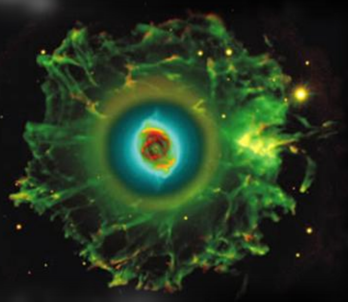
Dynamics and non-classicality of single-qubit laser



Sergei Kilin
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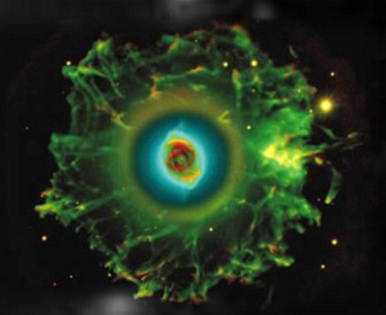
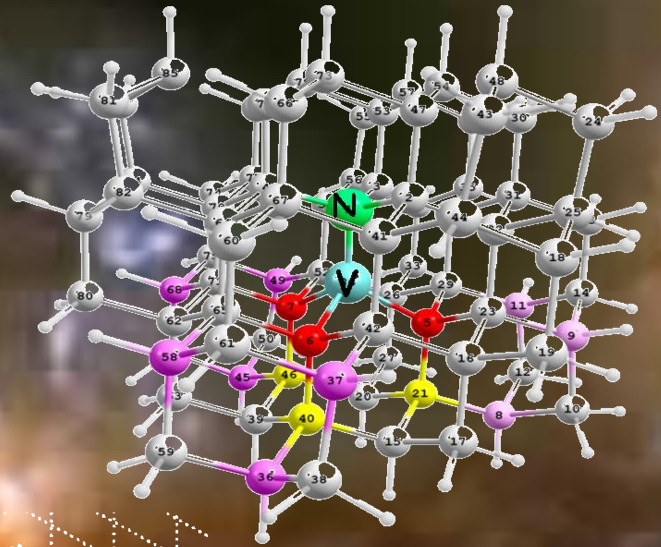
Olomouc, August 22, 2013

Quantum optics and QI in Minsk



Stepanov Institute of Physics, Minsk



NV color centers in diamond

-- appl. for QO &QI

QRNG

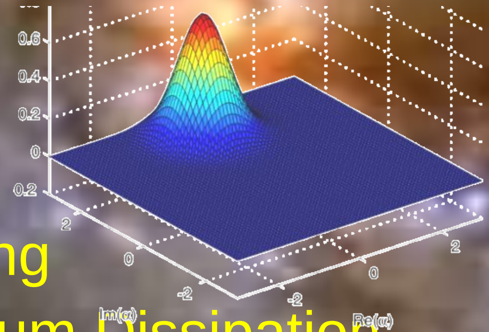
QKD systems

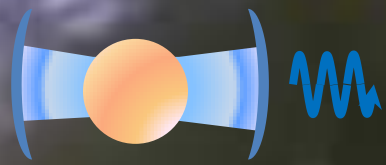
Quantum Tomography

Quantum States Engineering

Non-Markovianity of Quantum Dissipation

Single Emitters Quantum Physics (incl . CQED) →





ICQOQI – biannual international conference on quantum optics and quantum information launched in 1986 in Minsk

Outline

- 0. Introduction
- 1. History & Problems
- 2. Model
- 3. Stationary state
  - Equations
  - Parameterization
  - Solution in the form of nonlinear coherent states
  - Nonlinear transition probabilities
- 4. Quasi-distributions. Nonclassicality
- 5. Dynamics
  - Stages of evolution and regions of phase plane
  - Phase bistability
  - Approaching stationary state
- 6. Conclusions

- In this work we:
  - provide general unique and uniformly applicable solution for single-qubit laser (SQL) stationary state in terms of nonlinear coherent states;
  - introduce nonlinear transition probabilities, revealing quantum nature of single-qubit laser;
  - provide description in terms of quasi-distributions and prove nonclassicality of the stationary state;
  - investigate dynamics of the system.

0. Introduction

- One-atom-one-mode microlaser (or micromaser):
  - an extreme case of lasers;

[Y. Mu and C. M. Savage, Phys. Rev. A 46, 5944 (1992).
D. Meschede et al., PRL 54, 551 (1985); M. Brune et al., PRL 59, 1899 (1987);
K. An et al., PRL 73, 3375 (1994); J. McKeever et al., Nature 425, 268 (2003);
G. M. Meyer et al., PRA 56, R1099 (1997); Z. G. Xie et al., PRL 98, 117401 (2007);
O. Astafiev et al., Nature 449, 588 (2007)].

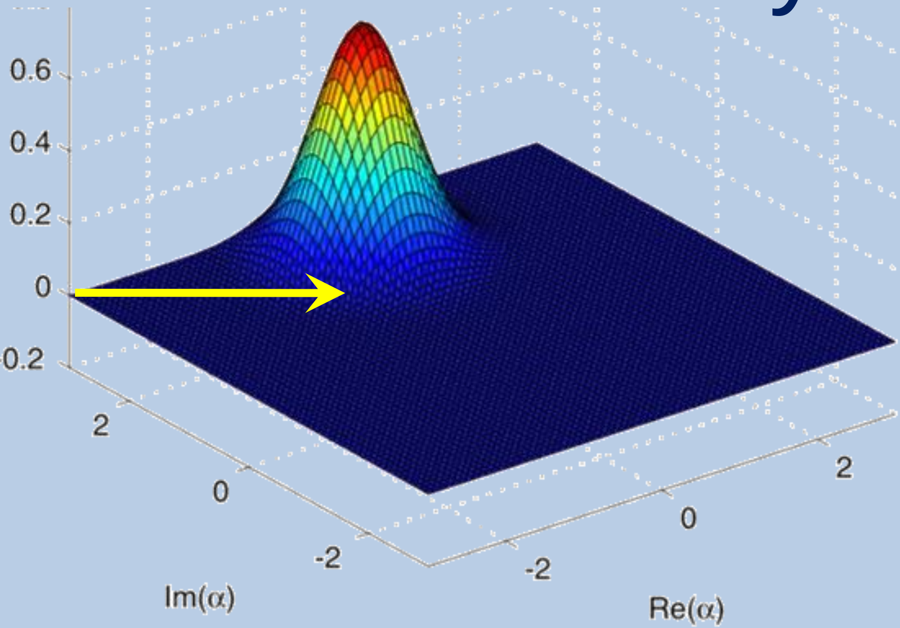
  - intrinsically quantum system with a number of properties strongly different from ordinary lasers;

[M. O. Scully and M. S. Zubairy, Quantum Optics(1997);
C. Cohen-Tannoudji et al., Photons and Atoms: Introduction to Quantum Electro-dynamics(1989);
P. R. Berman, Cavity Quantum Electrodynamics(1994);
R. J. Thompson et al., PRL 68, 1132 (1992);
A. Boca et al., PRL 93, 233603 (2004); J. H. Eberly et al., PRL 44, 1323 (1980);
G. Rempe et al., PRL 58, 353 (1987); K. M. Birnbaum et al., Nature 436, 87 (2005);
J. M. Raimond et al., Rev. Mod. Phys. 73, 565 (2001)].

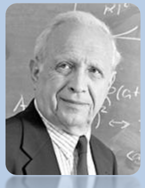
  - a source of nonclassical light.

[P. Filipowicz et al., PRA 34, 3077 (1986); J. McKeever et al., Science 303, 1992 (2004);
T. Wilk et al., Science 317, 488 (2007); J. Simon et al., PRL 98, 183601 (2007);
S.Ya. Kilin, T. B. Karlovich, JETP 95, 805 (2002)].

1.History & Problems

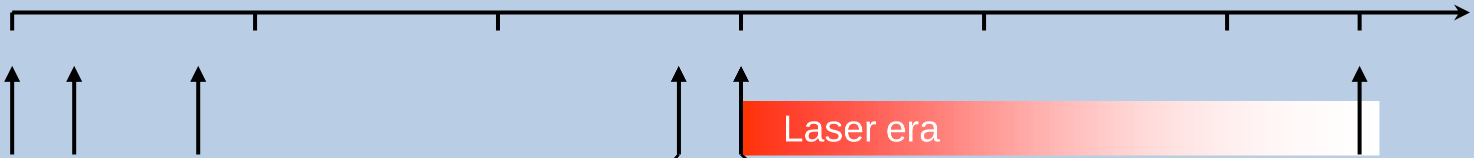


$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

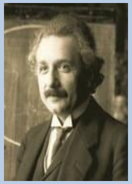
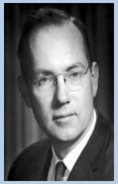


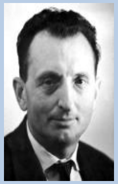
Glauber
1963

$$a|\alpha\rangle = \alpha|\alpha\rangle$$



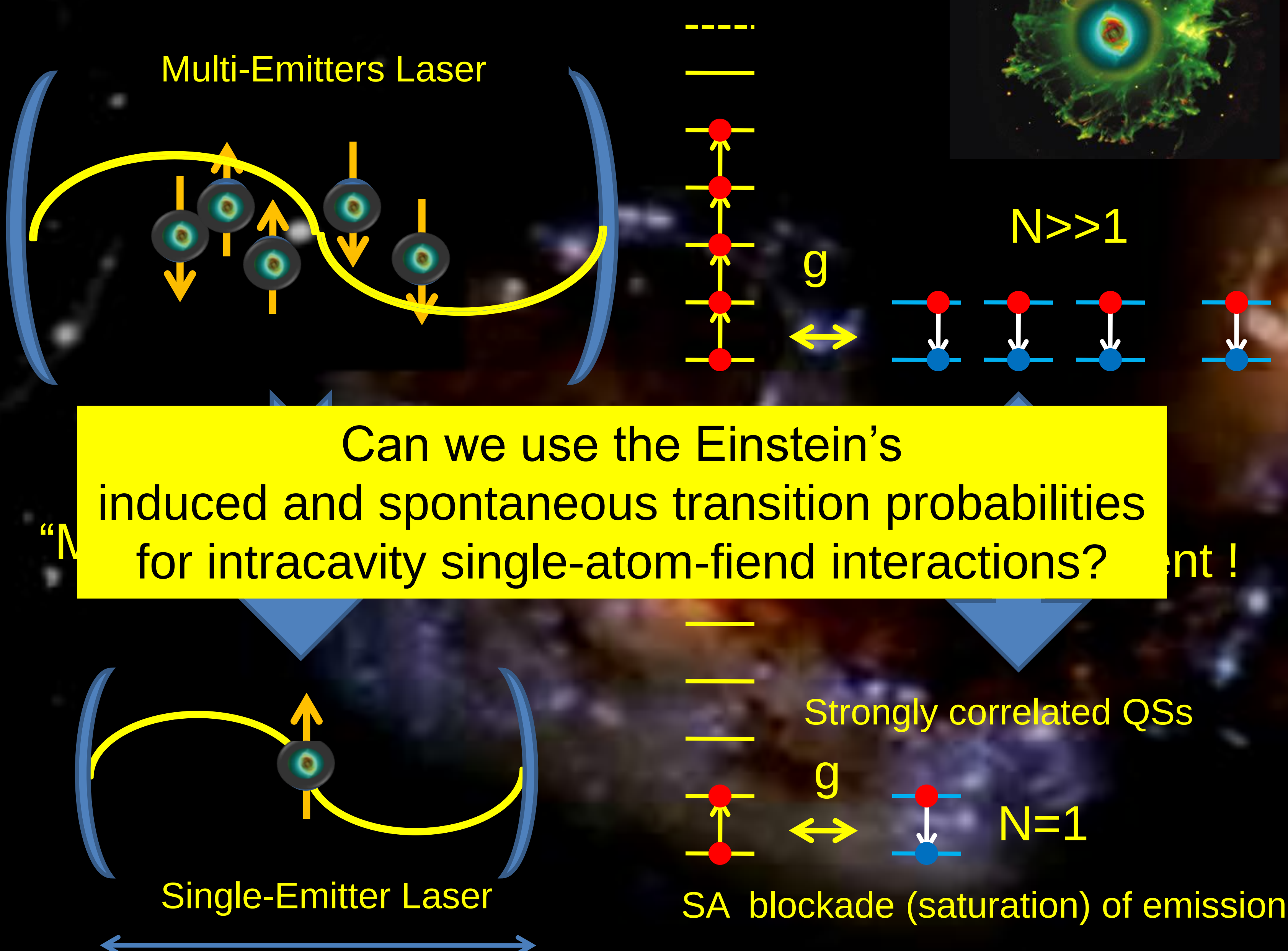
1900 1905 1916 1954 1960 2013




Planck Einstein Townes Basov Prokhorov Maiman

From Macro to Micro



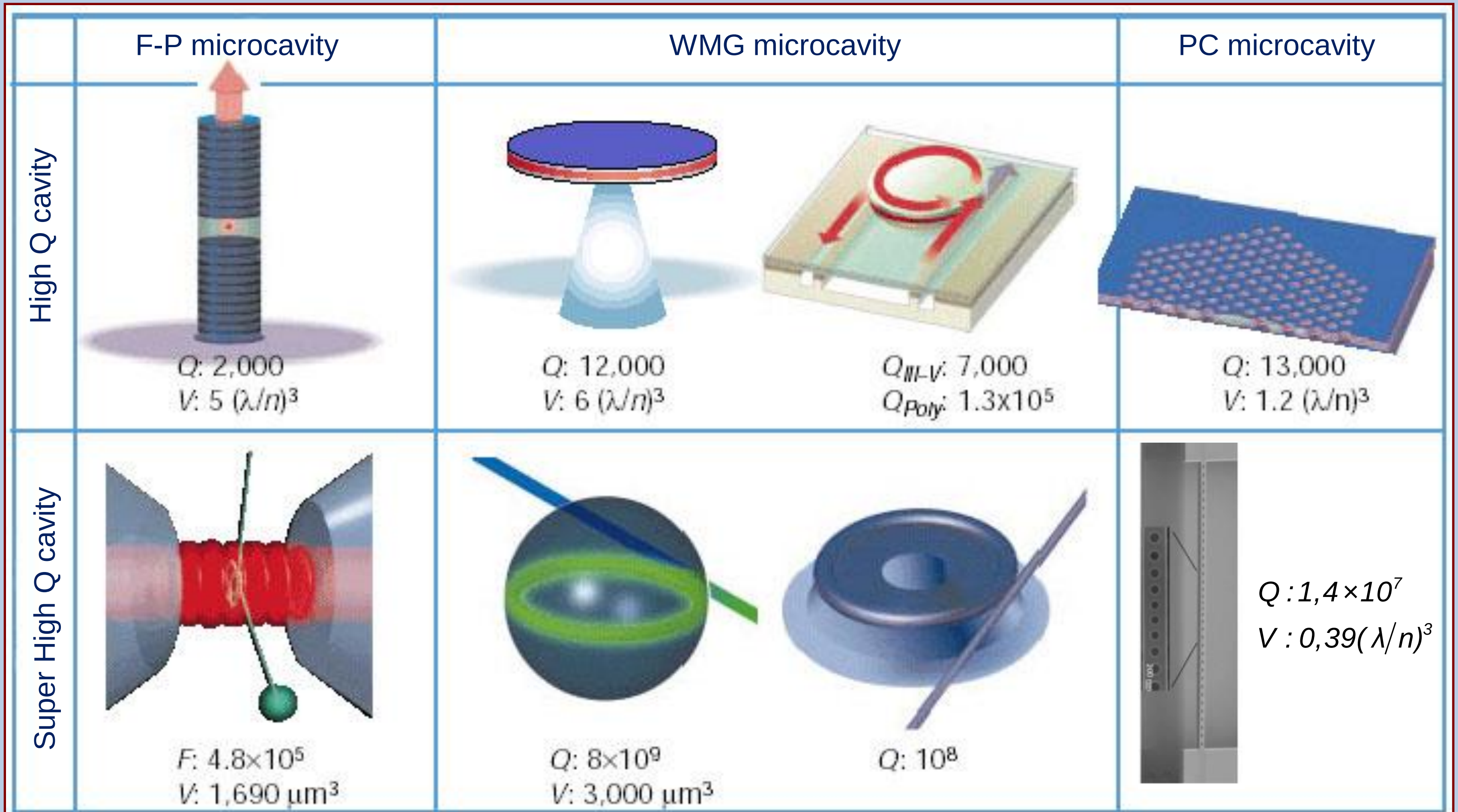
Can we use the Einstein’s
induced and spontaneous transition probabilities
for intracavity single-atom-fiend interactions?

OPTICAL MICROCAVITIES *

$$Q = \frac{\omega_f}{2k}, \qquad g = \frac{\mu |u(r)|}{\hbar}, \qquad |u(r)| = \sqrt{\frac{\hbar \omega_f}{\varepsilon_0 V}}$$

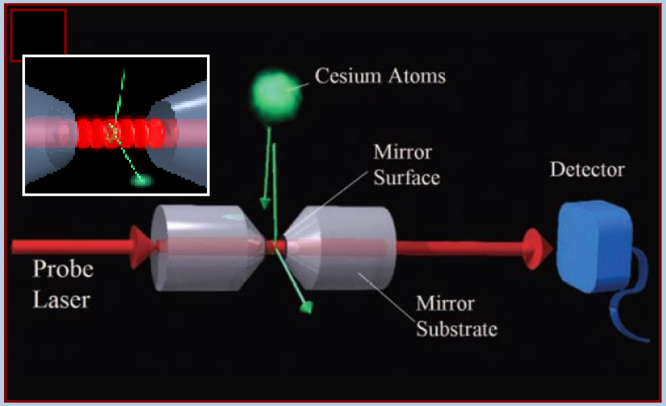
Q - Quality factor

V - Mode volume



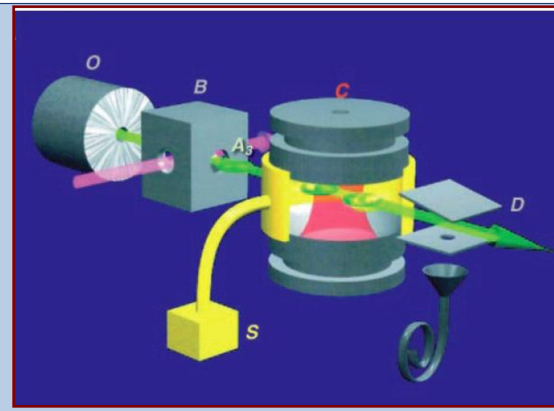
* K.J.Vahala Optical microcavities *Nature*, V.424, P.839-846 (2003), P.B.Deotare et.al., *APL*, V.94, 121106 (2009).

