Formal Concept Analysis of higher order

Ondrej Kridlo, Patrik Mihalcin, Lubomir Antoni, Stanislav Krajci

INVESTMENTS IN EDUCATION DEVELOPMENT

Motivation

$\mathcal{P}$	TV	W	Ce	R
Anna	•		•	•
Bob	∘	∘	•	•
Cyril	∘	•	•	∘
David	•	•	∘	
Erik	•	•	∘	•

Context of

- group of people
- hotel services
- accommodation requirements
- • - important, ∘ - less important, - no matter

Motivation

$\mathcal{P}$	TV	W	Ce	R
Anna	•		•	•
Bob	○	○	•	•
Cyril	○	•	•	○
David	•	•	○	
Erik	•	•	○	•

Context of

- group of people
- hotel services
- accommodation requirements
- • - important, ○ - less important, - no matter

Output:

- groups of people that have simmilar accommodation requirements
- How to add some more information?
- friendship relationships
- accommodation situation in the city

External data 1: Friendship relationship

$\mathcal{F}$	Anna	Bob	Cyril	David	Erik
Anna	•	∘	∘		
Bob	∘	•	•	∘	∘
Cyril	∘	•	•	•	•
David		∘	•	•	•
Erik		∘	•	•	•

- group of friends
- group of people simmilar to their simpathy to the others
- ideal (instant) situation (facebook like, etc.)

External data 2: Accommodation situation

$\mathcal{H}$	TV	W	C	R
H1	•	•	•	•
H2	•	•		•
H3			•	
H4	•		•	•

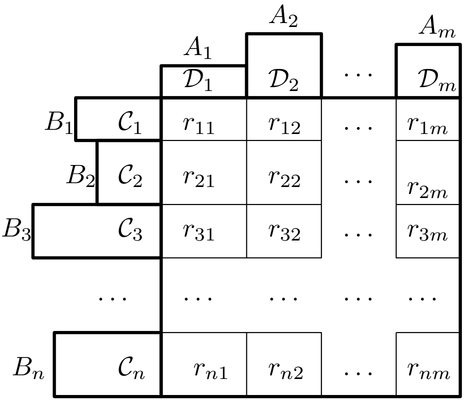
Output:

- subsets of hotels simmilar to services that they offer

Problem

Is it possible to advise to some groups of friends subsets of hotels that offers as much services common inside the group?
Solution: Second order FCA

Formal context of second order



- $$B = \bigcup_{i \in I} B_i$$
- $$A = \bigcup_{j \in J} A_j$$
- $$r = \bigcup_{(i,j) \in I \times J} r_{ij}$$
- $A_{i1} \cap A_{i2} = \emptyset$ for any $i1 \neq i2,\ i1, i2 \in I$
- $B_{j1} \cap B_{j2} = \emptyset$ for any $j1 \neq j2,\ j1, j2 \in J$

Formal context of second order

		A_1	A_2		A_m
		$\mathcal{D}_1$	$\mathcal{D}_2$	$\dots$	$\mathcal{D}_m$
B_1	$\mathcal{C}_1$	r_{11}	r_{12}	$\dots$	r_{1m}
B_2	$\mathcal{C}_2$	r_{21}	r_{22}	$\dots$	r_{2m}
B_3	$\mathcal{C}_3$	r_{31}	r_{32}	$\dots$	r_{3m}
	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
B_n	$\mathcal{C}_n$	r_{n1}	r_{n2}	$\dots$	r_{nm}

$$\left\langle \bigcup_{i\in I} B_i; \{\mathcal{C}_i | i \in I\}; \bigcup_{j\in J} A_j; \{\mathcal{D}_j | j \in J\}; \bigcup_{(i,j)\in I\times J} r_{ij} \right\rangle$$

Idea?

