Parameter estimation for different quantum systems

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Introduction

State of a quantum system

- A finite quantum state $\rho \in M_n(\mathbb{C})$ can be described with the following properties:

$$\mathrm{Tr}(\rho) = 1, \quad \rho \geq 0$$

- Let σ_i be generalized Pauli-matrices: orthonormal basis with respect to the Hilbert-Schmidt inner product:

$$\langle A, B \rangle = \mathrm{Tr}(A^* B)$$

- We use the Bloch parametrization

$$\rho(\theta) = \sum_{i=0}^{n^2-1} \theta_i \sigma_i.$$

State of a quantum system II.

$$\rho(\theta) = \sum_{i=0}^{n^2-1} \theta_i \sigma_i,$$

- $\mathrm{Tr}(\rho) = 1 \iff \theta_0 = \frac{1}{\sqrt{n}}$.
  State space can be parametrized with $\theta \in \mathbb{R}^{n^2-1}$
- $\rho \geq 0 \Longrightarrow \sum_{i=0}^{n^2-1} \theta_i^2 \leq 1$.
  Note that if $n = 2$ (qubit case) this is also a sufficient condition, so in that case we have the so-called Bloch ball as state space.

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Measurements

- $(E_1, E_2, \ldots, E_k)$ forms a positive operator valued measurement (POVM) if

$$\forall i : E_i \geq 0 \quad \text{and} \quad \sum_i E_i = I.$$

  - For $k = 2$: $(P, I - P)$ are projections (von Neumann meas.).

- The probability of observing an outcome related to E_i is

$$p_i = \mathrm{Tr}(\rho E_i).$$

- E.g., $A = \sum \lambda_i P_i$. Then $E_i := P_i$, while the outcome is λ_i.
- State after measurement:

$$\rho_i' = \frac{E_i \rho E_i}{\mathrm{Tr} E_i \rho E_i}$$

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Quantum tomography

- The state estimation process has the following steps:
  - Choose a set of measurements
  - Measure multiple times on identical copies of a quantum state
  - Construct an estimator from the measurement data
- Our choices:
  - Measurements
  - Estimator
  - Figure of merit for estimation efficiency

Standard method

- We measure in the 3 axis directions: $P_i = \frac{I+\sigma_i}{2}, \ (i = 1, 2, 3)$
- The probability of an outcome related to P_i:

$$p_i = \frac{1}{2}(1 + \theta_i)$$

- m measurements are performed in each direction

$$\nu_i := \frac{m_i}{m}, \quad \text{where } m_i \text{ is the number of outcomes related to } P_i$$

- Then the estimation on θ:

$$\Phi_m(\nu_1, \nu_2, \nu_3) = \begin{bmatrix} 2\nu_1 - 1 \\ 2\nu_2 - 1 \\ 2\nu_3 - 1 \end{bmatrix}$$

Standard method II.

- Φ_m is unbiased: $E(\Phi_m) = \theta$.
- Its covariance matrix is

$$\operatorname{Var}(\Phi_m) = \frac{1}{m}\begin{bmatrix} 1-\theta_1^2 & 0 & 0 \\ 0 & 1-\theta_2^2 & 0 \\ 0 & 0 & 1-\theta_3^2 \end{bmatrix}$$