SINGLE-QUBIT LASERS & MASERS



SQL on Cs atoms

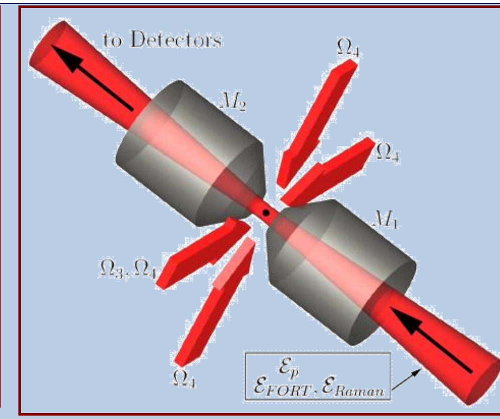
H. J. Kimble et.al. *Science,* (2004)



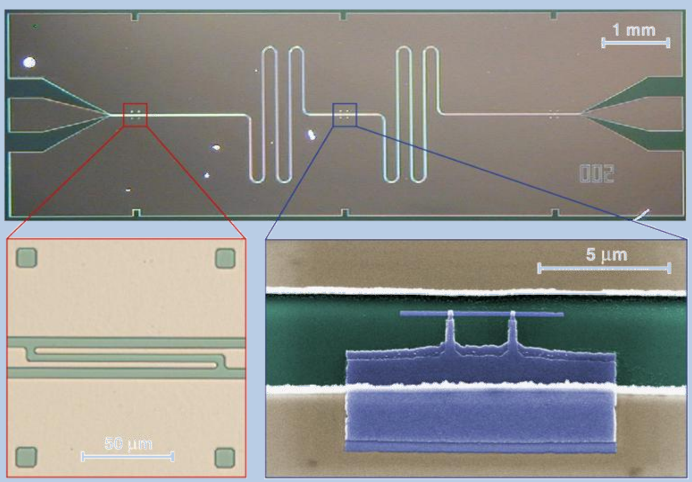
SQM on Rb atoms

H.Walther et.al. (2001),

S. Haroche et.al. *Rev. Mod. Phys.* V.73**,** P.565 (2001)

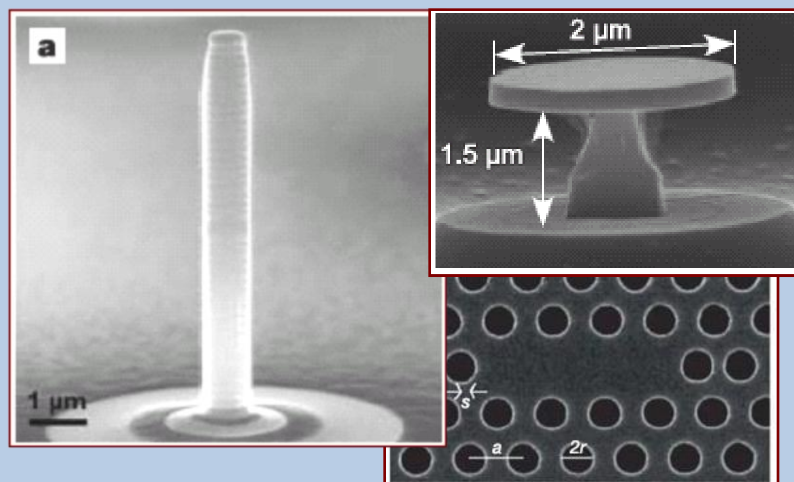


H.J.Kimble et.al. *PRL*, (2004)



Superconducting SQL

O. Astaf'ev, K. Inomata, et al., Nature 449,(2007).

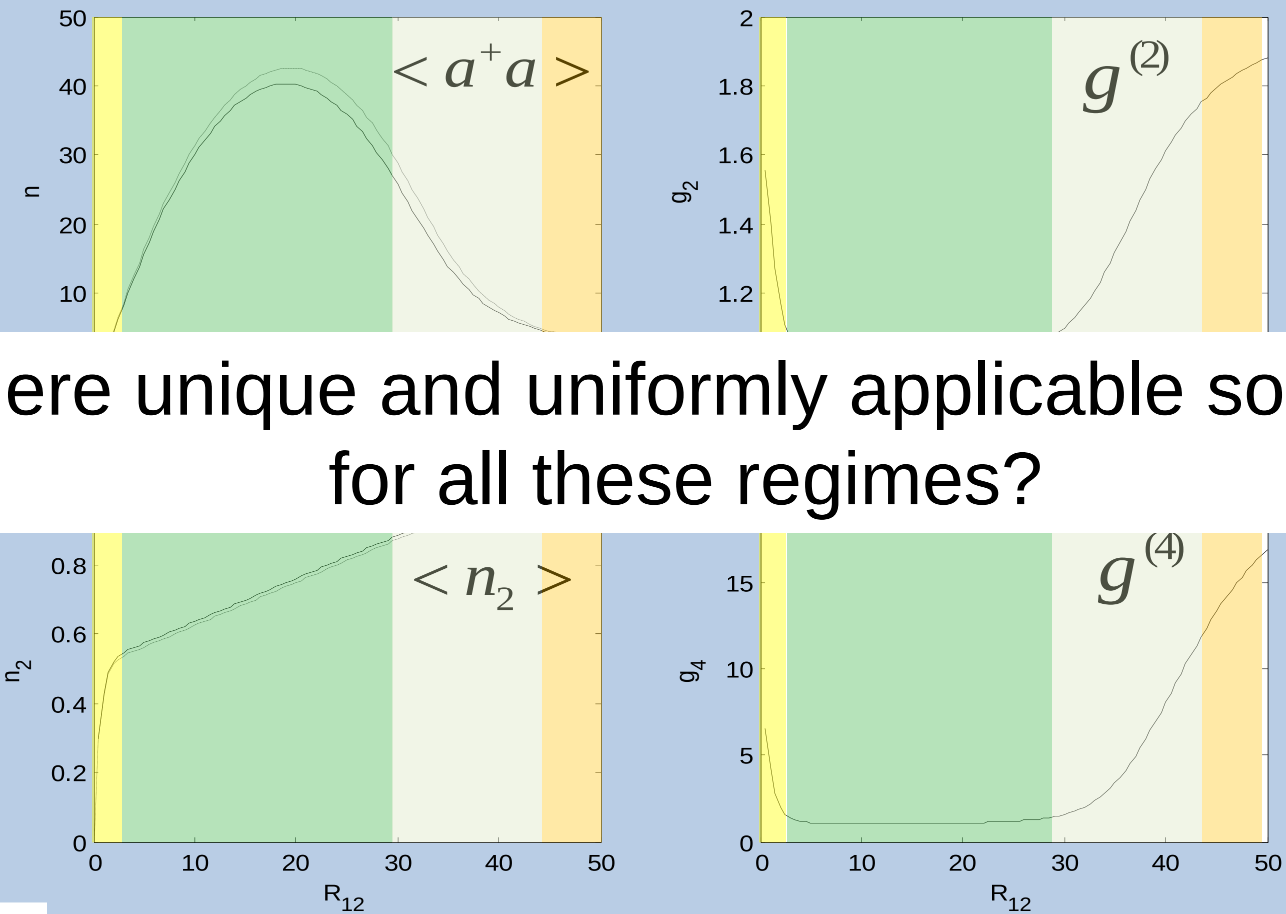


SQD lasers

Z. G. Xie, S. Gotzinger, et al., PRL 98, (2007).

History & Problems

Different regimes of SQ -Laser



Is there unique and uniformly applicable solution for all these regimes?

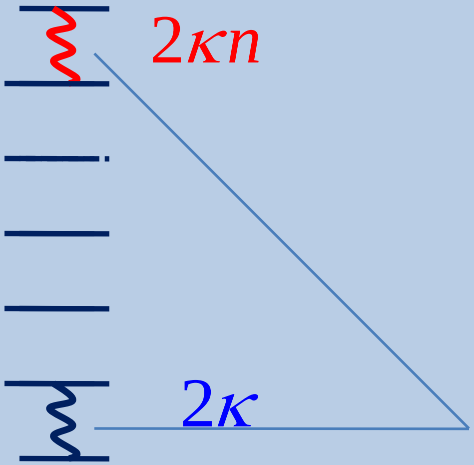
Quantum (linear & nonlinear)
Lasing
Self-quenching
Thermal

[Karlovich, Kilin Opt.&Spectr.(2001)
Mu, Savage, Phys. Rev. (1992).]

History & Problems

2014 -100th anniversary of Yakov Zel'dovich

Yakov Zel'dovich



Zel'dovich's paradox of ***damped quantum harmonic oscillator*** *(1969)*

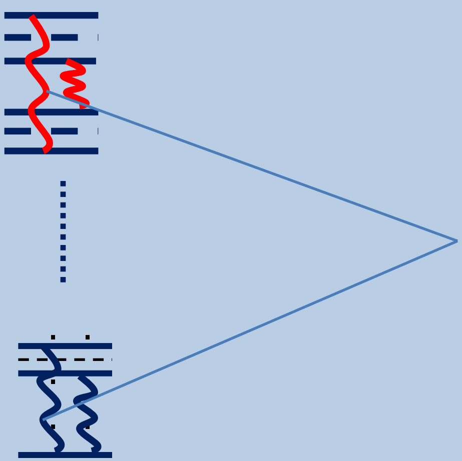
$$d\alpha / dt = -2\kappa\alpha$$

Where is $2\kappa n$ *?*

We can not distinguish the emitted photons!

2013 -50th anniversary of Jaynes –Cummings paper

Edwin Jaynes



Jaynes-Cummings model *(1963)*

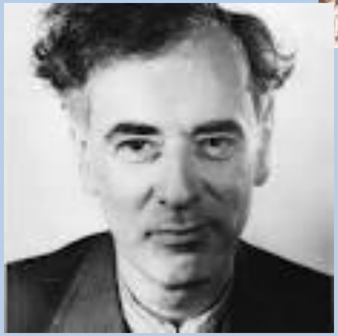
Photons become distinguishable!

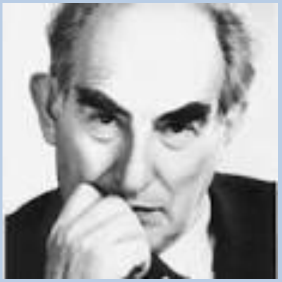
Fast decay $2\kappa n$ *appears.*

Soviet atomic project







Edwin Thompson Jaynes

Frederick W. Cummings

90 PROCEEDINGS OF THE IEEE January

1963 Jaynes and Cummings: Quantum and Semiclassical Radiation Theories 89

Comparison of Quantum and Semiclassical Radiation Theories with Application to the Beam Maser*

E. T. JAYNES† AND F. W. CUMMINGS‡

Summary—This paper has two purposes: 1) to clarify the relationship between the quantum theory of radiation, where the electromagnetic field-expansion coefficients satisfy commutation relations, and the semiclassical theory, where the electromagnetic field is considered as a definite function of time rather than as an operator; and 2) to apply some of the results in a study of amplitude and frequency stability in a molecular beam maser.

In 1), it is shown that the semiclassical theory, when extended to take into account both the effect of the field on the molecules and the effect of the molecules on the field, reproduces almost quantitatively the same laws of energy exchange and coherence properties as

* Received September 28, 1962.

† Washington University, St. Louis, Mo.

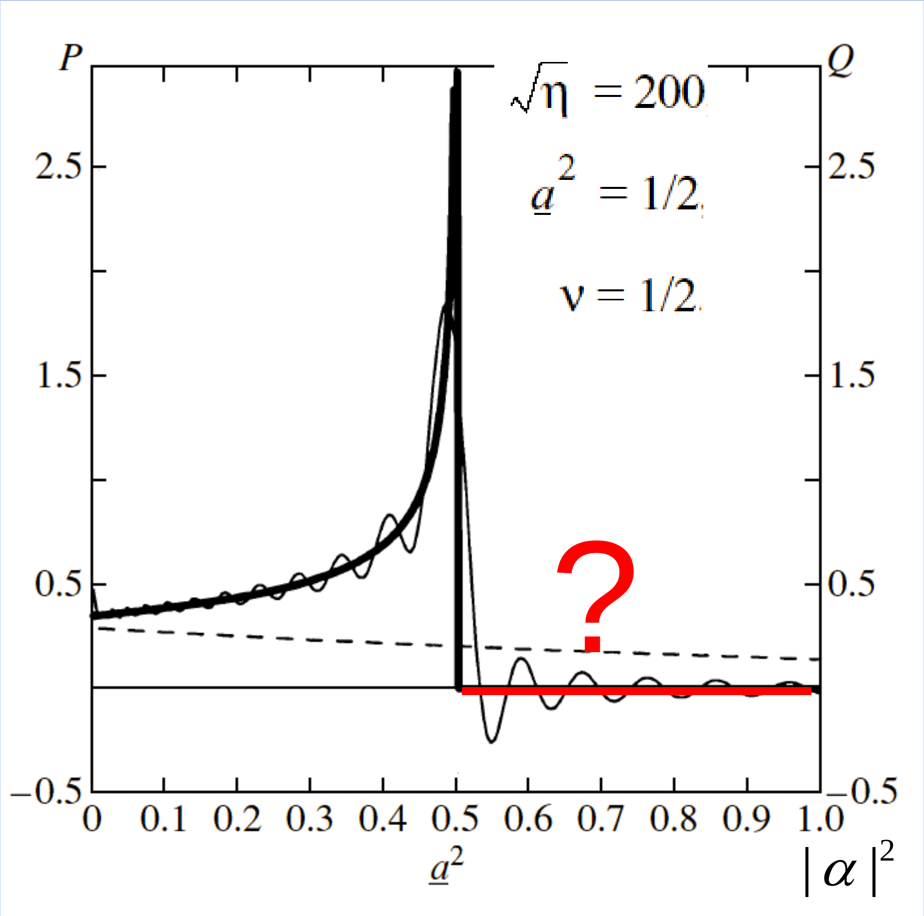
‡ Aeronutronic, Division of Ford Motor Co., Newport Beach, Calif.

1.History & Problems

History & Problems

"Phase" transition for P-function

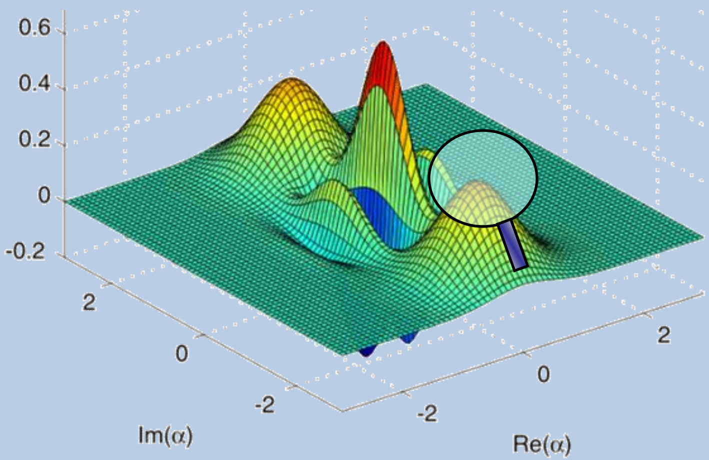
The P-function.
Classical
or quantum?



Black hole - like effect

[Kilin, Karlovich, JETP 95,(2002)].

