theta) \prec \mathbf{P}^{(k)}(\theta)$, just like for the TMS [1].
- $\mathbf{P}^{(k)}(\theta + \varepsilon) \prec \mathbf{P}^{(k)}(\theta)$ counter-intuitively only up to a limit, unlike the TMS [1].
- By examining specific examples, one may find more violations of majorization within the same regions of ordering or between different regions.
- Incomparable states resulting from different values of θ can be catalyzed with the help of an experimentally accessible state, such a single-photon path-entangled state.

[1] Phys. Rev. Lett. **108**, 110505 (2012).

Conclusion and further research

Further investigation:

- General solution concerning the catalysis process in the parameter varying case.
- More general inputs, e.g. $|m, N-m\rangle$.
- Majorization relations in complicated optical circuits.
- The study of phase transitions and critical phenomena under the prism of majorization.

Thank You!

arXiv:1301.5229