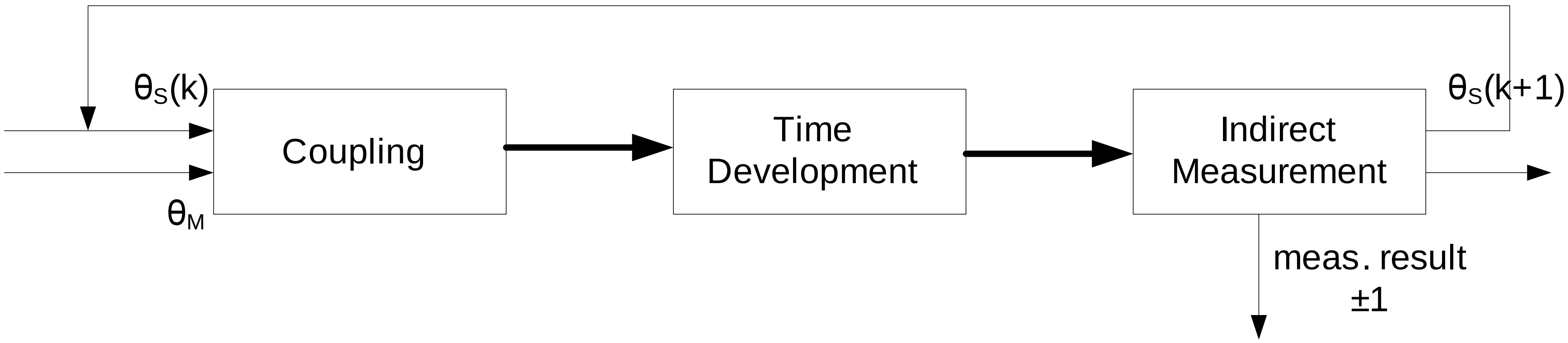
- If Ψ_m is an unbiased estimator, the Cramér-Rao inequality says

$$\operatorname{Var}(\Psi_m) \geq I_m(\theta)^{-1}.$$

For Φ_m we have equality, so Φ_m is efficient.

Weak measurements

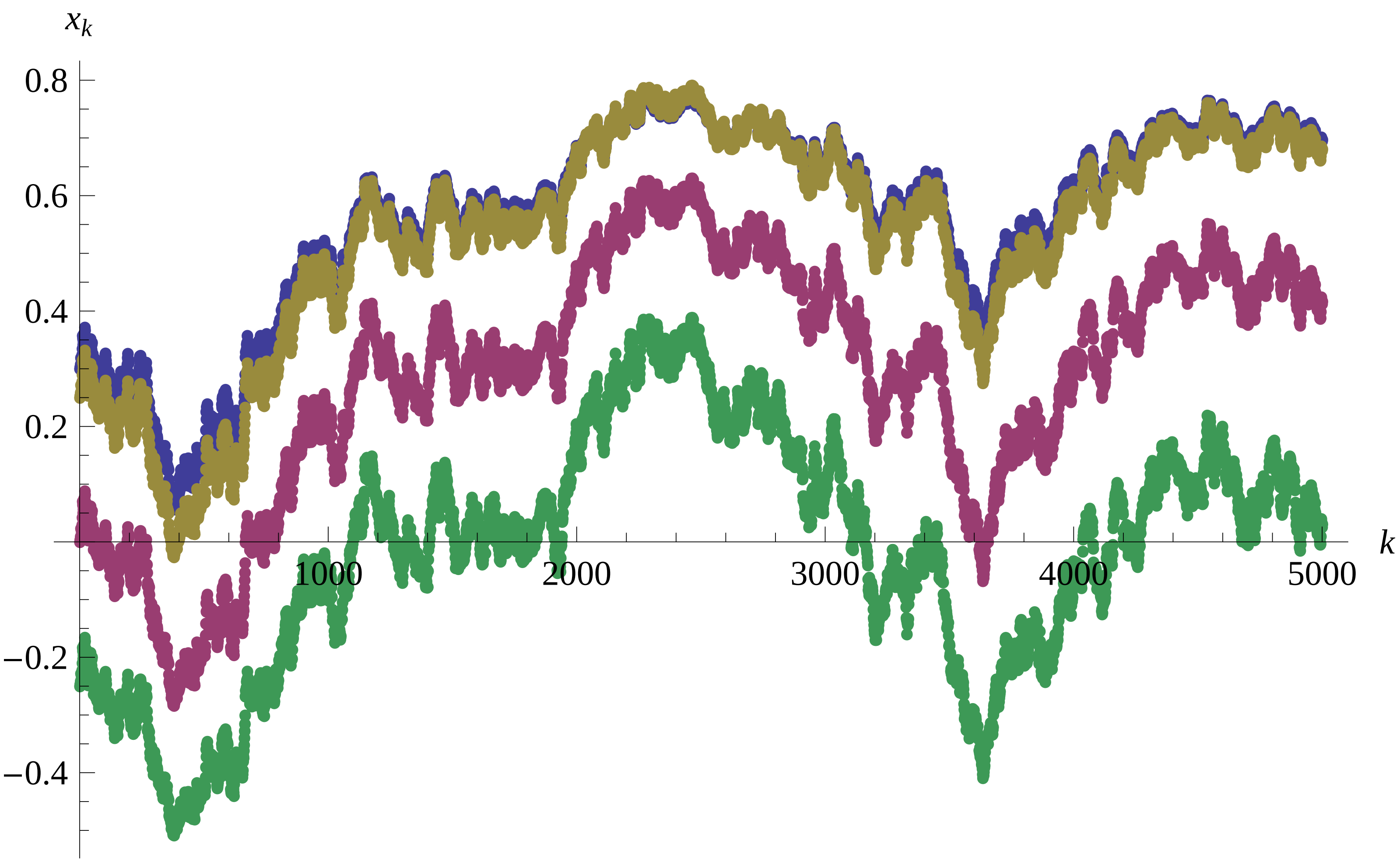
State evolution driven by weak measurements



