Representing Fuzzy Attribute Implications by Fuzzy Logic Programs

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INVESTMENTS IN EDUCATION DEVELOPMENT

Outline

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   - Complete residuated lattice
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   - Fuzzy data
   - Fuzzy attribute implication
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3. Fuzzy Logic Programming
   - Syntax
   - Declarative semantics
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   - Soundness and completeness
4. Representing FAIs by FLPs

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Complete residuated lattice

We use complete residuated lattices as structures of truth degrees.

Definition

A *complete residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ such that

- $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice,
- $\langle L, \otimes, 1 \rangle$ is a commutative monoid,
- $\otimes$ and $\rightarrow$ satisfies $a \otimes b \leq c$ iff $a \leq b \rightarrow c$, for all $a, b, c \in L$.

Complete residuated lattices on unit interval

Example

Łukasiewicz connectives:

$a \otimes_L b = \max(0, a + b - 1)$

$a \to_L b = \min(1, 1 - a + b)$

Gödel connectives:

$a \otimes_G b = \min(a, b)$

$a \to_G b = b$ for $a > b$ and $a \to_G b = 1$ for $a \leq b$

Goguen (product) connectives:

$a \otimes_P b = a \cdot b$

$a \to_P b = \frac{b}{a}$ for $a > b$ and $a \to_P b = 1$ for $a \leq b$

Fuzzy set and subsethood

Definition

An **L**-*set* A in universe U is a map $A\colon U \to L$. $A(u)$ is interpreted as "the degree to which u belongs to A".

L^U denotes the collection of all **L**-sets in U.

Definition

For **L**-sets $A, B \in L^U$, we define a *subsethood* degree of A in B by $S(A, B) = \bigwedge_{u \in U} \bigl(A(u) \to B(u)\bigr)$.

In addition, we write $A \subseteq B$ iff $S(A, B) = 1$.

Truth-stressing hedge

Sometime we equip the structue of truth degrees with an unary operation, which can be seen as a truth function of a logical connective “very true”.

Definition

A *truth-stressing hedge* * is an additional unary operation on L satisfying the following conditions:

- $1^* = 1$,
- $a^* \leq a$,
- $(a \to b)^* \leq a^* \to b^*$, and
- $a^{**} = a^*$ for all $a, b \in L$.

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Classical data table

Car	Age	Mileage	Fuel Economy	...
Audi A3	2 years	43.215 miles	24 MpG	...
Aston Martin	7 years	163.547 miles	13 MpG	...
BMW Z3	12 years	214.845 miles	20 MpG	...
Acura RDX	0.5 years	4.257 miles	22 MpG	...

Fuzzy data table

Car	IA	IM	hAT	hFE	hP
Audi A3	0.9	0.8	1	0.9	0.7
Aston Martin	0.2	0.1	0	0.1	0.3
BMW Z3	0	0	1	0.8	0.2
Acura RDX	1	1	1	0.9	0.8

Attributes abbreviations:

- Low Age – IA
- Low Mileage – IM
- Has Automatic Transmission – hAT
- High Fuel Economy – hFE
- High Price – hP

Fuzzy attribute implication

Definition

Let Y be a nonempty set of *attributes*. A *fuzzy attribute implication* (or shortly a FAI) is an expression $A \Rightarrow B$, where $A, B \in L^Y$.

Example

Given $Y = \{lA, lM, hAT, hFE, hP\}$ and $\mathbf{L}$ being Łukasiewicz structure.
$\{^{0.7}/lA, {}^{0.9}/lM\} \Rightarrow \{^{0.6}/hFE, {}^{0.9}/hP\}$ is an attribute implication saying that cars with low age (at least to degree 0.7) and low mileage (at least to 0.9) have also high fuel economy (at least to 0.6) and high price (at least to 0.9).

Semantics of FAIs

Definition

For an **L**-set $M \in L^Y$ of attributes, we define a degree $||A \Rightarrow B||_M \in L$ to which $A \Rightarrow B$ is true in M by $||A \Rightarrow B||_M = S(A, M)^* \to S(B, M)$.

Definition

Let T be an **L**-set T of FAIs (theory). M is a *model* of T if $T(A \Rightarrow B) \leq ||A \Rightarrow B||_M$ for all $A, B \in L^Y$.
The set of all models of T is denoted by $\mathrm{Mod}(T)$.

Definition

We define a degree $||A \Rightarrow B||_T$ to which $A \Rightarrow B$ *semantically follows from* T by $||A \Rightarrow B||_T = \bigwedge_{M \in \mathrm{Mod}(T)} ||A \Rightarrow B||_M$.