History & Problems



Local properties of Glauber P- function via measurement of "pointer" observable

$$\Phi(\sigma;\alpha)=\frac{1}{\pi\sigma^2}\int d^2\gamma P(\gamma)\exp\left(-\frac{|\alpha-\gamma|^2}{\sigma^2}\right)$$

s-Parameterized phase-space functions *[Ararwal, Wolf, 1968]*

$$\Phi(\sigma;\alpha)\equiv P(\alpha;s),\ with\ s=1-2\sigma^2$$ *[Cahill, Glauber 1969]*

$$\sigma^2=1/2\Rightarrow W;1\Rightarrow Q$$

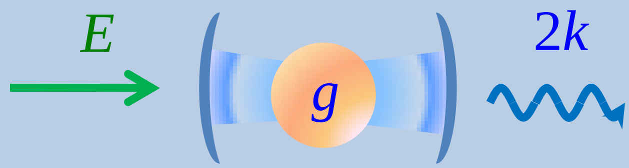
"Pointer" observable

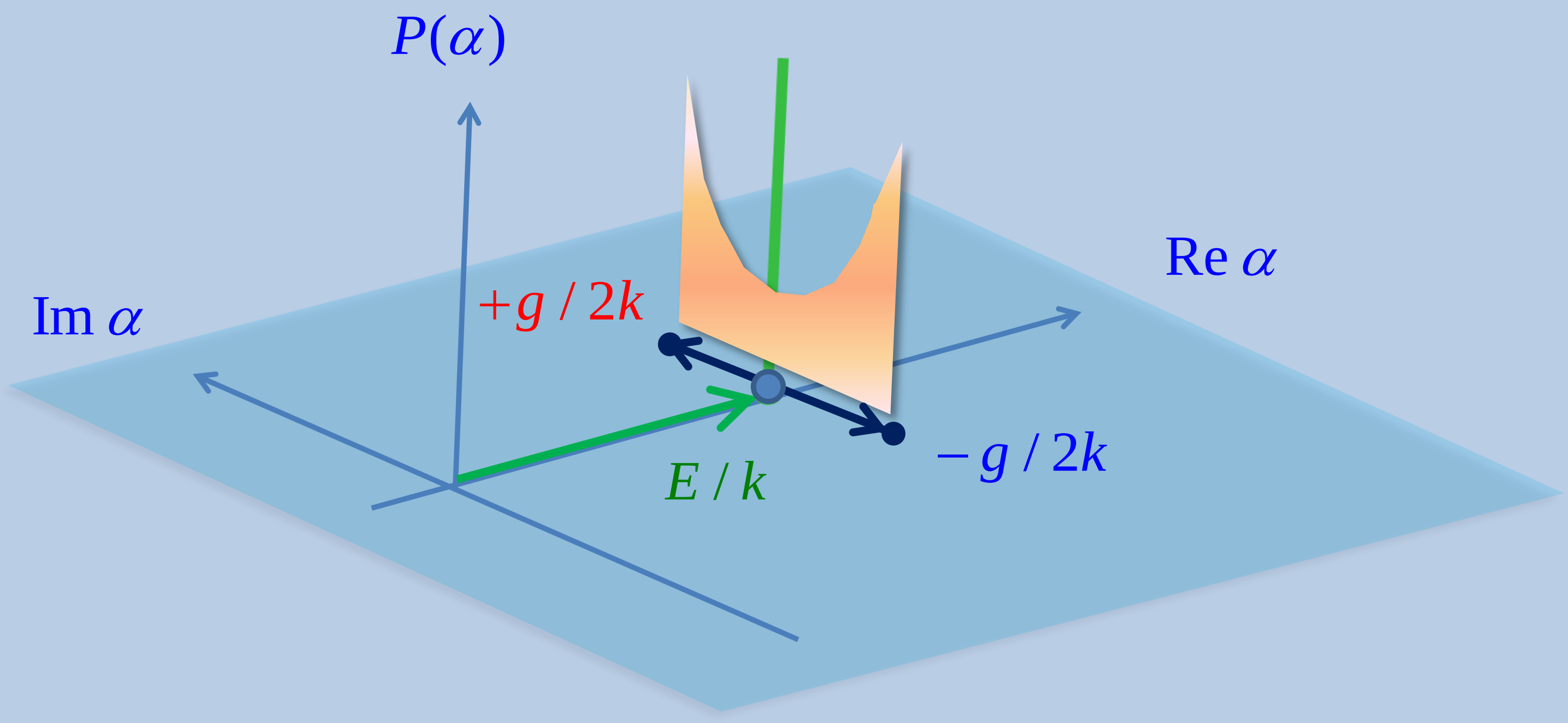
$$\hat{\Phi}(\sigma;\alpha)=\frac{1}{\pi\sigma^2}:\exp\left(-\frac{(\hat{a}^+-\alpha^*)(\hat{a}-\alpha)}{\sigma^2}\right):\equiv\hat{\delta}(\hat{a}-\alpha;s)$$

$$\Phi(\sigma;\alpha)=<\hat{\Phi}(\sigma;\alpha)>$$

History & Problems:

Single-Atom Phase Bistability in coherently driven J-C system



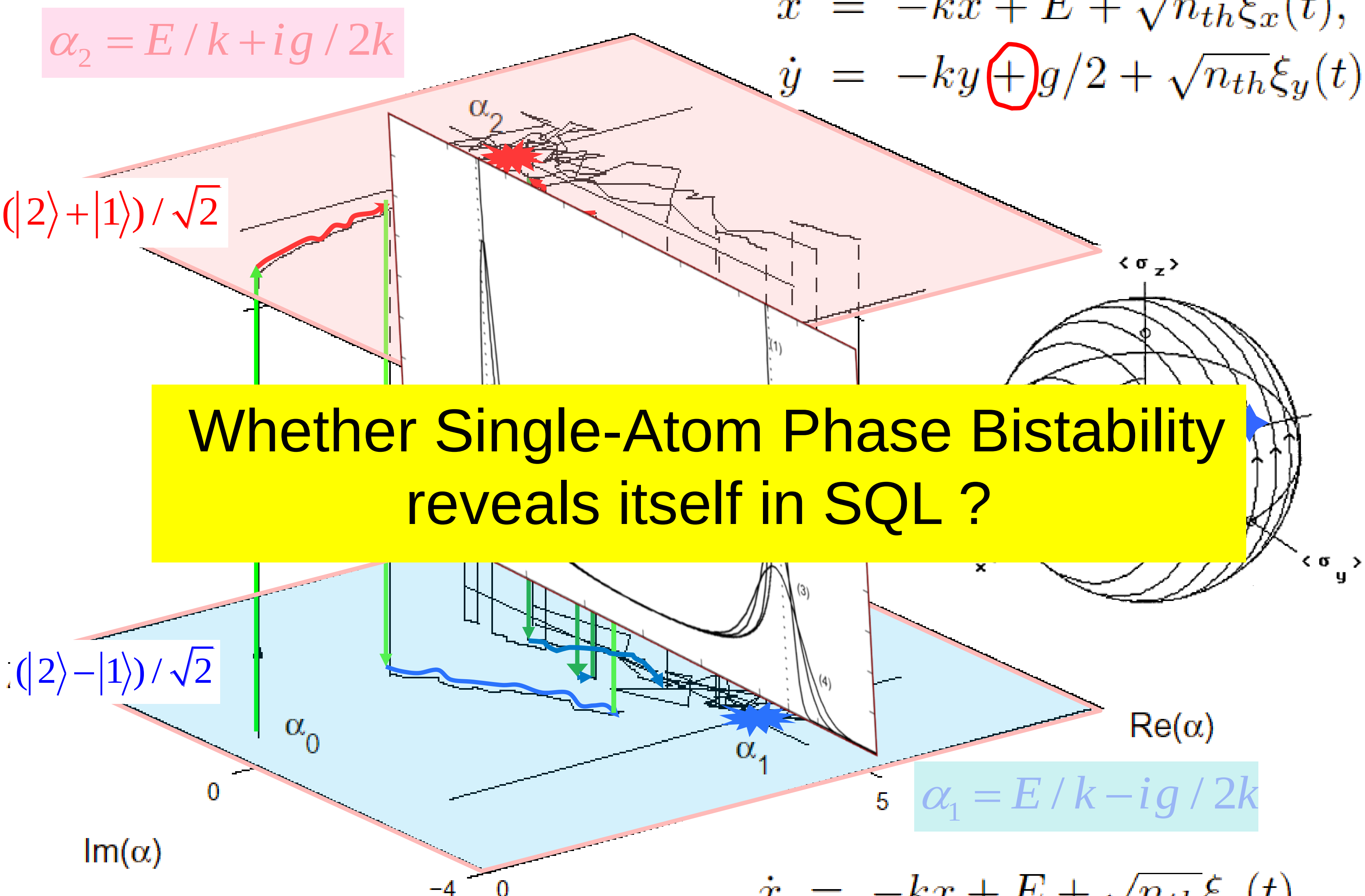


S.Kilin, T.Krinitskaya, JOSA B (1991)

History & Problems:

Single-Atom Phase Bistability in coherently driven J-C system.

$$\dot{x} = -kx + E + \sqrt{n_{th}}\xi_x(t),$$
$$\dot{y} = -ky + g/2 + \sqrt{n_{th}}\xi_y(t).$$



Whether Single-Atom Phase Bistability reveals itself in SQL ?

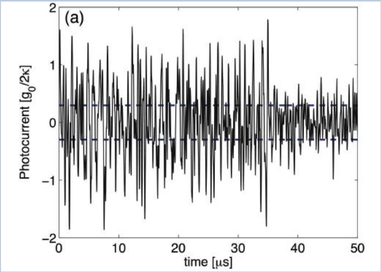
$$\dot{x} = -kx + E + \sqrt{n_{th}}\xi_x(t),$$
$$\dot{y} = -ky - g/2 + \sqrt{n_{th}}\xi_y(t),$$

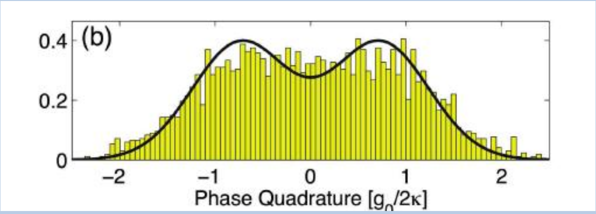
S.Kilin, T.Krinitskaya, JOSA B (1991)
M.Armen, A.Miller, H.Mabuchi PRL (2009)

History & Problems:

Single-Atom Phase Bistability in coherently driven J-C system.

THEORY (1991) S.Kilin, T.Krinitskaya, JOSA B (1991)

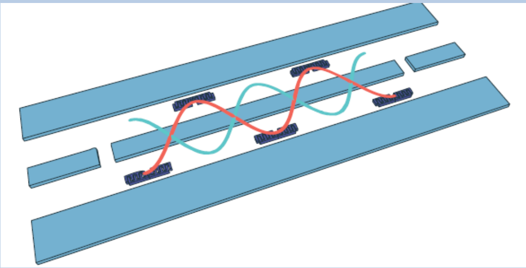




EXPERIMENT (2009)

M.Armen, A.Miller, H.Mabuchi PRL (2009)

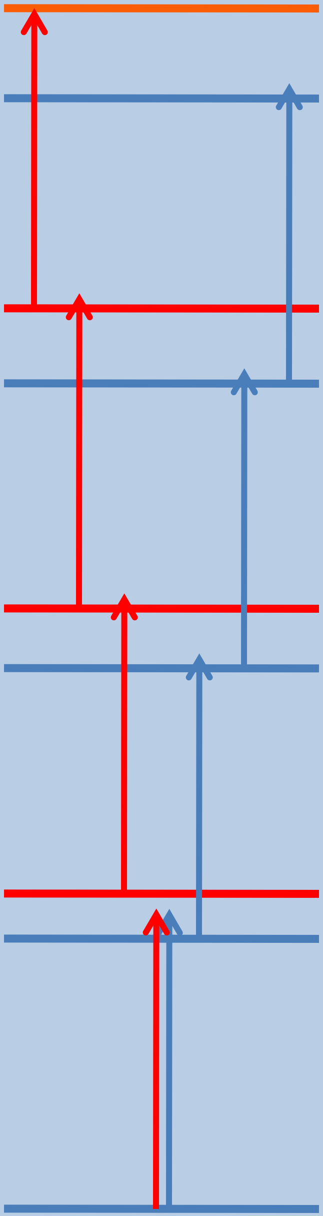
Cirquit QED (transmon qubits)



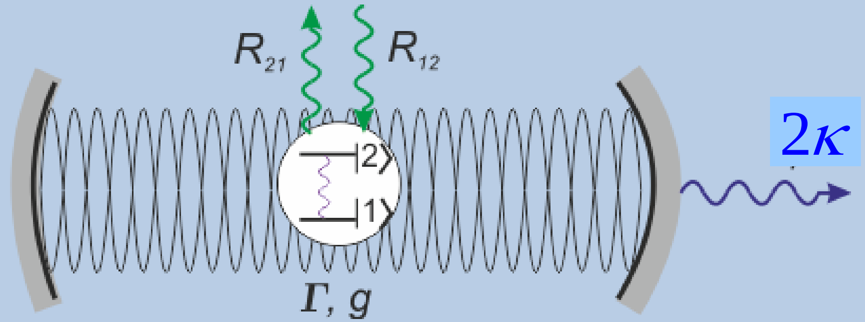
- *Ultralow energy optical switching*
- *Quantum feedback*
- *Qubit measurment*
- *….*

M Delanty[1], S Rebić[1] and J Twamley

New Journal of Physics **13** (2011) 053032



2. Model



- Master equation:

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + 2\kappa L_a\rho + R_{12}L_{\sigma_+}\rho + R_{21}L_{\sigma_-}\rho + \Gamma L_{\sigma_z}$$

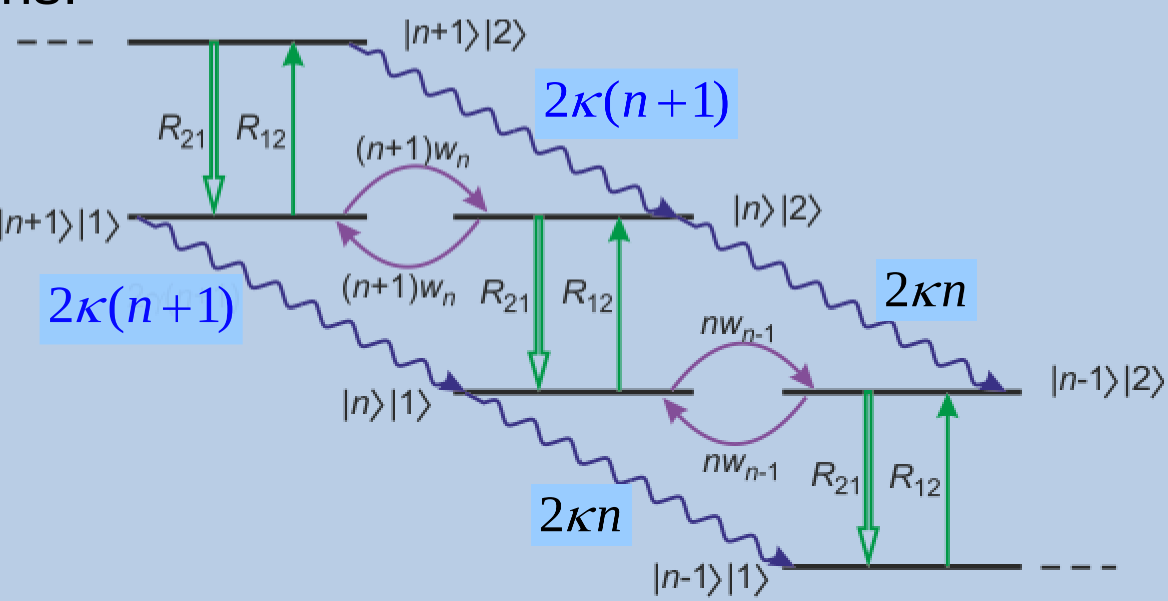
Jaynes-Cummings interaction

Field decay

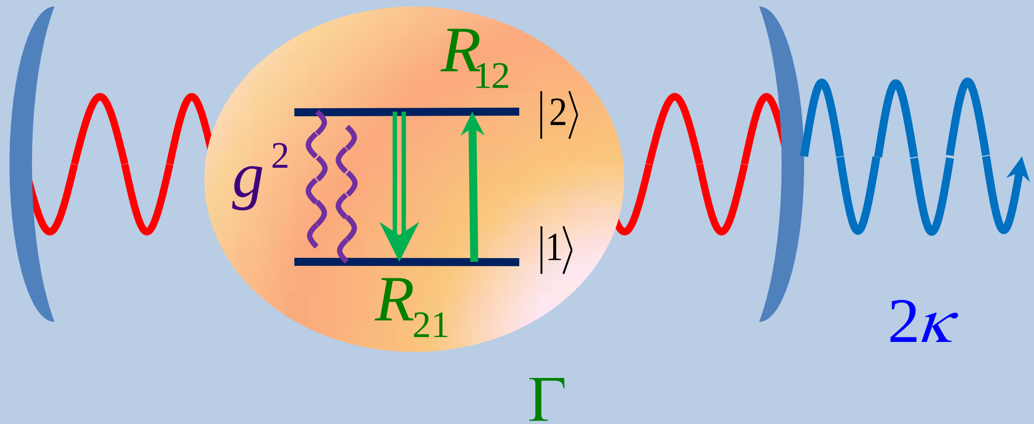
Atom: incoherent pump, decay and decoherence

$$H = g\hbar\left(a\sigma_+ + a^+\sigma_-\right), \quad 2L_X\rho = 2X,\rho X^+ - X^+X\rho - \rho X^+X$$

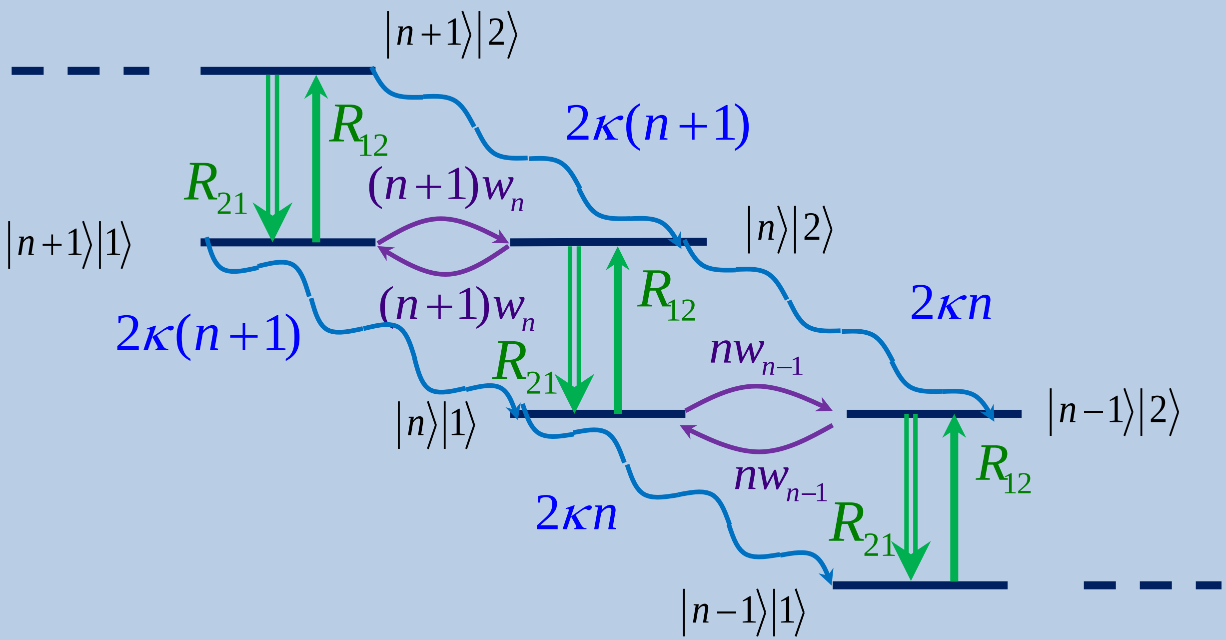
- Transitions:



2.1 Transitions



Scheme of transitions:



3. Stationary state

- Density matrix presentation:

$$\rho = |1\rangle\langle 1| \otimes \rho_{11} + |2\rangle\langle 2| \otimes \rho_{22} - i|1\rangle\langle 2| \otimes \rho_{12} + i|2\rangle\langle 1| \otimes \rho_{21}$$

qubit states conditional field mode states

- Invariance of the ME with respect to the simultaneous phase shift of the field and the polarization of the qubit:
  - Off-diagonal elements are uniquely expressed in terms of diagonal:

$$\rho_{12} = \frac{\rho_f a^+}{g/\kappa}, \quad \rho_{21} = \frac{a\rho_f}{g/\kappa}, \quad \rho_f = \rho_{11} + \rho_{22}$$

  - Conditional field mode states are diagonal in Fock basis:

$$\rho_{11} = \rho_{11}(\hat{n}), \quad \rho_{22} = \rho_{22}(\hat{n})$$

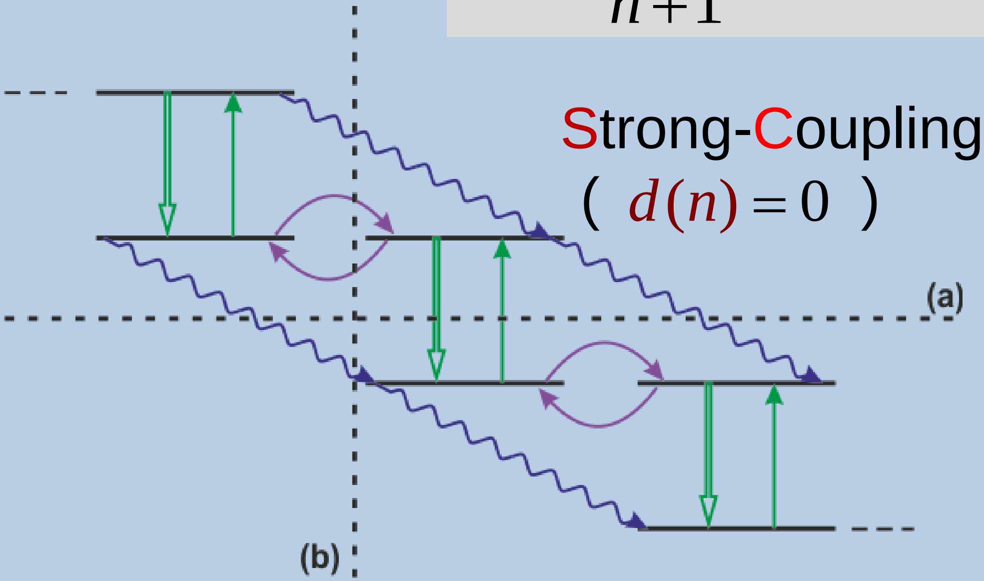
3.1 Thermodynamics of SQL

- 1) Balance of transitions between (*n*+1) and *n* excitations– **(a)**:

$$\left(2\nu_0 + n + 1\right)\rho_{22} = \left(2a_0^2 - J\right)\rho_{11}$$

- 2) Balance of transitions between (*n*+1) and *n* photons – **(b)**:

$$\rho_{22} = \frac{1}{n+1} J\left\{1 + d(n)\right\}\rho_{11}$$



Strong-Coupling Limit

($d(n) = 0$) → $\rho_{22}(n) = \rho_{11}(n+1)$

Deviation from SCL:

$$d(n)\rho_{11} = \frac{2}{\eta}\left(\mu_0 + n - J\right)\rho_f$$

Pump ↓ Losses ↓ Dephasing ↓ ICSpE ↓

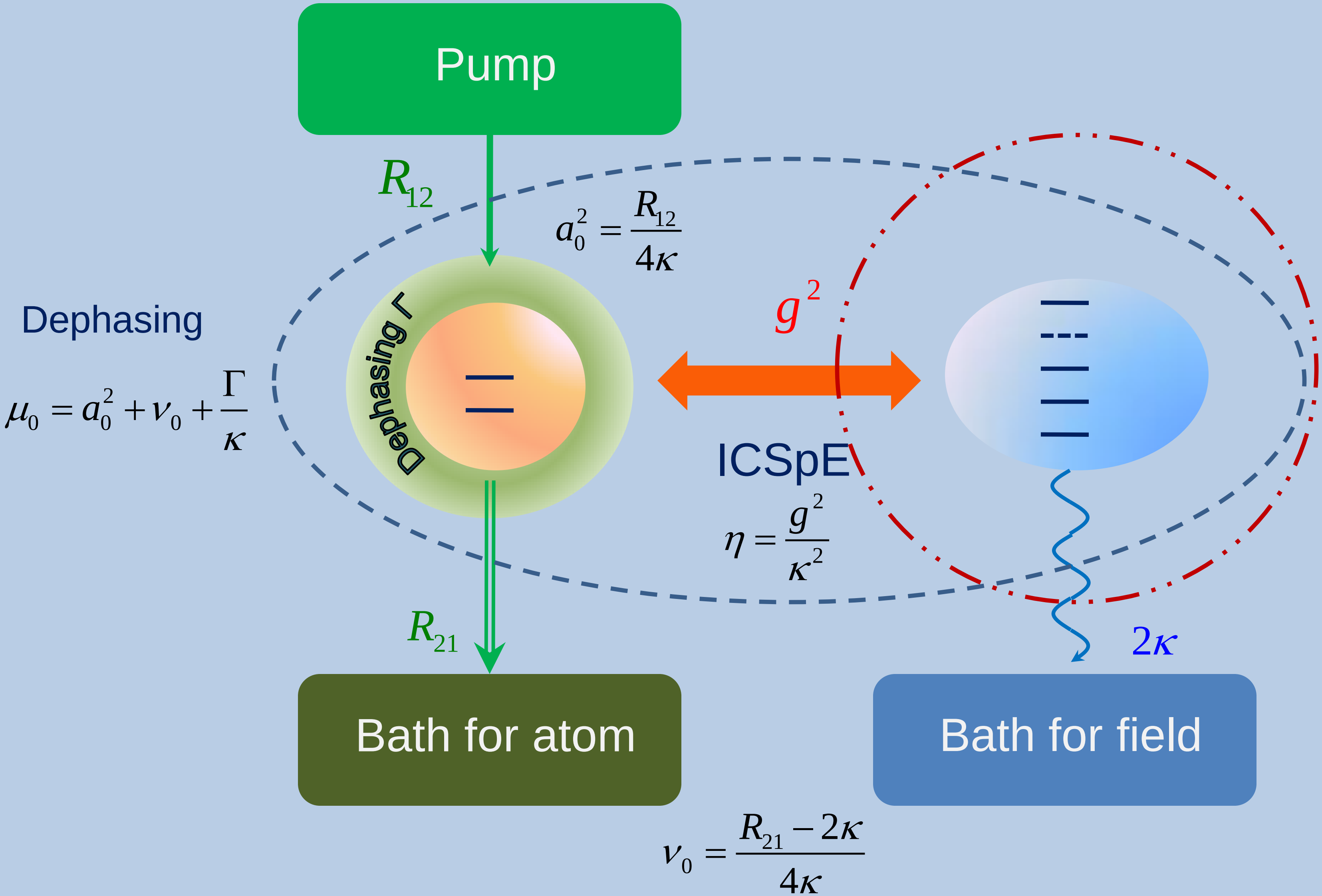
$$a_0^2 = \frac{R_{12}}{4\kappa}, \quad \nu_0 = \frac{R_{21} - 2\kappa}{4\kappa}, \quad \mu_0 = a_0^2 + \nu_0 + \frac{\Gamma}{\kappa}, \quad \eta = \frac{g^2}{\kappa^2}$$

$$J\rho = a\rho a^+, \quad n\rho = a^+ a\rho$$

3. Stationary state

3.1 Thermodynamics of SQL

Detailed balance principle



3.2 Nonlinear Coherent States

- Definition: deformed annihilation operator and nonlinear coherent states (NCSs): [R. L. de Matos Filho, W. Vogel, PRA 54, 4560 (1996); V. I. Man'ko et al., Phys.Scr. 55, 528 (1997) V V Dodonov , *J. Opt. B: Quant.Semiclass. Opt.* **4 (2002)**].

$$A_F = \sqrt{F(aa^+)}\,a, \quad A_F\left|\alpha;F\right\rangle = \alpha\left|\alpha;F\right\rangle$$

- Equation for ρ_{11}:

$$A_{F_{11}}\rho_{11}A_{F_{11}}^+ = a_0^2\rho_{11}$$

- Solution: phase-averaged NCSs of a specific form

$$\rho_{11} = diag\left(\left|a_0;F_{11}\right\rangle\left\langle a_0;F_{11}\right|\right), \quad F_{11}(n) = \frac{1}{2} + \left(\frac{1}{2} + \frac{\nu_0}{n}\right)\left\{1 + d(n)\right\}$$

The solution is unique!

For all regimes the stationary state of a single-qubit laser belongs to the found specific class of NCSs

$$d(n)\rho_{11} = \frac{2}{\eta}\left(\mu_0 + n - J\right)\rho_f$$

$$a_0^2 = \frac{R_{12}}{4\kappa}, \quad \nu_0 = \frac{R_{21} - 2\kappa}{4\kappa}$$

3. Stationary state

SQL generates NCS of the specific form:

$$\sqrt{F(aa^{+})}\,a\left|a_0, F\right\rangle = a_0\left|a_0, F\right\rangle$$

$$F_{11}(n) = \frac{1}{2} + \left(\frac{1}{2} + \frac{\nu_0}{n}\right)\left\{1 + d(n)\right\}$$

FOCK STATES REPRESENTATION

$$\left|a_0; F\right\rangle = const \cdot \sum_{n=0}^{\infty} \left|n\right\rangle \frac{a_0^n}{\sqrt{n!}} \prod_{m=1}^{n} \frac{1}{\sqrt{F(m)}}$$

$$a_0^2 = \frac{R_{12}}{4\kappa}, \quad \nu_0 = \frac{R_{21} - 2\kappa}{4\kappa}$$

3.3 Special cases

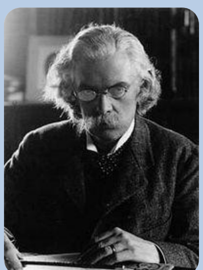
$$\rho_{11} = diag\left(\left|a_0;F_{11}\right\rangle\left\langle a_0;F_{11}\right|\right), \quad F_{11}(n) = \frac{1}{2} + \left(\frac{1}{2} + \frac{\nu_0}{n}\right)\{1 + d(n)\}$$

- Ordinary coherent states:

$$\nu_0 = 0,\ d(n) = 0 \Rightarrow \left|a_0;F_{11}\right\rangle = \left|a_0\right\rangle$$

- Strong-coupling regime ($\nu_0 \neq 0,\ d(n) = 0$) → Mittag-Leffler states *[S.Ya. Kilin, T. B. Karlovich, JETP (2002)].*

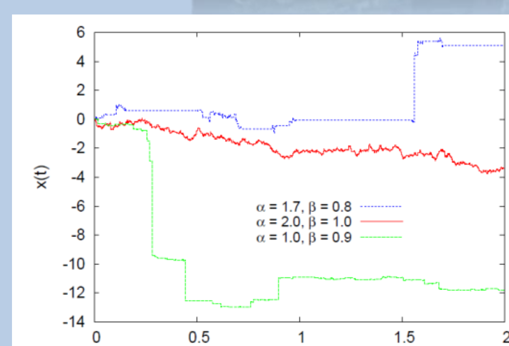
$$\left|a_0;F_{11}\right\rangle = const \cdot \sum_{n=0}^{\infty} \left|n\right\rangle \frac{a_0^n}{\sqrt{\Gamma(n+\nu_0)}}$$



Gösta Mittag-Leffler

INSTITUT MITTAG-LEFFLER
THE ROYAL SWEDISH ACADEMY OF SCIENCES
"bringing together the past and present for the future of mathematics"

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad z \in \mathbb{C},$$



Continuous-time random walks (CTRW)
Fractional diffusion equations (FDE)
Fractional Fokker-Planck eqs. → anomalous diffusion

Other density matrix elements:

$$\rho_{22} = diag\left(\left|a_0; F_{22}\right\rangle\left\langle a_0; F_{22}\right|\right), \quad F_{22}(n) = F_{11}(n)\varphi(n)/\varphi(n+1),$$

$$\rho_f = diag\left(\left|a_0; F_f\right\rangle\left\langle a_0; F_f\right|\right), \quad F_f(n) = F_{11}(n)[1+\varphi(n)]/[1+\varphi(n+1)],$$

$$\varphi(n) = a_0^2/[F_{11}(n)n_d(n)], \qquad n_d(n) = n/[1+d(n)]$$

3.4 Recurrent equation. Superconvergence

- Recurrent equation for the deviation function:

$d(n) \leftarrow d(n+1) \longleftarrow d(n+2)$

$n_d(n+1) = (n+1)/[1+d(n+1)],$

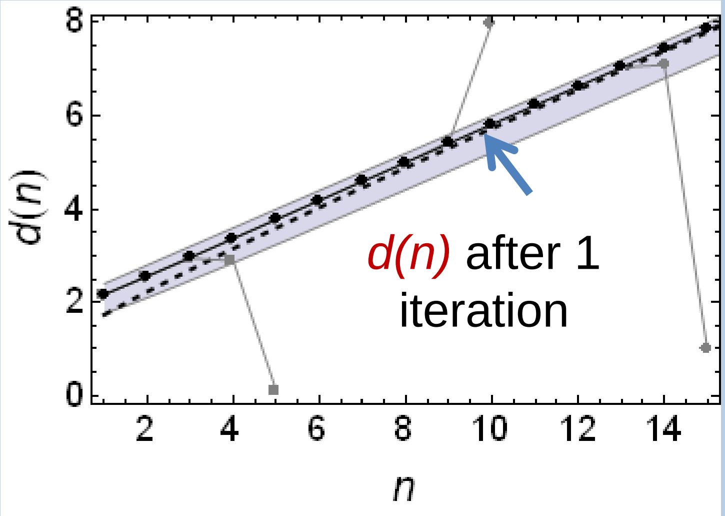
$\varphi(n+1) = a_0^2/[F_{11}(n+1)n_d(n+1)]$

$\varphi(n+2) = a_0^2/[F_{11}(n+2)n_d(n+2)]$

$$d(n) = \frac{2}{\eta}\Big[(\mu_0+n)\{1+\varphi(n+1)\} - n_d(n+1)\varphi(n+1)\{1+\varphi(n+2)\}\Big]$$

- The expression for *d(n)* is a tightly bounded function of *d(n+1)* and *d(n+2)*.

For any starting values of *d(n+1)* and *d(n+2)* very accurate value of *d* is obtained after just several iterations



Iterations for different starting values *d(n+1)*, *d(n+2)*

$$d(n)\rho_{11} = \frac{2}{\eta}(\mu_0 + n - J)\rho_f$$

Asymptotic behavior (n>>1)

$$d(n) = \frac{2}{\eta}\left[\mu_0 + n + a_0^2 \frac{2n}{n+\eta} + O\left(\frac{1}{n}\right)\right]$$

$$\left|a_0; F_{ii}\right\rangle \Box \sum_{n=0}^{\infty} \left|n\right\rangle \frac{a_0^n \eta^n}{n!}$$

$$\rho_f(n) \sim 1/(n!)^2$$

→ Nonclassicality

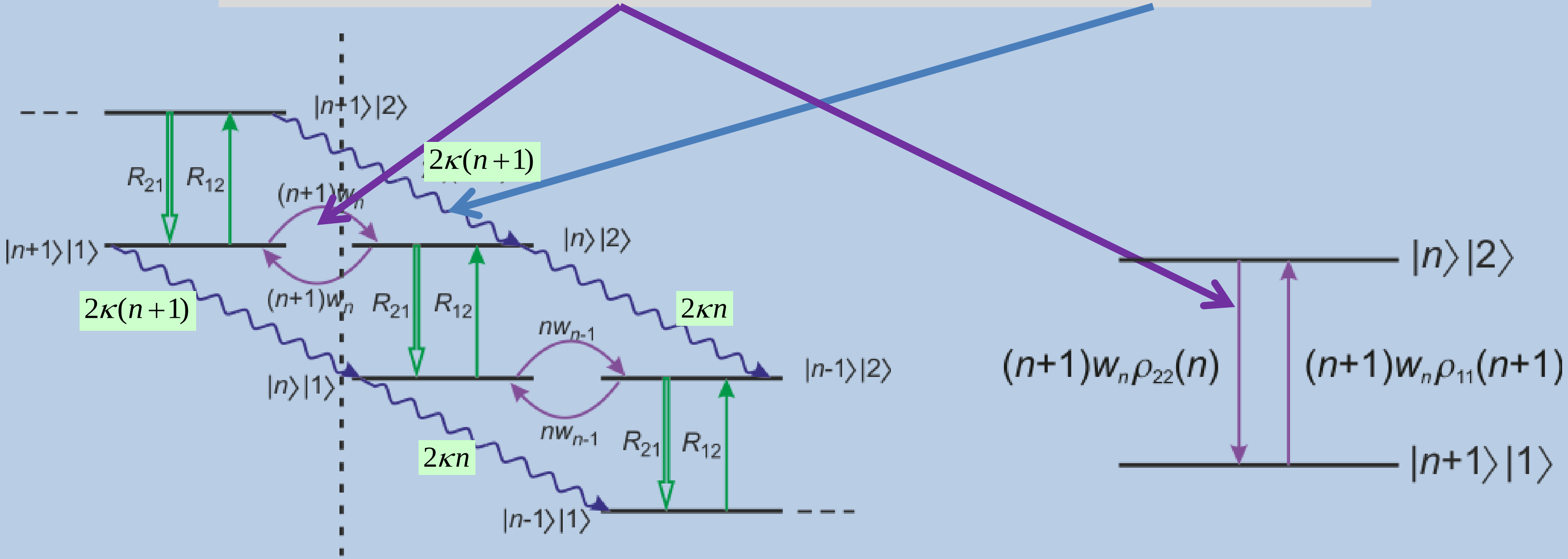
3.5 ISpE Transition Probabilities:

- Second equation for ρ_{11} and ρ_{22}:

$$\rho_{22} = \frac{1}{n+1} J \left\{1 + d(n)\right\} \rho_{11}$$

- Interpretation as balance of transitions between states with (n+1) and n photons:

$$(n+1) w_n \left\{\rho_{22}(n) - \rho_{11}(n+1)\right\} = 2\kappa(n+1)\rho_f(n+1)$$



3.6 ISpE Transition probabilities: nonlinearity

$$(n+1)w_n\{\rho_{22}(n)-\rho_{11}(n+1)\} = 2\kappa(n+1)\rho_f(n+1)$$

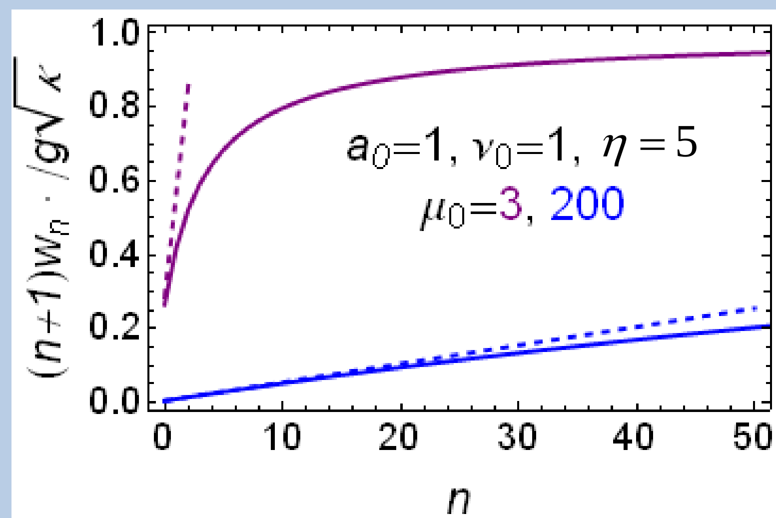
$$w_n = \frac{g^2}{\kappa}\left[\mu_0 + (n+1)\left\{1-\frac{(n+2)\rho_f(n+2)}{(n+1)\rho_f(n+1)}\right\}\right]^{-1}$$

$$w_n \neq const$$

$$w_n \sim \frac{g^2}{\kappa n}, \quad (n+1)w_n \to const$$

Quantumness manifestation!

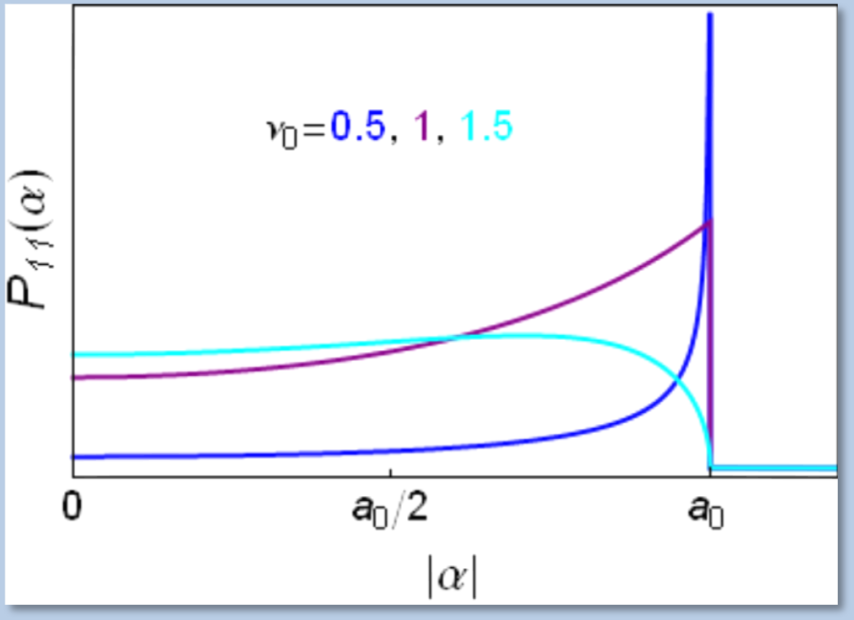
Ordinary transition probability: $w_n = const$ *(dashed lines in Fig.)*



$$a_0^2 = \frac{R_{12}}{4\kappa}, \quad \nu_0 = \frac{R_{21}-2\kappa}{4\kappa}, \quad \mu_0 = a_0^2 + \nu_0 + \frac{\Gamma}{\kappa}, \quad \eta = \frac{g^2}{\kappa^2}$$

4. Quasi-distribution. Nonclassicality

- Stationary state Glauber function in Strong-Coupling Limit:
  - for $\nu_0 > 0$ - positive; border at $|\alpha| = a_0$
  - for $\nu_0 \le 0$ - generalized function, nonclassical state.

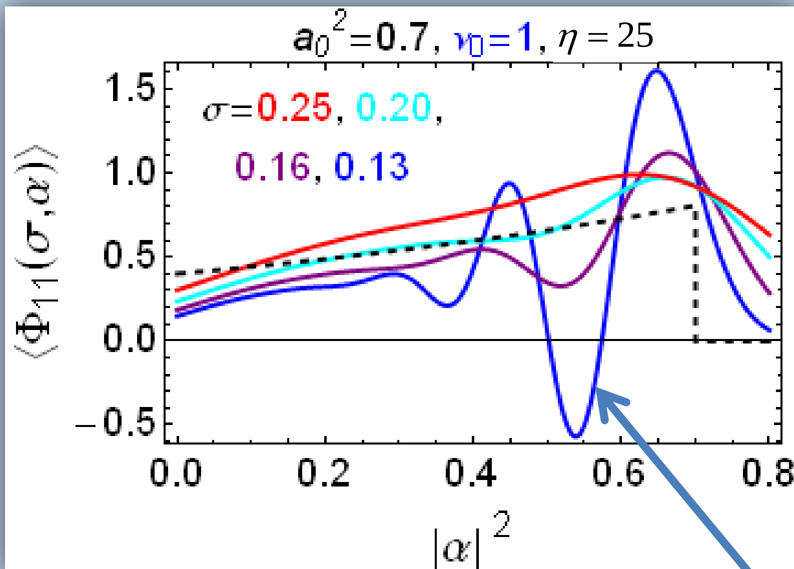


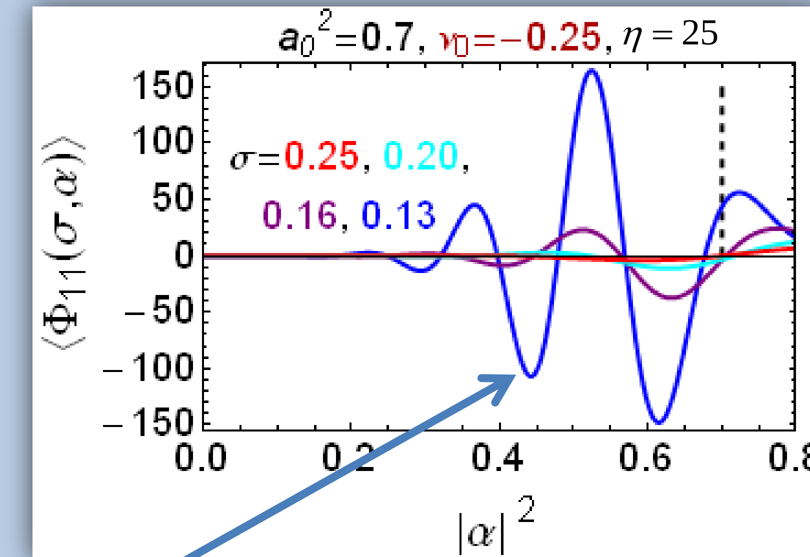
[S.Ya. Kilin, T. B. Karlovich, JETP 95, 805 (2002)].

- Solution of exact equations:
  - Glauber function is always a **generalized function**, and the stationary state is **always nonclassical.**

$$a_0^2 = \frac{R_{12}}{4\kappa}, \quad \nu_0 = \frac{R_{21} - 2\kappa}{4\kappa}$$

4.1 “Pointer” observable





Manifestation of nonclassicality

Local properties of Glauber P- function via measurement of “pointer” observable

$$\Phi_{ij}(\sigma;\alpha) = \frac{1}{\pi\sigma^2}\int d^2\gamma P_{ij}(\gamma)\exp\left(-\frac{|\alpha-\gamma|^2}{\sigma^2}\right)$$

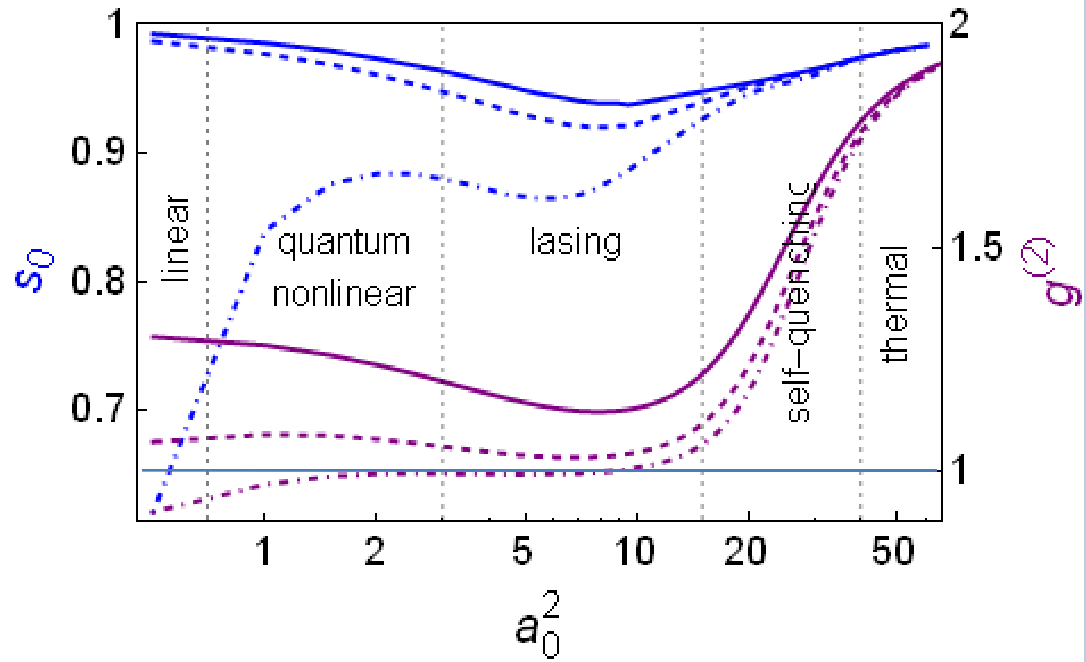
Dashed lines – Glauber function in strong-coupling limit

$$a_0^2 = \frac{R_{12}}{4\kappa}, \quad \nu_0 = \frac{R_{21}-2\kappa}{4\kappa}$$

4.2 Nonclassicality order

[C. T. Lee, Phys. Rev. A 44, R2775 (1991). N. Lutkenhaus and S. M. Barnett, Phys. Rev. A 51, 3340 (1995).]

- Negativity of *s*-parameterized phase space function → manifestation of nonclassicality.
- Minimal *s*, for which $P(\alpha,s)$ is not strictly positive, → nonclassicality order s_0.



$\nu_0 = 1$ (solid lines), 0 (dashed), -0.5 (dot-dashed)

$$\Phi(\sigma;\alpha) \equiv P(\alpha;s), \text{ with } s = 1 - 2\sigma^2$$

5. Quantum dynamics

- We describe dynamics by the smoothed observables:

$$\rho_{ij}(t) \rightarrow \Phi_{ij}(\sigma;\alpha,t)$$

- Equations have the form

$$\frac{\partial}{\partial t}\Phi_{ij}(\sigma;x,y,t)=F_{ij}\left(x,y,\Phi_{kl},\left\{\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial^2}{\partial x^2},\frac{\partial^2}{\partial y^2},\frac{\partial^2}{\partial x\partial y}\right\}\Phi_{kl}\right)$$

$$\alpha = x+iy$$

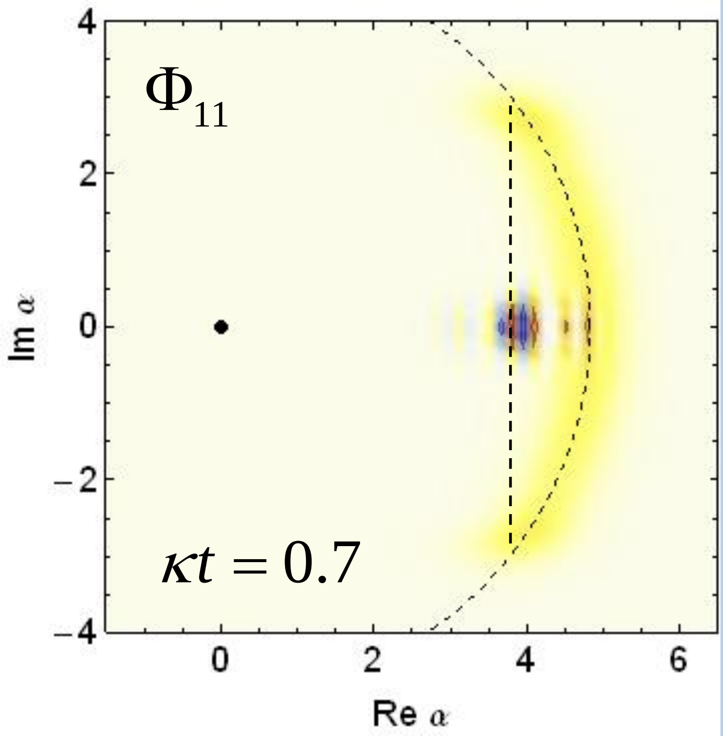
- Additional definition:

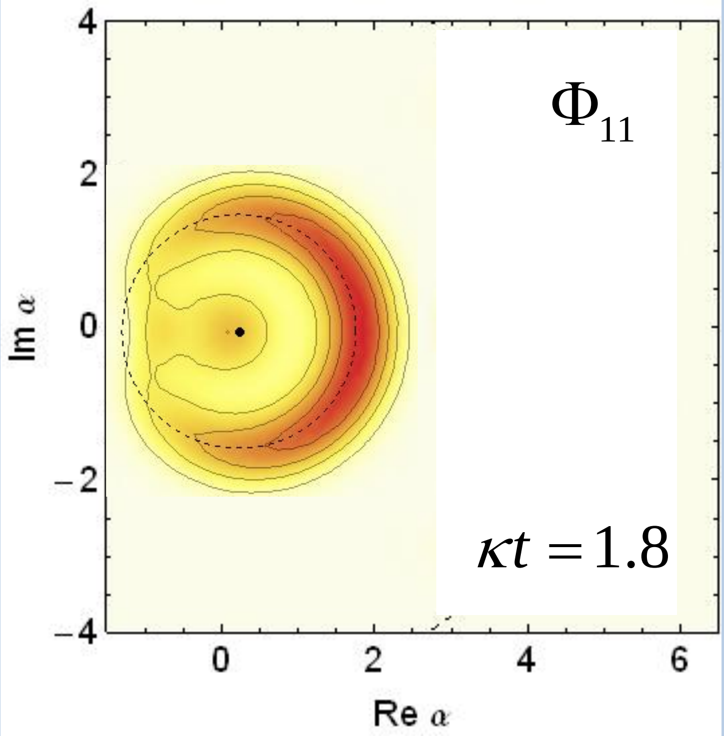
$$\Phi_{12r}=\mathrm{Re}\,\Phi_{12}=\mathrm{Re}\,\Phi_{21},\quad \Phi_{12i}=\mathrm{Im}\,\Phi_{12}=-\mathrm{Im}\,\Phi_{21}$$

- Initial state: ground state of atom, coherent state of field

$$\rho(0)=|1\rangle\langle 1|\otimes|\alpha_0\rangle\langle\alpha_0|$$

5.1. Two stages of quantum dynamics





I. High amplitude (coherent) regime:

$$|\alpha| >> a_0$$

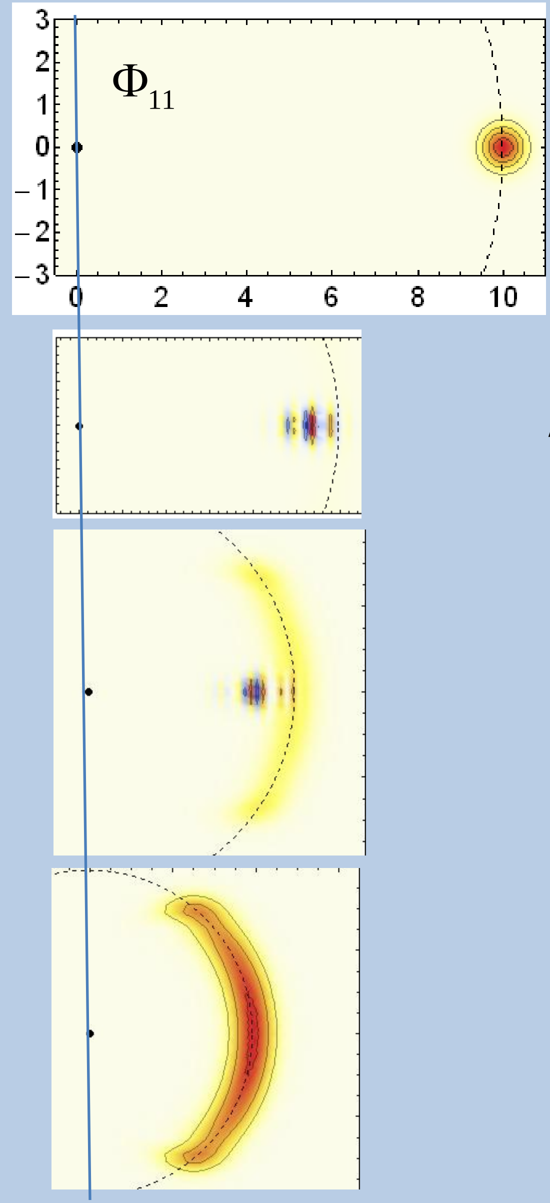
II. Approaching stationary state

$$|\alpha| \sim a_0$$

$$a_0 = 1, \quad \nu_0 = 1, \quad \mu_0 = 13, \quad \eta = 1.3 \cdot 10^2, \quad \sigma = 0.5$$

5.2. High amplitude regime $|\alpha| >> a_0$

- Two regions with different behavior:



$\kappa t = 0$

$\kappa t = 0.52$

$\kappa t = 0.73$

$\kappa t = 0.94$

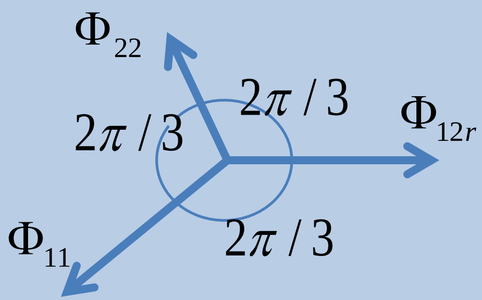
  - oscillations region (nonclassical interference terms);
  - “arc” (superposition of coherent states with specific amplitude distribution).