- State evolution:

$$x_{k+1} = \begin{cases} \frac{x_k + c}{1 + c x_k}, \text{ with probability } \frac{1 + c x_k}{2}: \text{ +1 measurement} \\ \\ \frac{x_k - c}{1 - c x_k}, \text{ with probability } \frac{1 - c x_k}{2}: \text{ -1 measurement} \end{cases}$$

Example: State evolution for different x_0-s



Estimation of the initial state

- Aim: Estimation of the initial state x_0
- Result: We gave 3 working methods
  - Histogram
  - Bayesian
  - **Martingale**
- Martingale property: $\mathbb{E}(x_{k+1}) = x_k$
- For fixed value u, v, we run the process until $u < x_k < v$.
- Doob's optional stopping theorem: $\mathbb{E}(x_T) = x_0$, so

$$\mathbb{E}(x_T) = pu + (1-p)v = x_0 \quad \Rightarrow \quad \hat{x}_0 = \hat{p}u + (1-\hat{p})v$$

Estimation of the process

- Aim: Estimation of the process x_k (filtering)
- Kalman filter:
  - State evolution: $x_{k+1} = Ax_k + w_k$
  - Measurement: $y_k = Hx_k + v_k$
  - w_k and v_k are independent noises with probability distribution: $w \sim \mathcal{N}(0, Q), \quad v \sim \mathcal{N}(0, R)$
  - Kalman filter:

$$\hat{x}_{k+1} = A\hat{x}_k + K_k\Big(y_k - H\hat{x}_k\Big)$$

  - Task: optimal choice of K_k to minimize:

$$\mathbb{E}(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T \to \text{min.}$$

Obtaining the state space model

- State evolution:

$$x_{k+1} = x_k + Nc^2 x_k (1 - x_k^2) + \omega_k \cdot c(1 - x_k^2)$$

- Measurements:

$$y_k = Ncx_k + \omega_k,$$

with $\omega_k \sim \mathcal{N}(0, N)$.

- Comparison to the classical Kalman filter settings:
  - State evolution: non-linear
  - Measurement: linear
  - Noise: not independent (measurement feedback) and additional non-linear factor

Related publications

[1] L. Ruppert, A. Magyar, K.M. Hangos, *Compromising non-demolition and information gaining for qubit state estimation*, Quantum Probability and Related Topics, World Scientific, p. 212-224, 2008.

[2] L. Ruppert, K.M. Hangos: *Martingale approach in quantum state estimation using indirect measurements*, Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems, p. 2049-2054, 2010.