Example

Example

- Let $L = \{0, 0.5, 1\}$ with Łukasiewicz connectives and * being identity and $Y = \{a, b, c\}$.
- Let $T = \{{}^{1}/\{{}^{0.5}/a\} \Rightarrow \{{}^{0.5}/b, {}^{1}/c\}, {}^{1}/\{{}^{0.5}/c\} \Rightarrow \{{}^{1}/a\}\}$.
- Cardinalities: $|L^Y| = 3^3 = 27$, $|\mathrm{Mod}(T)| = 5$.
- $\mathrm{Mod}(T) = \{\emptyset, \{{}^{0.5}/b\}, \{{}^{1}/b\}, \{{}^{1}/a, {}^{0.5}/b, {}^{1}/c\}, \{{}^{1}/a, {}^{1}/b, {}^{1}/c\}\}$
- Computation of $||\{{}^{1}/c\} \Rightarrow \{{}^{1}/b\}||_T$:
  $||\{{}^{1}/c\} \Rightarrow \{{}^{1}/b\}||_T = \bigwedge_{M \in \mathrm{Mod}(T)} ||\{{}^{1}/c\} \Rightarrow \{{}^{1}/b\}||_M = \bigwedge_{M \in \mathrm{Mod}(T)} \{1, 1, 1, 0.5, 1\} = 0.5$

The computation of a degree to which a FAI follows from a theory is demanding. On the other hand, there is more suitable notion of provability of an implication from a theory.

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Language

Definition

Language of a fuzzy logic program (FLP) is given by

- a finite nonempty set $\mathcal{R}$ of *relation symbols*,
- a finite set $\mathcal{F}$ of *function symbols*,
- *arities* of these symbols,
- a denumerable set $\mathcal{V}$ of *variables*,
- symbols for binary logical connectives
  - $\&_1, \&_2, \dots$ (fuzzy conjunctions),
  - $\vee_1, \vee_2, \dots$ (fuzzy disjunctions),
  - $\leftarrow_1, \leftarrow_2, \dots$ (fuzzy implications),
- and symbols for aggregations $@_1, @_2, \dots$

Term and formula

Definition

For given language of FLP, *term* is defined recursively:

- Each variable $\mathbb{X} \in \mathcal{V}$ is a term.
- If $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is term for each functor $f \in \mathcal{F}$.

Definition

For given language of FLP, *formula* is defined as follows:

- If $t_1, \ldots, t_n$ are terms, then $p(t_1, \ldots, t_n)$ is an *atomic formula* for each predicate $p \in \mathcal{P}$.
- If $f_1, \ldots, f_n$ are formulas, then $(f_1 \&_i f_2)$, $(f_1 \vee_i f_2)$, $(f_1 \leftarrow_i f_2)$, $@_i(f_1, \ldots, f_n)$ are formulas.

Multi-adjoint lattice

Definition

A *complete multi-adjoint lattice* is an algebra $\langle L, \wedge, \vee, \otimes_1, \leftarrow_1, \dots, \otimes_n, \leftarrow_n, 0, 1 \rangle$, where

- $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice,
- $\langle L, \otimes_i, 1 \rangle$ is a commutative monoid for each $i \in \{1, \dots, n\}$,
- each adjoint pair $\langle \otimes_i, \leftarrow_i \rangle$ satisfies $a \otimes_i b \leq c$ iff $a \leq c \leftarrow_i b$ for all $a, b, c \in L$.

Fuzzy logic program

Definition

A *fuzzy logic program* for a given language with values from a given multi-adjoint lattice is a finite set P containing *rules* in the form of $\langle A \leftarrow_i B, \vartheta \rangle$ and *facts* in the form of $\langle A, \vartheta \rangle$, where

- the *head* A is an atomic formula,
- the *tail* B is a formula without any implication
- and $\vartheta \in L$.

Herbrand universe and base

Definition

We define *Herbrand universe* as a set of all ground terms (with no free occurrences of variables), it is denoted by $\mathcal{U}_P$.

Herbrand base is defined as a set of all atomic ground formulas and it is denoted by $\mathcal{B}_P$.

Structure for FLP

- A *structure* for FLP P is any L-set in $\mathcal{B}_P$.
- If M is a structure for P, $M(\varphi)$ is interpreted as a degree to which the atomic ground formula φ is true under M.
- This notion can be extended to all formulas. First, lets define $M^\sharp$ as an **L**-set of all ground formulas by
  - $M^\sharp(\varphi) = M(\varphi)$ if φ is a ground atomic formula,
  - $M^\sharp(\varphi \&_i \psi) = M^\sharp(\varphi) \otimes_j M^\sharp(\psi)$ where both φ and ψ are ground and $\otimes_j$ is a truth function interpreting $\&_i$,
  - analogously for the other binary connectives and aggregators.
- Then, we define $M^\sharp_\forall$ to extend the notion to all formulas by $M^\sharp_\forall(\varphi) = \bigwedge\{M^\sharp(\varphi\theta) \mid \theta$ is a substitution and $\varphi\theta$ is ground$\}$.