5.3. Oscillations region

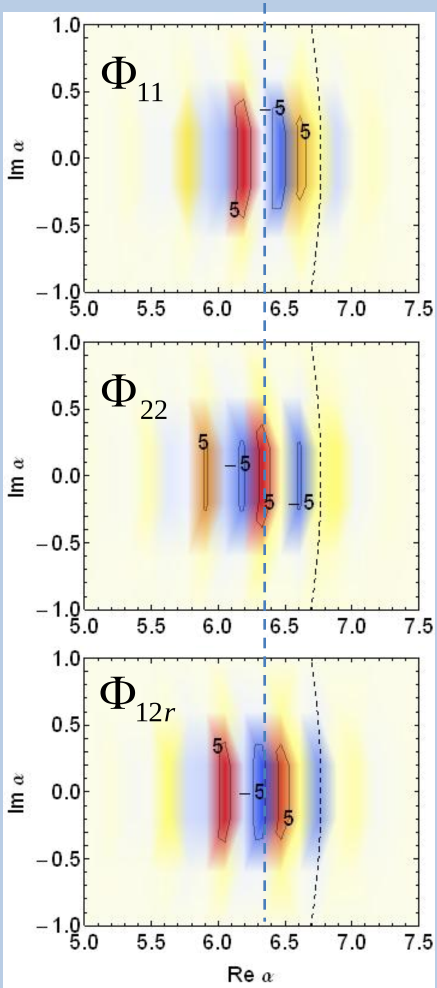
$|\alpha| >> a_0$

- Relations between parameters: $g, x, \dfrac{\partial \Phi}{\partial x}, \dfrac{\partial \Phi}{\partial t} >> 1$
- Zero-order approximate solution:

$$\Phi_\beta^{(0)}(x,y,t) = a_\beta \cos(\omega t + kx + \varphi_\beta),\ \beta = 11, 22, 12r;$$
$$\Phi_{12i}^{(0)}(x,y,t) = 0$$



Phase diagram for
$a_0 = 1, \nu_0 = -0.3, \mu_0 = 6,$
$\eta = 1.3 \cdot 10^2, \sigma = 0.5, x \approx 6.5$



- Solution of Eqs. for oscillation parameters:

$$k \approx \frac{2g}{\kappa\sigma}\sqrt{\frac{2}{3} - 2\sigma^2}, \quad \omega \approx 3\kappa(kx)$$

- Rapid oscillations correspond to highly nonclassical field state.
- Oscillations region exists only for strongly localized “pointer” oservable (weak smoothing): $\sigma^2 < 1/3$

$R = R_{12} + R_{21}$

5.4. Oscillations region: envelope

- First order approximate solution:

$$\Phi_\beta(x,y,t) = u(x,y,t)\Phi_\beta^{(0)}(x,y,t)$$

- Equation for envelope function *u*:

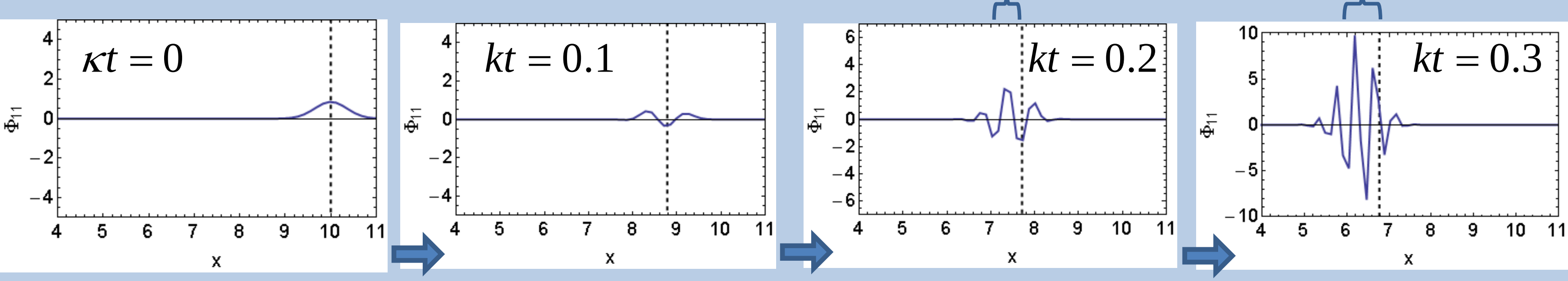
$$\frac{\partial u}{\partial t} \approx \left\{-\frac{R}{2} + \eta(x,k,\omega)\right\}u + \kappa\frac{\partial}{\partial y}(yu) + \frac{\partial}{\partial x}\{\kappa x u + \zeta(x,k,\omega)u\}$$

Exponential growth of envelope function at earlier stages of evolution

Faster drift towards vacuum state, than observed for ordinary linear loss

Rapid transformation of classical starting state into highly nonclassical state

Rapid decoherence of the nonclassical state



Dashed line – amplitude for ordinary linear loss: $x = \alpha_0 e^{-\kappa t}$

$R = R_{12} + R_{21}$

5.5. "Arc" region: equations

$$|\alpha| >> a_0$$

- Polar coordinates: $x = r\cos\theta, \quad y = r\sin\theta$
- Relations between parameters: $g, r, \dfrac{\partial\Phi}{\partial t} >> 1$
- Zeroth order approximate solution:

$$\Phi_{22} = \Phi_{11}, \quad \Phi_{12r} = \Phi_{12i}\tan\theta$$

- First order: radial and angular parts of eqs.

$$\frac{\partial}{\partial t}\Phi_{11} \approx \left(\frac{\partial}{\partial t}\Phi_{11}\right)_\theta + L_r\Phi_{11}, \quad \frac{\partial}{\partial t}\Phi_{12i} \approx \left(\frac{\partial}{\partial t}\Phi_{12i}\right)_\theta + L_r\Phi_{12i}$$

  - Radial part is the same for all components.
  - Angular parts differ – consequence of atom-field interaction.

$$\left(\frac{\partial}{\partial t}\Phi_{11}\right)_\theta = \frac{g}{2r}\frac{\partial}{\partial\theta}\left(\frac{\Phi_{12i}}{\cos\theta}\right),$$

$$\left(\frac{\partial}{\partial t}\left(\frac{\Phi_{12i}}{\cos\theta}\right)\right)_\theta = \frac{g}{2r}\frac{\partial}{\partial\theta}\Phi_{11} - \frac{R}{2}\frac{\Phi_{12i}}{\cos\theta}$$

$$R = R_{12} + R_{21}$$

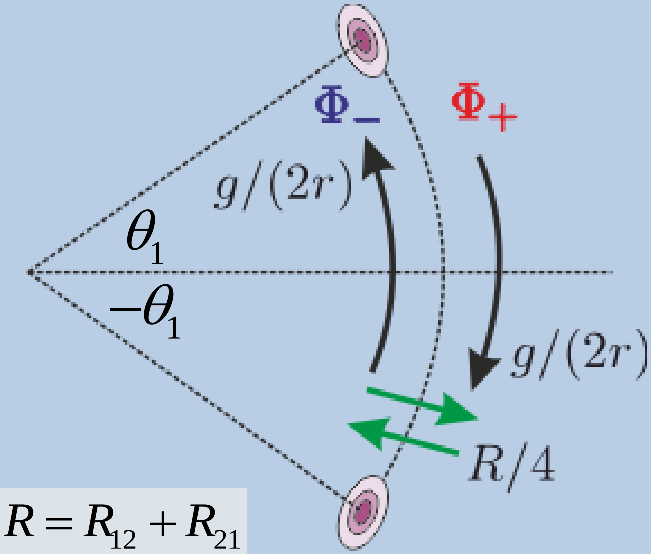
5.6 . “Arc” region: phase “bistability”

- New dependent functions: $\Phi_{\pm} = \Phi_{11} \pm \left(\Phi_{12i} / \cos\theta\right)$

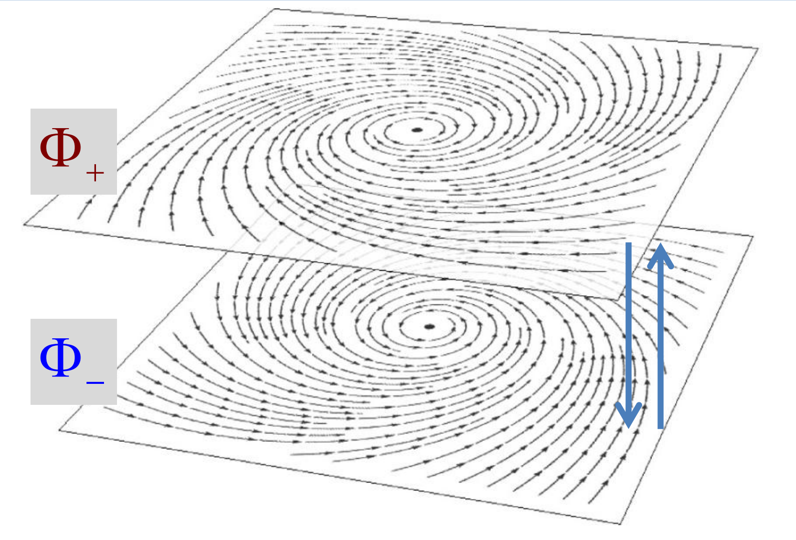
$$\left(\frac{\partial}{\partial t}\Phi_{+}\right)_{\theta} = \frac{g}{2r}\frac{\partial}{\partial\theta}\Phi_{+} - \frac{R}{4}\Phi_{+} + \frac{R}{4}\Phi_{-},$$

$$\left(\frac{\partial}{\partial t}\Phi_{-}\right)_{\theta} = -\frac{g}{2r}\frac{\partial}{\partial\theta}\Phi_{-} + \frac{R}{4}\Phi_{+} - \frac{R}{4}\Phi_{-}$$

Angular drift (the $\frac{g}{2r}\frac{\partial}{\partial\theta}$ terms) — “Jumps” (the $\frac{R}{4}$ terms)



$R = R_{12} + R_{21}$



← Phase space flow

5.7. “Arc” region: eigenfunctions

- New dependent functions: $\Phi_{\pm} = \Phi_{11} \pm \left(\Phi_{12i} / \cos\theta\right)$
- Eigenstates of Jaynes-Cummings Hamiltonian:

$$\left|\Psi_{n,\pm}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|2\right\rangle\left|n\right\rangle \pm \left|1\right\rangle\left|n+1\right\rangle\right) = \frac{1}{\sqrt{2}}\underbrace{\left(\frac{\sigma_{+}a}{\sqrt{n+1}} \pm 1\right)}_{\text{“Entangling” operator}}\left|1\right\rangle\left|n+1\right\rangle$$

- Phase-space functions (smoothed observables):

$$\left\langle 1\right|\rho(t)\left|1\right\rangle \rightarrow \Phi_{11}(\sigma;\alpha,t),$$
$$\left\langle 2\right|\rho(t)\left|2\right\rangle \rightarrow \Phi_{22}(\sigma;\alpha,t)$$

- Conditional field states

$$\frac{1}{2}\left\langle 1\right|\left(\frac{\sigma_{+}a}{\sqrt{n+1}} \pm 1\right)\rho(t)\left(\frac{\sigma_{+}a}{\sqrt{n+1}} \pm 1\right)^{+}\left|1\right\rangle \rightarrow \Phi_{\pm}(\sigma;\alpha,t)$$

- Introduced functions correspond to atom-field states, entangled by Jaynes-Cummins interaction.

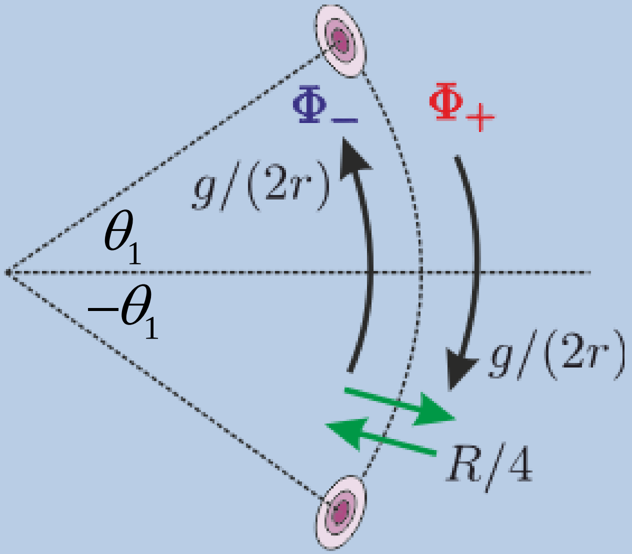
5.8. “Arc” region: limits

$$\Phi_{\pm} = \Phi_{11} \pm \left(\Phi_{12i} / \cos\theta \right)$$

$$\left(\frac{\partial}{\partial t} \Phi_{+} \right)_{\theta} = \frac{g}{2r} \frac{\partial}{\partial \theta} \Phi_{+} - \frac{R}{4} \Phi_{+} + \frac{R}{4} \Phi_{-},$$

$$\left(\frac{\partial}{\partial t} \Phi_{-} \right)_{\theta} = -\frac{g}{2r} \frac{\partial}{\partial \theta} \Phi_{-} + \frac{R}{4} \Phi_{+} - \frac{R}{4} \Phi_{-}$$

Angular drift “Jumps”



- Strong Coupling Limit: $g / r >> R$
  - low “jumps” probability; 2 separate peaks at

$$\theta = \pm\theta_1, \quad \theta_1 = \frac{g}{2\kappa} \left(\frac{1}{r(t)} - \frac{1}{r(0)} \right) = \frac{g}{2\kappa\alpha_0} \{ \exp(\kappa t) - 1 \}$$