[3] K.M. Hangos, L. Ruppert: *State estimation methods using indirect measurements*, Quantum Probability and Related Topics, World Scientific, p. 163-180, 2011

[4] L. Ruppert, K. M. Hangos, J. Bokor, *Possibilities of Quantum Kalman Filtering*, submitted for publication

Channel tomography

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Complementarity

- Quantum channel: $M_n(\mathbb{C}) \to M_n(\mathbb{C})$ CPTP map
- The basis $e_1, e_2, \dots, e_n$ is complementary to the basis $f_1, f_2, \dots, f_n$ (also called mutually unbiased bases) if

$$|\langle e_i, f_j \rangle|^2 = \frac{1}{n} \qquad (1 \le i, j \le n).$$

- Generalization for POVMs $(1 \le i \le k,\ 1 \le j \le m)$:

$$\left(\mathrm{Tr}\, E_i F_j = \frac{1}{n} \mathrm{Tr}\, E_i \mathrm{Tr}\, F_j \right) \Leftrightarrow \left(E_i - \frac{\mathrm{Tr}\, E_i}{n} I \ \perp\ F_j - \frac{\mathrm{Tr}\, F_j}{n} I \right)$$

- We can generalize quasi-orthogonality for subspaces:

$$\mathcal{A}_1 \ominus \mathbb{C}I \perp \mathcal{A}_2 \ominus \mathbb{C}I,$$

Parameter estimation of Pauli channels

- Let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_N$ be a complementary decomposition of $M_n(\mathbb{C})$:

$$A_i - \frac{\mathrm{Tr} A_i}{n} I \perp A_j - \frac{\mathrm{Tr} A_j}{n} I, \qquad \forall A_i \in \mathcal{A}_i, A_j \in \mathcal{A}_j \ (i \neq j)$$

- Pauli channel: contractions with λ_i on traceless part of $\mathcal{A}_i$.
- Example: $\mathcal{A}_i := span\{I, \sigma_i\}, i \in \{1, 2, \dots, 2^n - 1\}$

$$\mathcal{E} : \rho = \frac{1}{n}\left(I + \sum_{i=1}^{2^n-1} \theta_i \sigma_i\right) \mapsto \mathcal{E}(\rho) = \frac{1}{n}\left(I + \sum_{i=1}^{2^n-1} \lambda_i \theta_i \sigma_i\right).$$

- Aim: Select input state, send through channel, measure the output, repeat many times $\Rightarrow$ estimate λ_i

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Parameter estimation of Pauli channels II.

- Figure of merit: Fisher information matrix of the parameters λ_i

$$F_{ij} = \sum_{\alpha} \frac{1}{p_\alpha} \frac{\partial p_\alpha}{\partial \lambda_i} \frac{\partial p_\alpha}{\partial \lambda_j}$$

- Optimization:

$$\forall i: \; F_{ii} \to \max. \quad \text{(independently)}$$

- Result: Input and measurement in the direction of $\mathcal{A}_i$. It depend on the algebraic structure of $\mathcal{A}_i$.

Unknown channel directions

- Another problem: What if σ_i are unknown too?
- We gave an efficient method for the qubit case.
- channel matrix: $A : \theta_{in} \to \theta_{out}$

$$A(\lambda_1, \lambda_2, \lambda_3, \phi_z, \phi_y, \phi_x) = R_z R_y R_x \Lambda R_x^{-1} R_y^{-1} R_z^{-1}$$

1. $\mathbb{E}\|\hat{A} - A\|^2 \to$ min.: in the channel directions (equivalent to the average squared distance of ρ_{out} and $\hat{\rho}_{out}$)
2. $\mathbb{E}\sum(\hat{\lambda}_i - \lambda_i)^2 \to$ min.: in the channel directions
3. $\mathbb{E}\sum(\hat{\phi}_i - \phi_i)^2 \to$ min.: NOT in the channel directions

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[1] L. Ruppert, D. Virosztek and K.M. Hangos *Optimal parameter estimation of Pauli channels*, Journal of Physics A: Math. Theor. **45**, 265305, 2012.

[2] D. Virosztek, L. Ruppert and K. M. Hangos, *Pauli channel tomography with unknown channel directions*, submitted for publication

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State tomography

Complementarity and DACM

- Wooters and Fields proved in 1989 the optimality of complementary measurements
- Petz, Hangos and Magyar used in 2007 the optimization

$$\det \langle \mathrm{Var}\,(\hat{\theta}) \rangle \to \min\,.$$

for proving the optimality of complementary measurements in the qubit case.