Model and correct answer

Definition

Structure M is called a *model* for program P if $P(\chi) \leq M^{\sharp}_{\forall}(\chi)$ for each formula χ where $P(\chi) = a$ if $\langle \chi, a \rangle \in P$ and $P(\chi) = 0$ otherwise.
The collection of all models of P will be denoted by $\mathrm{Mod}(P)$.

Definition

A pair $\langle a, \theta \rangle$ consisting of $a \in L$ and a substitution θ is a *correct answer* for a definite program P and an atomic formula φ (called a query) if $M^{\sharp}_{\forall}(\varphi\theta) \geq a$ for each $M \in \mathrm{Mod}(P)$.

Admissible rules

A computation for program P and query φ starts with $\langle\varphi, \emptyset\rangle$. Then, following rules can be used.

1. Overwrite $\langle \alpha A \beta, \Theta\rangle$ to $\langle \alpha(B\Theta' \otimes_i \vartheta)\beta, \Theta \circ \Theta'\rangle$, when
   - A is an atomic formula,
   - Θ' is the most general unifier of A and A',
   - there is a rule $\langle A' \leftarrow_i B, \vartheta\rangle$ in P.
2. Overwrite $\langle \alpha A \beta, \Theta\rangle$ to $\langle \alpha\vartheta\beta, \Theta \circ \Theta'\rangle$, when
   - A is an atomic formula,
   - Θ' is the most general unifier of A and A',
   - there is a fact $\langle A', \vartheta\rangle$.
3. Overwrite $\langle \alpha A \beta, \Theta\rangle$ to $\langle \alpha 0 \beta, \Theta\rangle$, when A is an atomic formula.
4. Compute the truth value of formula and let substitution remain the same.

Computed answer

Definition

A pair $\langle a, \Theta \rangle$, where Θ is a substitution and $a \in L$, is called a *computed answer* for query φ and program P, if there is a sequence $G_0, \ldots, G_n$ such that

- $G_0 = \langle \varphi, \emptyset \rangle$,
- each G_{i+1} we get from G_i by one of the admissible rules,
- $G_n = \langle a, \Theta' \rangle$,
- Θ is Θ' restricted to variables which occur in φ.

Soundness and completeness

Theorem (Soundness)

Each computed answer for fuzzy logic program P and query φ is a correct answer for the same program and query.

Theorem (Completeness)

For every correct answer $\langle a, \Theta \rangle$ for program P and query φ, there exist a sequence of elements $a_i \in L$ such that

- $a \leq \bigvee_i a_i$
- *and for an arbitrary i_0 there exists a computed answer $\langle b, \Theta \rangle$ such that $a_{i_0} \leq b$.*

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Main results

Theorem

For each set T of FAIs and $A \Rightarrow B$ there is a definite program P such that $||A \Rightarrow B||_T = a$ iff for each attribute y such that $B(y) > 0$, the pair $\langle a \otimes B(y), \emptyset \rangle$ is a correct answer for the program P and y.

Corollary

For each set T of FAIs and $A \Rightarrow B$ there is a definite program P such that $||A \Rightarrow B||_T$ is the supremum of all degrees $a \in L$ for which $\langle a \otimes B(y), \emptyset \rangle$ is a correct answer for P and all y satisfying $B(y) > 0$.

Sketch of the proof 1/2

- Consider a language (of FLP) without functors and with only nullary relation symbols $R = \{y_1, y_2, \ldots, y_k\}$ that correspond to attributes which appear in FAIs from T to a nonzero degree.
- Clealy, R is a finite set and Herbrand base of any program is equal to R.
- Moreover, we consider the following logical connectives:
  - implication $\Leftarrow$ (interpreted by the residuum $\rightarrow$),
  - conjunction & (interpreted by the infimum $\wedge$),
  - a unary aggregation ts (interpreted by the hedge *, i.e. $M(ts(\varphi)) = M(\varphi)^*$),
  - for each rational $a \in (0, 1]$ a unary aggregation sh_a called an a-shift aggregation (interpreted by $M(sh_a(\varphi)) = a \rightarrow M(\varphi)$).