- Intense atom pump/decay: $g / r << R$
  - high “jumps” probability, analog of random walk; Gaussian peak with half-width

$$\theta_2 \sim \frac{g}{\kappa R \alpha_0} \sqrt{\exp(2\kappa t) - 1}$$

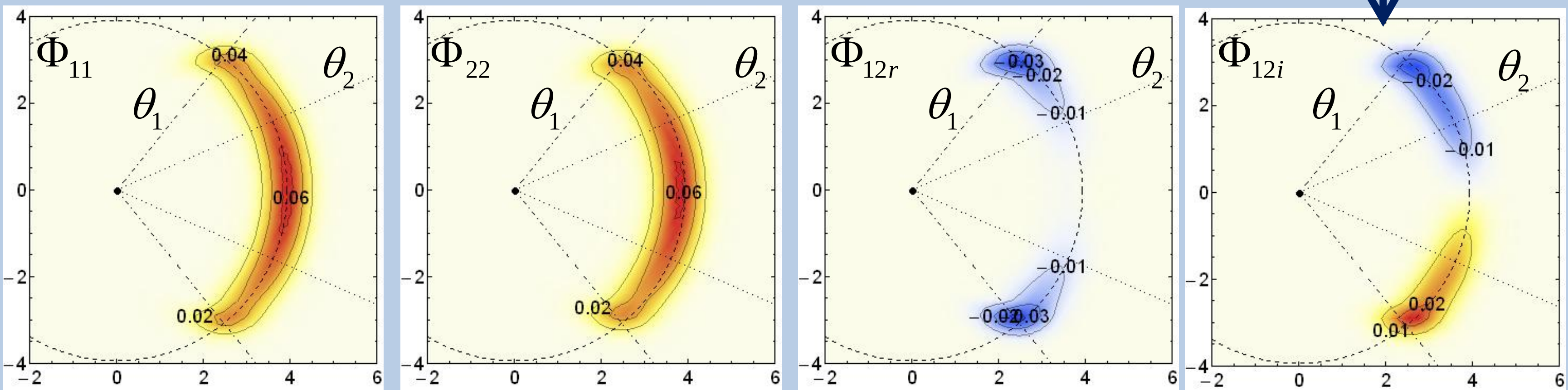
$$R = R_{12} + R_{21}$$

5.9. “Arc” region: examples

$$\Phi_{12i} = \left(\Phi_{+} - \Phi_{-}\right)\cos\theta$$

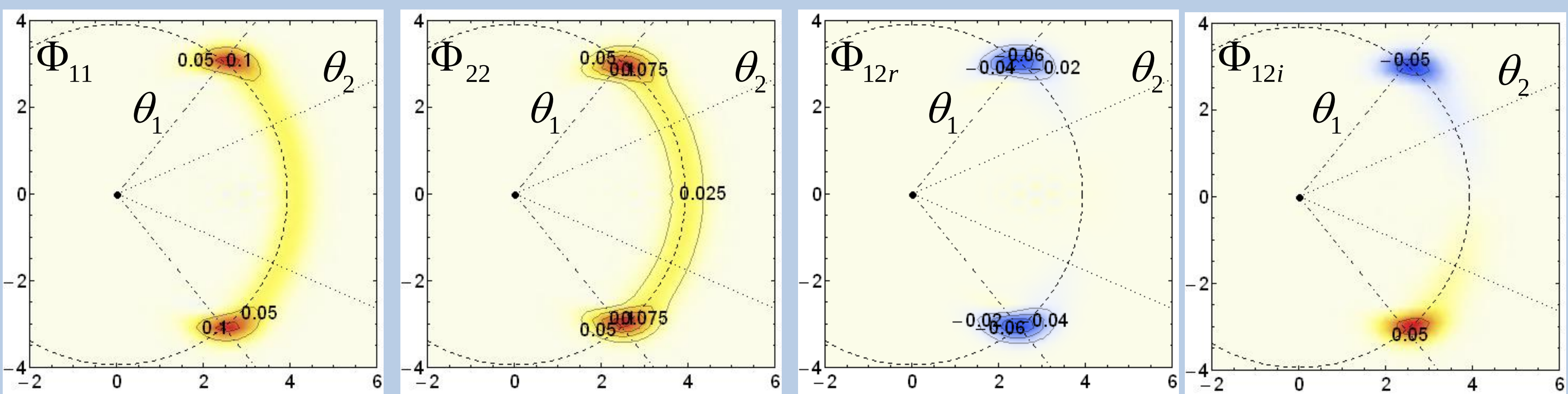
- odd function of y

- Frequent jumps (motional narrowing):



$$a_0 = 1, \quad \nu_0 = 1, \quad \mu_0 = 13, \quad \eta = 1.3\cdot 10^2, \quad \sigma = 0.5, \quad \kappa t = 0.94$$

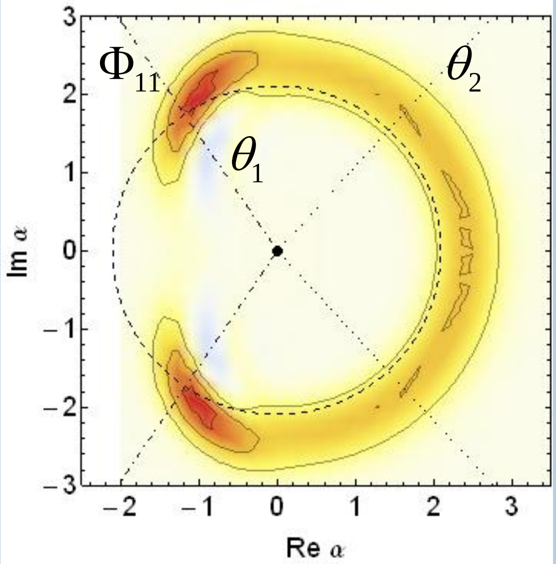
- Slow jumps (phase bistability) :



$$a_0 = 1, \quad \nu_0 = -0.3, \quad \mu_0 = 6, \quad \eta = 1.3\cdot 10^2, \quad \sigma = 0.5, \quad \kappa t = 0.94$$

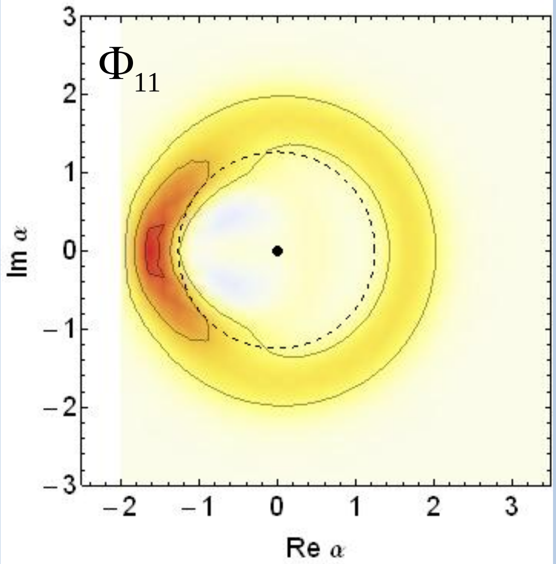
5.10. Approaching stationary state

$|\alpha| \sim a_0$



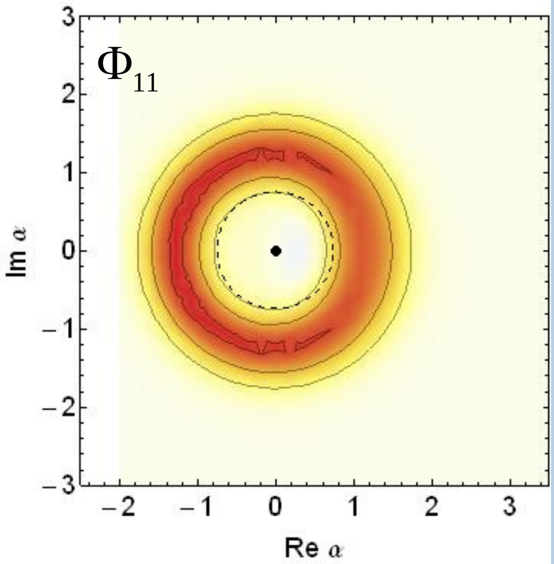
“Arc”

$\kappa t = 1.6$



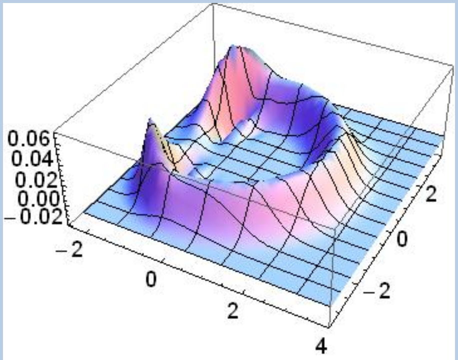
Closing “arc”

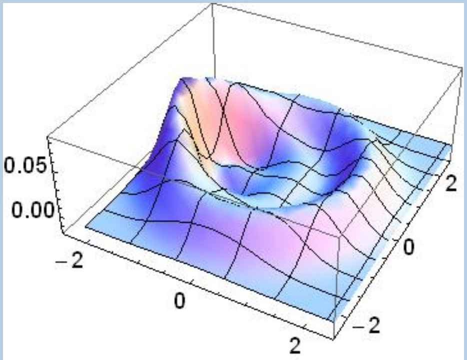
$\kappa t = 2$

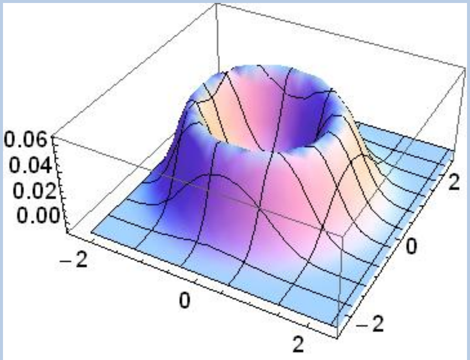


Stationary state

$\kappa t = 2.6$







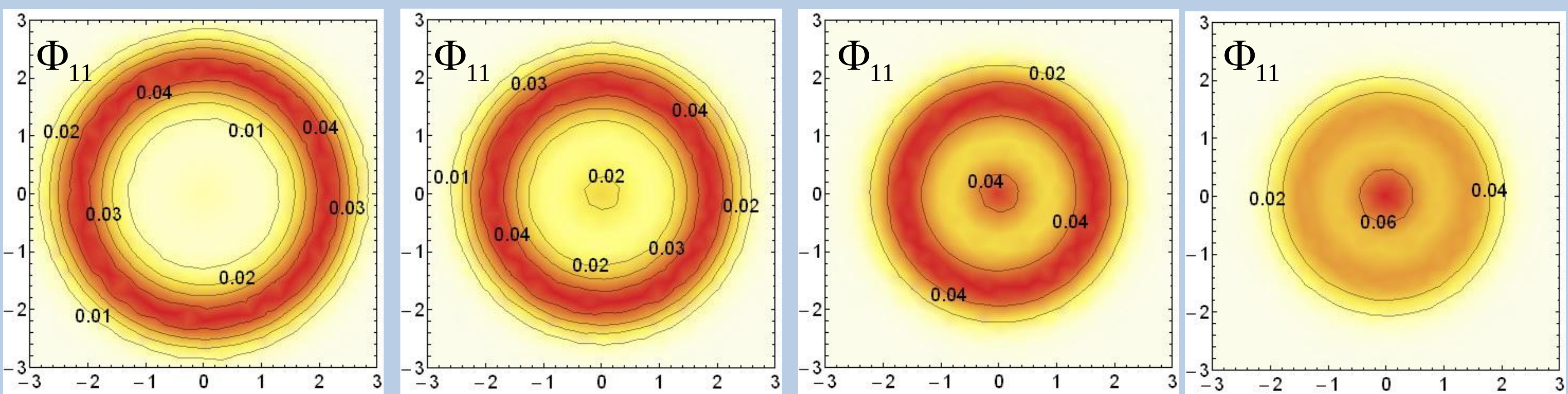
$a_0 = 1, \quad \nu_0 = -0.3, \quad \mu_0 = 6, \quad \eta = 1.3 \cdot 10^2, \quad \sigma = 0.5$

5.11. Approaching stationary state: amplitude "jump"

- The state is approximately phase-independent:

$$\Phi_{12r} = \Phi_{12}\cos\theta, \quad \Phi_{12i} = -\Phi_{12}\sin\theta.$$

- Quasi-probability amplitude effectively "jumps" from *r*~1 to *r*~0.1:

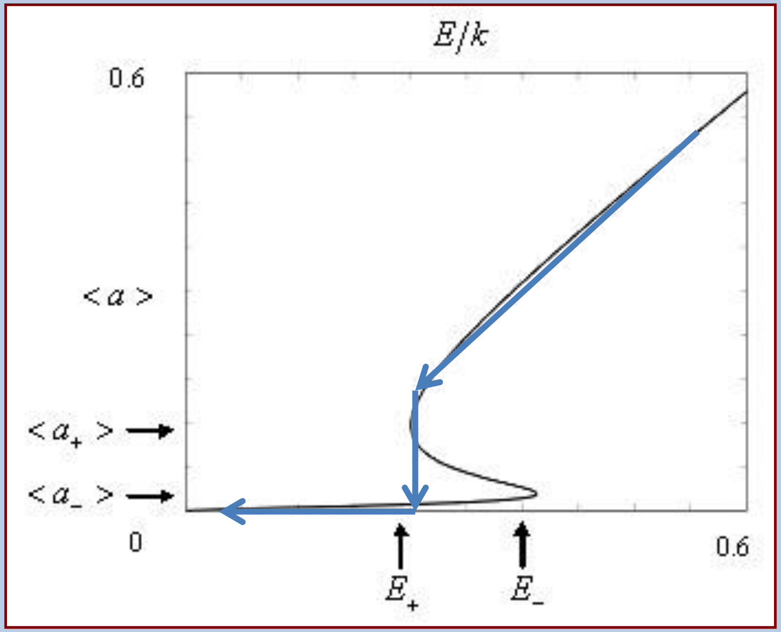


$\kappa t = 0.9$ $\kappa t = 1.0$ $\kappa t = 1.1$ $\kappa t = 1.2$

$$a_0 = 1, \quad \nu_0 = 1, \quad \mu_0 = 13, \quad \eta = 1.3\cdot 10^2, \quad \sigma = 0.5$$

5.12 . Approaching stationary state: amplitude "bistability"

- Similar "jump" (amplitude bistability) is characteristic for a stationary state of a coherently driven SQL :



The steering parameter (coherent excitation strength) is assumed to be changed adiabatically

Semi-classical approximation:

$$<a^+\sigma_- > \approx < a^+ >< \sigma_- >$$

$$\langle \dot{a} \rangle = E - k\langle a \rangle - ig\langle \sigma_- \rangle$$

$$\langle \dot{\sigma}_- \rangle = -i\Delta\omega\langle \sigma_- \rangle + ig\langle \sigma_z \rangle\langle a \rangle - \gamma\langle \sigma_- \rangle/2$$

$$\langle \dot{\sigma}_z \rangle = -2ig\left(\langle \sigma_+ \rangle\langle a \rangle - \langle \sigma_- \rangle\langle a^+ \rangle\right) - \gamma\left(\langle \sigma_z \rangle + 1\right)$$

Stationary state: $\langle \dot{a} \rangle = \langle \dot{\sigma}_- \rangle = \langle \dot{\sigma}_z \rangle = 0$

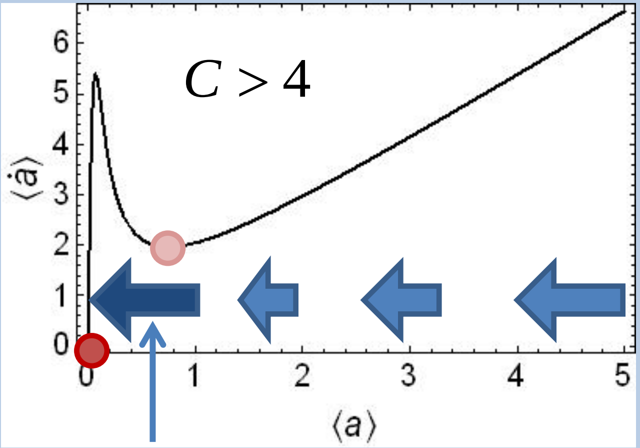
$$E = \kappa\langle a \rangle + \frac{2g^2}{R_{21} - R_{12}} \frac{\langle a \rangle}{1 + \frac{8g^2}{(R_{12} - R_{21})^2}\langle a^+ \rangle\langle a \rangle}$$

- SQL without coherent driving:
  - Adiabatic elimination of qubit: $\langle \dot{\sigma}_- \rangle = \langle \dot{\sigma}_z \rangle = 0$
  - Equation for field amplitude:

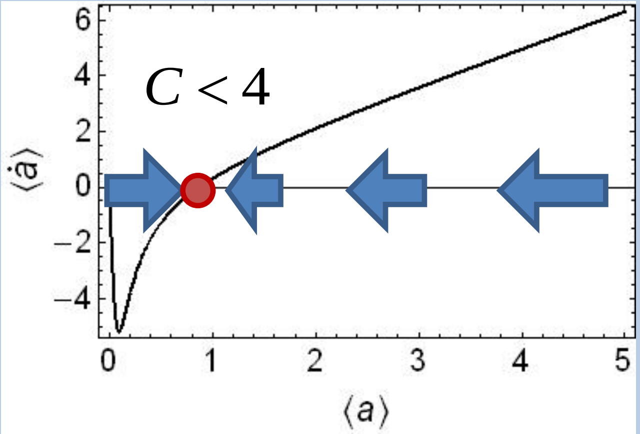
$$\langle \dot{a} \rangle = -\kappa \langle a \rangle - \frac{2g^2}{R_{21} - R_{12}} \frac{\langle a \rangle}{1 + \frac{8g^2}{(R_{21} - R_{12})^2} \langle a^+ \rangle \langle a \rangle}$$

  - Amplitude decay rate for SQL without coherent driving equals driving strength for stationary state of SQL with coherent driving for the same value of field amplitude.