- Baier and Petz used this quantity in 2010 to prove the optimality in a more general setting.

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Symmetric measurements

- The Bloch vector has $n^2 - 1$ parameters, so we have at least n^2 elements in POVM.
- Symmetric informationally complete POVM (SIC-POVM):

$$E_i = \frac{1}{n} P_i, \qquad \mathrm{Tr}\, P_i P_j = \frac{1}{n+1} \quad (i \neq j,\ 1 \leq i, j \leq n^2),$$

  where P_i is a rank-one projection.
- Rehacek, Englert and Kaszlikowski used in 2004 the 2-dimensional SIC-POVM for state tomography.
- Scott used in 2006 the average squared Hilbert-Schmidt distance for proving the optimality of SIC-POVMs.

Multiple von Neumann measurements

- We have a decomposition

$$M_n(\mathbb{C}) = \mathbb{C}I \oplus \mathcal{A} \oplus \mathcal{B},$$

  where $\mathcal{A} \to$ known, $\mathcal{B} \to$ unknown parameters.

- If $\mathcal{B}$ has l dimensions, then we have the measurements

$$(F_1, I - F_1),\ (F_2, I - F_2),\ \dots,\ (F_l, I - F_l)$$

Theorem. *If the positive contractions $F_1, \dots, F_l$ have the same spectrum, then the determinant of the average covariance matrix is minimal if the operators $F_1, \dots, F_l$ are complementary to each other and to $\mathcal{A}$.*

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Single POVMs

- We have once again the decomposition

$$M_n(\mathbb{C}) = \mathbb{C}I \oplus \mathcal{A} \oplus \mathcal{B},$$

  with $\dim(\mathcal{B}) = k - 1$, then we have the measurement

$$(E_1, E_2, \dots, E_k)$$

- In the n dimensional case we can obtain results if $k = n^2$, i.e. all parameters are unknown.

Theorem. *If a symmetric informationally complete system exists, the optimal POVM is described by its projections P_i as $E_i = P_i/n$ $(1 \le i \le n^2)$.*

Single POVMs II.

- In the conditional case there are some technical issues, which we barely overcame in the qubit case.

Theorem. *The optimal POVM for the unknown Bloch parameters* θ_1 *and* θ_2 *can be described by projections* P_i, $1 \leq i \leq 3$*:*

$$E_i = \frac{2}{3} P_i, \quad \mathrm{Tr} P_i P_j = \frac{1}{4}\,(i \neq j), \quad \textit{and} \quad \mathrm{Tr}\sigma_3 P_i = 0,$$

- We get that the optimal POVM is symmetrical and complementary to the subspace of the known parameters $\Rightarrow$ generalization of SIC-POVM

Numerical algorithm

- I show the first non-trivial example of a conditional SIC-POVM

$$E_1 = \frac{1}{7}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, E_2 = \frac{1}{7}\begin{bmatrix} 1 & \varepsilon^6 & \varepsilon^2 \\ \varepsilon & 1 & \varepsilon^3 \\ \varepsilon^5 & \varepsilon^4 & 1 \end{bmatrix}, E_3 = \frac{1}{7}\begin{bmatrix} 1 & \varepsilon^2 & \varepsilon^3 \\ \varepsilon^5 & 1 & \varepsilon \\ \varepsilon^4 & \varepsilon^6 & 1 \end{bmatrix},$$

$$E_4 = \frac{1}{7}\begin{bmatrix} 1 & \varepsilon^4 & \varepsilon^6 \\ \varepsilon^3 & 1 & \varepsilon^2 \\ \varepsilon & \varepsilon^5 & 1 \end{bmatrix}, E_5 = E_2^\intercal,\ E_6 = E_3^\intercal,\ E_7 = E_4^\intercal, \text{ with } \varepsilon = \exp\left(\frac{2\pi i}{7}\right).$$

- There is a conditional SIC-POVM containing the diagonal matrix units.
- There is an example for conditional SIC-POVMs that contains projections of rank 2.
- There is an example where no conditional SIC-POVM exists.