Sketch of the proof 2/2

- For any $C \Rightarrow D \in T$ and $y \in Y$ such that $D(y) > 0$ and all attributes $z \in Y$ satisfying $C(z) > 0$ being exactly $z_1, \ldots, z_n$, consider a rule
$$\langle y \Leftarrow ts\big(sh_{A(z_1)}(z_1)\& \cdots \& sh_{A(z_n)}(z_n)\big), D(y)\rangle.$$
- The fuzzy logic program P_T generated by T consists only all these rules.
- The proof then continues by observing that $||A \Rightarrow B||_T = a > 0$ iff $||A \Rightarrow a{\otimes}B||_T = 1$ iff $||\emptyset \Rightarrow a{\otimes}B||_{T\cup\{\emptyset\Rightarrow A\}} = 1$ iff $a \otimes B(y) \leq ||\emptyset \Rightarrow \{^1/y\}||_{T\cup\{\emptyset\Rightarrow A\}}$ for all $y \in Y$ such that $B(y) > 0$.
- The latter is true iff for each $y \in Y$ such that $B(y) > 0$, the pair $\langle a \otimes B(y), \emptyset\rangle$ is a correct answer for the program $P_{T\cup\{\emptyset\Rightarrow A\}}$ and query y.

Example 1/3

- Let **L** be the standard Łukasievicz structure of truth degrees with * being the identity.
- Consider a set of attributes $Y = \{IA, IM, hAT, hFE, hP\}$.
- Let T being a set containing the following FAIs over Y:
  - $\{{}^{0.7}/IA, {}^{0.9}/IM, {}^{0.4}/hAT\} \Rightarrow \{{}^{0.6}/hFE, {}^{0.9}/hP\}$,
  - $\{{}^{0.8}/IA\} \Rightarrow \{{}^{0.7}/IM\}$.
- Using the presented Theorem, we can find a FLP P_T that corresponds to FAIs from T:
  - $hFE \overset{0.6}{\Leftarrow} ts(sh_{0.7}(IA) \& sh_{0.9}(IM) \& sh_{0.4}(hAT))$,
  - $hP \overset{0.9}{\Leftarrow} ts(sh_{0.7}(IA) \& sh_{0.9}(IM) \& sh_{0.4}(hAT))$,
  - $IM \overset{0.7}{\Leftarrow} ts(sh_{0.8}(IA))$.
- Obviously, the aggregator *ts* interpreted by identity can be omitted.

Example 2/3

- All aggregation interpreting $sh_a(y)$ as well as the function $\wedge$ interpreting conjunctor & are left-semicontinuous in this case. Thus, we can characterize $||A \Rightarrow B||_T$ using computed answers for program $P_{T \cup \{\emptyset \Rightarrow A\}}$ and queries $y \in Y$ with $B(y) > 0$.
- For example, someone can ask "How much expensive are quite new cars with automatic transmission?", i.e., more precisely "To which degree $a \in L$, is the FAI $\{{}^{0.6}/lA, {}^{1}/hAT\} \Rightarrow \{{}^{a}/hP\}$ true in T?".
- First, expand P_T to $P_{T \cup \{\emptyset \Rightarrow A\}}$ by adding facts:
  - $lA \overset{0.6}{\Leftarrow}$,
  - $hAT \overset{1}{\Leftarrow}$.

Example 3/3

- Then, we can compute an answer to query *hP* using the usual admissible rules of FLPs:

$$\langle hP, \emptyset\rangle$$
$$\langle 0.9 \otimes \big(sh_{0.7}(lA)\&sh_{0.9}(lM)\&sh_{0.4}(hAT)\big), \emptyset\rangle,$$
$$\langle 0.9 \otimes \big(sh_{0.7}(lA)\&sh_{0.9}(0.7 \otimes sh_{0.8}(lA))\&sh_{0.4}(hAT)\big), \emptyset\rangle,$$
$$\langle 0.9 \otimes \big(sh_{0.7}(0.6)\&sh_{0.9}(0.7 \otimes sh_{0.8}(0.6))\&sh_{0.4}(1)\big), \emptyset\rangle,$$
$$\langle 0.9 \otimes \big(0.7 \to 0.6 \wedge 0.9 \to (0.7 \otimes (0.8 \to 0.6)) \wedge 0.4 \to 1\big), \emptyset\rangle,$$
$$\langle 0.5, \emptyset\rangle.$$

- Using this computed answer $\langle 0.5, \emptyset\rangle$, we immediately get $||\{{}^{0.6}/lA, {}^{1}/hAT\} \Rightarrow \{{}^{1}/hP\}||_T = 0.5$, i.e., $||\{{}^{0.6}/lA, {}^{1}/hAT\} \Rightarrow \{{}^{0.5}/hP\}||_T = 1$.

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