- Bistability condition : $C = \frac{g^2}{\kappa (R_{21} - R_{12})} > 4$



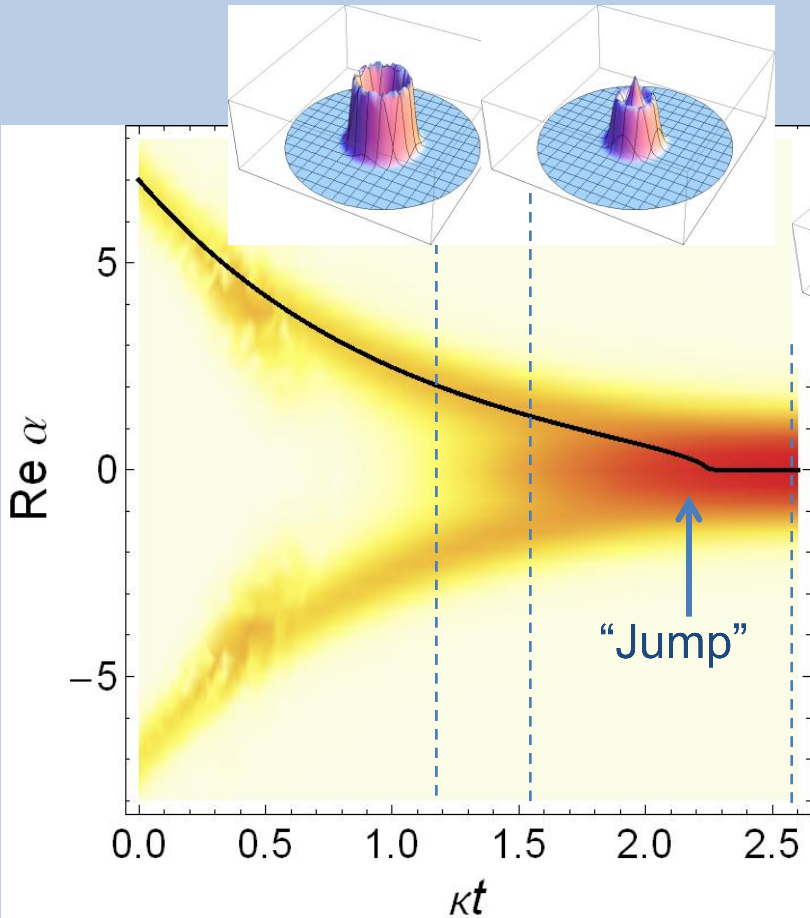
"Jump"

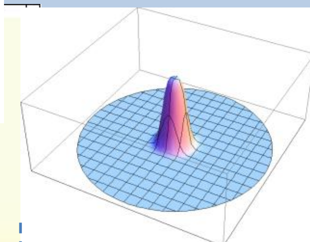


5.13. Approaching stationary state: examples

- Bistability condition is satisfied

$$a_0 = 1, \quad \nu_0 = 1, \quad \mu_0 = 13, \quad \eta = 1.3 \cdot 10^2, \quad \sigma = 0.5, \quad C > 4$$





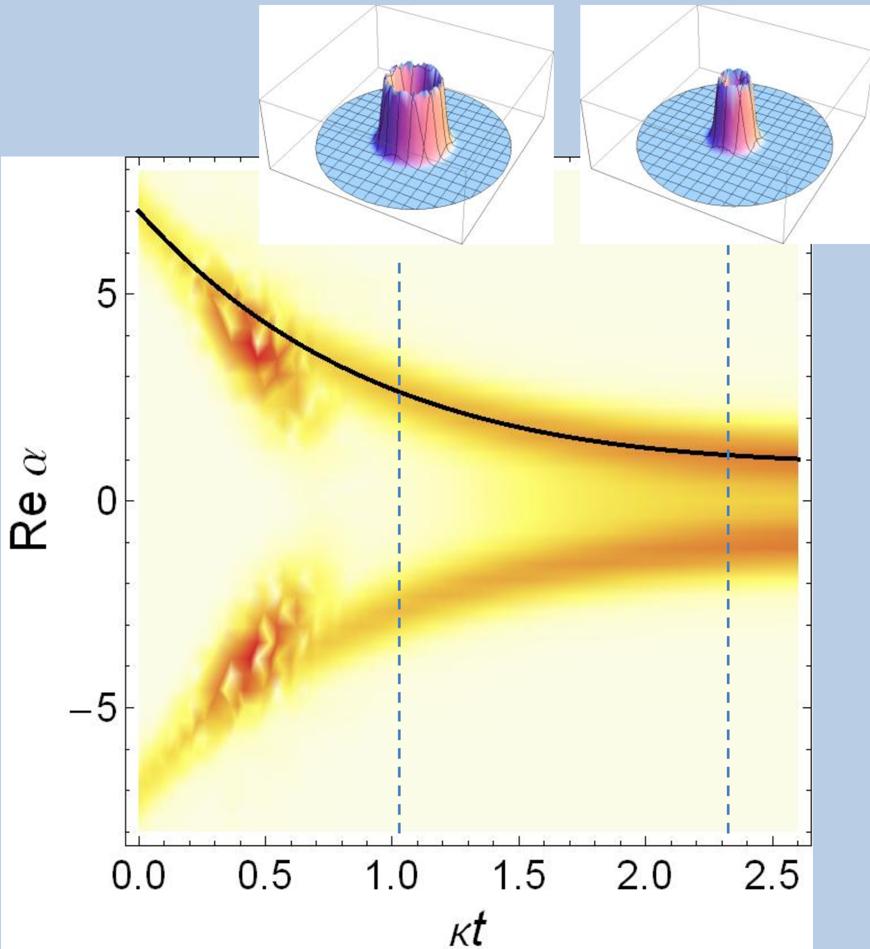
Black line – solution of semi-classical differential equation

Starting state – phase-averaged coherent state with amplitude $\alpha_0 = 7$

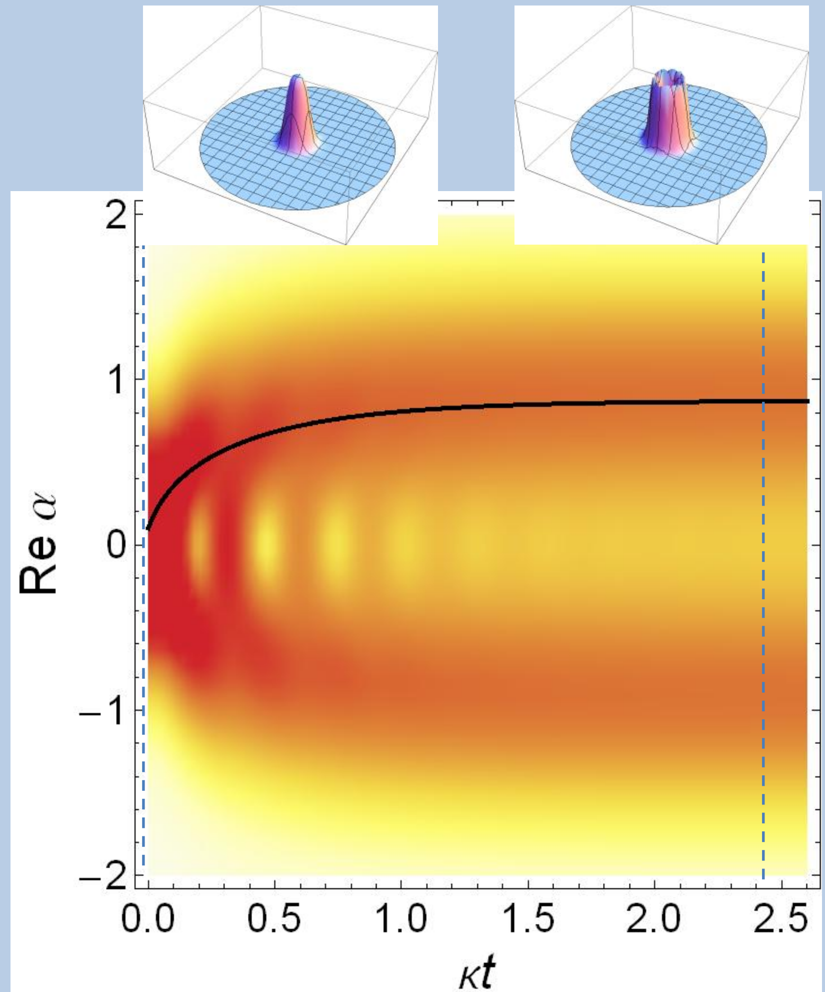
- Bistability condition **is not satisfied**

$$a_0 = 1, \quad \nu_0 = -0.3, \quad \mu_0 = 6, \quad \eta = 1.3\cdot 10^2, \quad \sigma = 0.5, \quad C < 4$$

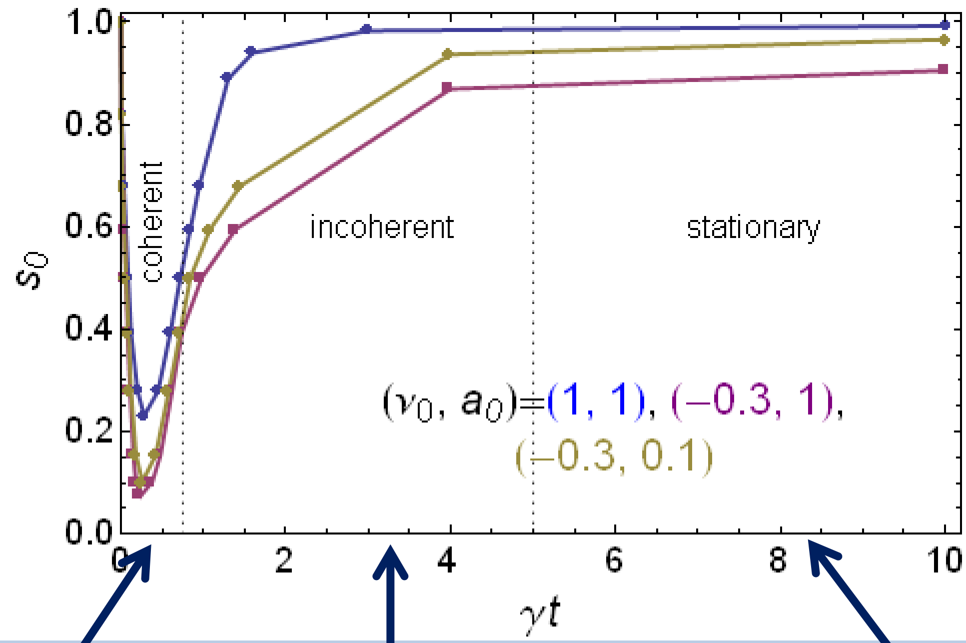
Starting state – phase-averaged coherent state with amplitude $\alpha_0 = 7$

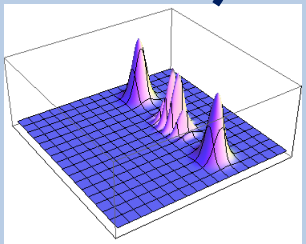


Starting state – vacuum state

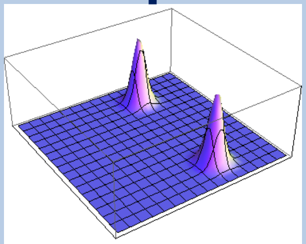


5.14. Dynamics of nonclassicality order

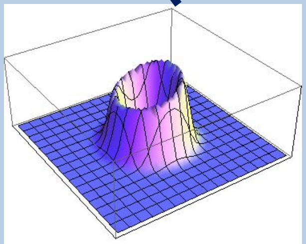




Coherent regime



Incoherent regime



Approaching stationary state

Thank you for attention!

5.14 Approaching stationary state: phase-space flow

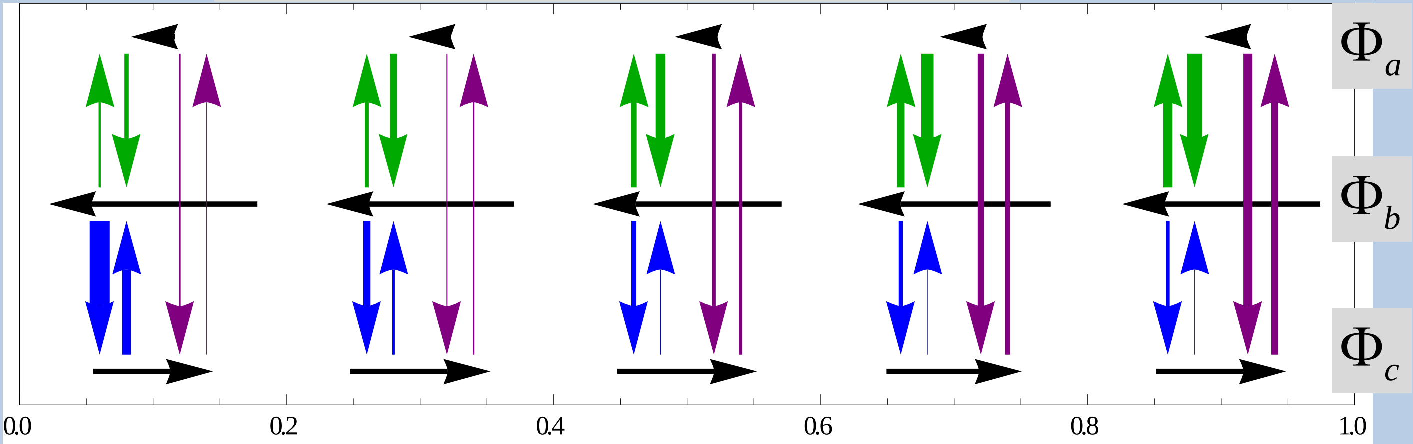
- Eigenfunctions of flow operator (for σ=0.5):

$$\Phi_a = -\frac{\Phi_{11}}{3} + \Phi_{22}, \quad \Phi_{b,c} = \pm\frac{\Phi_{11}}{2\sqrt{3}} \pm \frac{2\Phi_{22}}{\sqrt{3}} + \Phi_{12}.$$

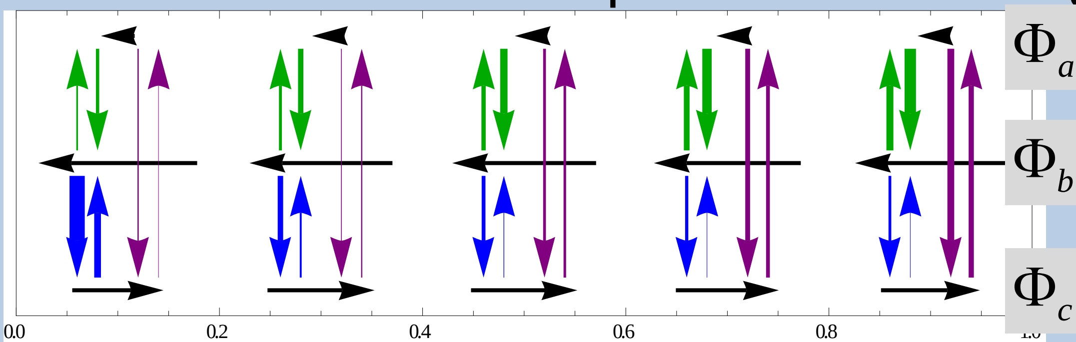
- Flow rates for eigenfunctions:

$$\frac{\partial}{\partial t}\Phi_a \sim -\left(\gamma r + \frac{\gamma}{8r}\right)\frac{\partial}{\partial r}\Phi_a,$$

$$\frac{\partial}{\partial t}\Phi_{b,c} \sim -\left(\gamma r\frac{\gamma}{8r} \pm \frac{\sqrt{3}g}{4}\right)\frac{\partial}{\partial r}\Phi_{b,c}.$$

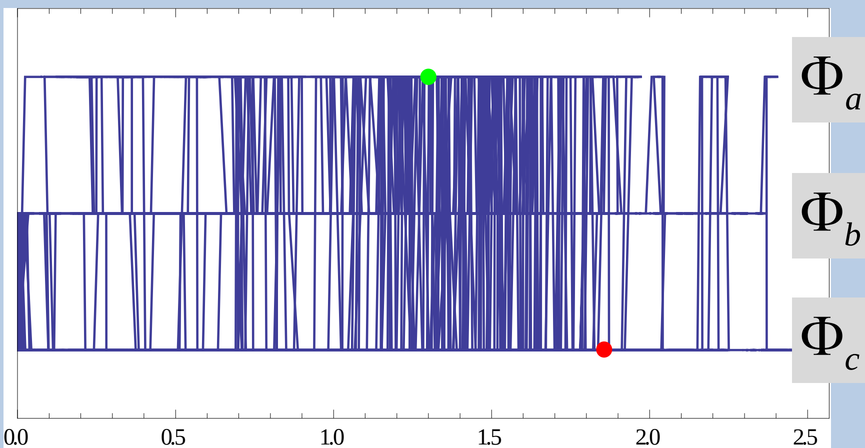


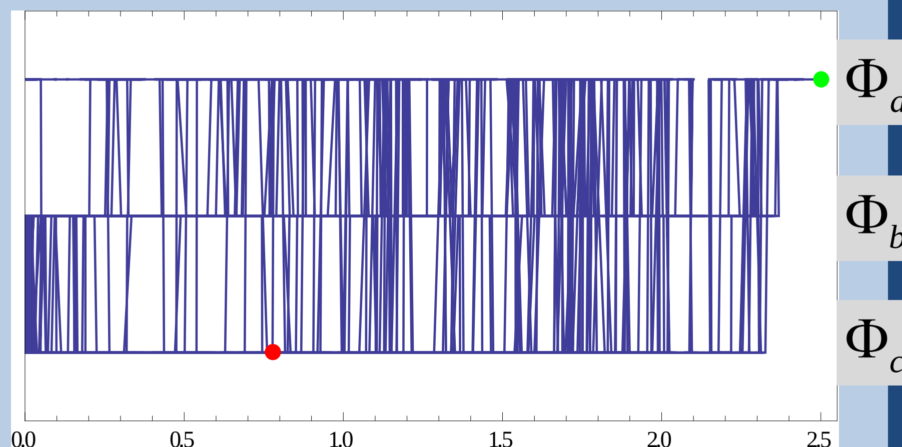
- Flow rates and transition probabilities for eigenfunctions:



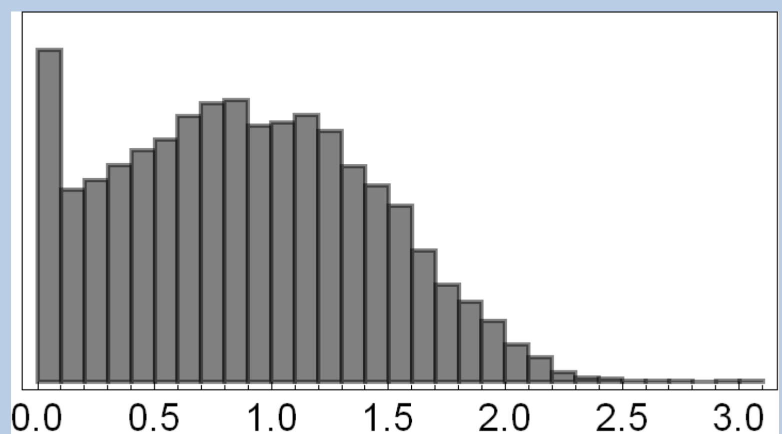
$$\Phi_a = -\frac{\Phi_{11}}{3} + \Phi_{22},$$
$$\Phi_{b,c} = \pm\frac{\Phi_{11}}{2\sqrt{3}} \pm \frac{2\Phi_{22}}{\sqrt{3}} + \Phi_{12}.$$

- Trajectories (green – start, red – position after γt=10):





- Simulated probability distribution:



$$a_0 = 1, \quad \nu_0 = 1, \quad \mu_0 = 13, \quad \eta = 1.3\cdot 10^2, \quad \sigma = 0.5$$