Conditional SIC-POVM

- From these results we obtain the precise definition of conditional SIC-POVMs:

  Definition (Conditional SIC-POVM)

  $(E_1, E_2, \ldots, E_k)$ *forms a conditional SIC-POVM if there is a set of projections* $P_i,\ 1 \leq i \leq k$, *such that*

  $$E_i = \frac{1}{\lambda} P_i \quad \textit{and} \quad \mathrm{Tr} P_i P_j = \mu \quad (i \neq j).$$

  and E_i*-s are complementary to the subspace of known parameters.*

- We get a SIC-POVM in the special case when $k = n^2$, $\lambda = n$ and $\mu = 1/(n+1)$.

Conditional SIC-POVM II.

- Instead of the determinant of the average covariance matrix, minimize the square of the Hilbert-Schmidt distance.

Theorem. *In the conditional case, the elements of the optimal POVM can be described as multiples of rank-one projections with the following properties* $(1 \leq i, j \leq k)$*:*

$$E_i = \frac{n}{k} P_i, \quad \mathrm{Tr} P_i P_j = \frac{k-n}{n(k-1)} \quad (i \neq j)$$

$$\textit{and} \quad \mathrm{Tr} \sigma_l P_i = 0 \quad (\forall l : \sigma_l \in \mathcal{A}).$$

- So the conditional SIC-POVM is the optimal with rank-one projections, and constants $\lambda = \frac{k}{n}$, $\mu = \frac{k-n}{n(k-1)}$.

Example for existence

- Let us assume that the diagonal part of $\rho \in M_n(\mathbb{C})$ is known
- The number of POVM elements: $k = n^2 - n + 1$

Definition (Difference set)**.** *The set* $G := \{0, 1, \ldots, k-1\}$ *is an additive group modulo* k*. The subset* $D := \{\alpha_i : 1 \leq i \leq n\}$ *forms a difference set with parameters* (k, n, λ) *if the set of differences* $\alpha_i - \alpha_j$ *contains every nonzero element of* G *exactly* λ *times.*

- A few examples for difference sets with parameters $(k, n, 1)$:

$$n = 2, k = 3 : \ D = \{0, 1\}, \quad n = 3, k = 7 : \ D = \{0, 1, 3\}, \quad n = 4, k = 13 : \ D = \{0, 1, 3, 9\}.$$

Theorem. *We set* $|\phi\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |e_i\rangle$, $q = e^{2\pi i/k}$, $U = Diag(q^{\alpha_1}, q^{\alpha_2}, q^{\alpha_3}, \ldots q^{\alpha_n})$*. If* $(\alpha_1, \alpha_2, \ldots \alpha_n)$ *forms a difference sets with parameters* $(k, n, 1)$*, then*

$$P_i := |U^i \phi\rangle\langle U^i \phi|, \quad (i = 1, 2, \ldots, k)$$

will be an appropriate conditional SIC-POVM.

Application of Conditional SIC-POVM

- SIC-POVM is the BLE of a quantum state
- Conditional SIC-POVM is the BLE of a subsystem of a quantum state
- Let $\mathcal{A}_1, \ldots, \mathcal{A}_N$ be a complementary decomposition of $M_n(\mathbb{C})$
- $E^{(i)}$ is the conditional SIC-POVM for $\mathcal{A}_i \Rightarrow$ BLE for subsystems
- Best candidates:
  - $N = 1$, $\mathcal{A}_1 = M_n$: SIC-POVM
  - $N = n + 1$, $\mathcal{A}_1 = ... = \mathcal{A}_{n+1} = \mathbb{C}^n$: MUB

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[1] D. Petz, K.M. Hangos and L. Ruppert, *Quantum state tomography with finite sample size*, in Quantum Bio-Informatics, eds. L. Accardi, W. Freudenberg, M. Ohya, World Scientific, p. 247-257, 2008.

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