- Back to problem: advise a subset of hotels to a group of friends due to their accommodation requirements
- Translation: to find a sufficient connection between concept lattices of external contexts due to internal data

Krídlo, Ojeda-Aciego: Linking L-Chu correspondences and completely lattice L-ordered sets

- categorical equivalence between category $L\text{-}\mathrm{ChuCors}$ of L-contexts and L-Chu correspondences completely lattice L-ordered sets and L-fuzzy Galois connections
- functor $\Gamma : L\text{-}\mathrm{ChuCors} \to L\text{-}\mathrm{CLLOS}$
-
  - $\Gamma(\mathcal{C}) = L\text{-}\mathrm{FCL}(\mathcal{C})$ for any L-context $\mathcal{C}$
  - $\Gamma(\varphi) = \langle \lambda_L, \lambda_R \rangle$

$$\langle f_1, g_1 \rangle \preceq_1 \lambda_R(\langle f_2, g_2 \rangle) \Leftrightarrow \langle f_2, g_2 \rangle \preceq_2 \lambda_R(\langle f_1, g_1 \rangle)$$

for any $\varphi \in L\text{-}\mathrm{ChuCors}(\mathcal{C}_1, \mathcal{C}_2)$

Back to problem:

Now the problem is reduced by looking for a sufficient L-Chu correspondence.

Easy solution: Lets have external L-contexts $\mathcal{C}_i$ and $\mathcal{D}_j$ for any $(i,j) \in I \times J$:

- complete lattice (L-CLLOS) of $L\text{-ChuCors}(\mathcal{C}_i, \mathcal{D}_j)$ is dually isomorphic to $L\text{-Bonds}(\mathcal{C}_i, \mathcal{D}_j)$
- lets choose

$$\rho_{ij} = \bigvee \{\beta \in L\text{-Bonds}(\mathcal{C}_i, \mathcal{D}_j) | (\forall (o,a) \in B_i \times A_j) \beta(o)(a) \leq r_{ij}(o,a)\}$$

- create L-Chu correspondence φ_{ij}
  - $\varphi_{ijL}(o_i) = \downarrow_j (\rho_{ij}(o_i))$
  - $\varphi_{ijR}(a_j) = \uparrow_i (\rho_{ij}(a_j))$
- use a Galois connection $\Gamma(\varphi_{ij}) = \langle \lambda_{ijL}, \lambda_{ijR} \rangle$

Derivation operators

- $$\Uparrow\colon \prod_{i\in I} L\text{-FCL}(\mathcal{C}_i) \to \prod_{j\in J} L\text{-FCL}(\mathcal{D}_j)$$
- $$\Downarrow\colon \prod_{j\in J} L\text{-FCL}(\mathcal{D}_j) \to \prod_{i\in I} L\text{-FCL}(\mathcal{C}_i)$$
- $$\Uparrow(\Phi)(j) = \bigwedge_{i\in I} \lambda_{ijL}(\Phi(i))$$
- $$\Downarrow(\Psi)(i) = \bigwedge_{j\in J} \lambda_{ijR}(\Psi(j))$$
- for any $\Phi \in \prod_{i\in I} L\text{-FCL}(\mathcal{C}_i)$ and $\Psi \in \prod_{j\in J} L\text{-FCL}(\mathcal{D}_j)$

Solution of motivation problem:

	concepts of $\mathcal{F}$	concepts of $\mathcal{H}$
1	$\{\circ/A, \bullet/B, \bullet/C, \circ/D, \circ/E\}$ $\{\circ/A, \bullet/B, \bullet/C, \circ/D, \circ/E\}$	$\{\circ/TV, \circ/W, \bullet/Ce, \circ/R\}$ $\{\bullet/H1, \ /H2, \circ/H3, \circ/H4\}$
2	$\{\bullet/A, \bullet/B, \bullet/C, \bullet/D, \bullet/E\}$ $\{\ /A, \circ/B, \circ/C, \ /D, \ /E\}$	$\{\ /TV, \ /W, \circ/Ce, \ /R\}$ $\{\bullet/H1, \circ/H2, \bullet/H3, \bullet/H4\}$
3	$\{\bullet/A, \bullet/B, \bullet/C, \circ/D, \circ/E\}$ $\{\circ/A, \circ/B, \circ/C, \ /D, \ /E\}$	$\{\circ/TV, \ /W, \bullet/Ce, \circ/R\}$ $\{\bullet/H1, \ /H2, \circ/H3, \bullet/H4\}$
4	$\{\bullet/A, \circ/B, \circ/C, \ /D, \ /E\}$ $\{\bullet/A, \circ/B, \circ/C, \ /D, \ /E\}$	$\{\bullet/TV, \ /W, \bullet/Ce, \bullet/R\}$ $\{\bullet/H1, \ /H2, \ /H3, \bullet/H4\}$
5	$\{\circ/A, \circ/B, \circ/C, \ /D, \ /E\}$ $\{\bullet/A, \bullet/B, \bullet/C, \circ/D, \circ/E\}$	$\{\bullet/TV, \circ/W, \bullet/Ce, \bullet/R\}$ $\{\bullet/H1, \ /H2, \ /H3, \circ/H4\}$
6	$\{\ /A, \circ/B, \circ/C, \ /D, \ /E\}$ $\{\bullet/A, \bullet/B, \bullet/C, \bullet/D, \bullet/E\}$	$\{\bullet/TV, \bullet/W, \bullet/Ce, \bullet/R\}$ $\{\bullet/H1, \ /H2, \ /H3, \ /H4\}$

Simplification:

- Nice property of bonds: Let $\beta \in L\text{-Bonds}(\mathcal{C}_i, \mathcal{D}_j)$
- derivation operators $\langle \uparrow_\beta, \downarrow_\beta \rangle$ based on relation β forms a Galois connection between $L\text{-Ext}(\mathcal{C}_i)$ and $L\text{-Int}(\mathcal{D}_j)$
- $$\mathcal{K} = \left\langle \bigcup_{i \in I} B_i; \{\mathcal{C}_i | i \in I\}; \bigcup_{j \in J} A_j; \{\mathcal{D}_j | j \in J\}; \bigcup_{(i,j) \in I \times J} r_{ij} \right\rangle$$
- $$\widehat{\mathcal{K}} = \left\langle \bigcup_{i \in I} B_i; \bigcup_{j \in J} A_j, \bigcup_{(i,j) \in I \times J} \rho_{ij} \right\rangle$$

Generalisation of standard FCA: sigleton connection

For any L-fuzzy formal context $\mathcal{C} = \langle B, A, r \rangle$

$$\mathcal{C}^1 = \left\langle \bigcup_{b \in B} \{b\}, \{\perp_b : b \in B\}, \bigcup_{a \in A} \{a\}, \{\perp_a^* : a \in A\}, \bigcup_{(b,a) \in B \times A} r(b,a) \right\rangle$$

where $\perp_x = \langle \{x\}, L, \lambda \rangle$ such that $L\text{-FCL}(\perp_x) \cong L$.
Concept lattices of $\mathcal{C}$ and $\mathcal{C}^1$ are isomrphic.

Generalisation of standard FCA: $\neq$ connection

For any L-fuzzy formal context $\mathcal{C} = \langle B, A, r \rangle$ where L is closed under double negation law, lets define a second order formal context $\mathcal{C}^{\neq} = \langle B, \mathcal{B}, A, \mathcal{A}, r \rangle$ where for any set X there is an L-context $\mathcal{X} = \langle X, X, \neq \rangle$.

Concept lattices of $\mathcal{C}$ and $\mathcal{C}^{\neq}$ are isomorphic.

S. Krajči, et al.: On heterogeneous formal contexts

Definition

Heterogeneous formal context is a tuple $\langle B, A, \mathcal{P}, R, \mathcal{U}, \mathcal{V}, \odot \rangle$, where

- B and A are non-empty sets,
- $\mathcal{P} = \{\langle P_{b,a}, \leq_{P_{b,a}} \rangle : (b, a) \in B \times A\}$ is a system of posets,
- R is a mapping from $B \times A$ such that $R(b, a) \in P_{b,a}$ for any $b \in B$ and $a \in A$,
- $\mathcal{U} = \{\langle U_b, \leq_{U_b} \rangle : b \in B\}$ and $\mathcal{V} = \{\langle V_a, \leq_{V_a} \rangle : a \in A\}$ are systems of complete latices,
- $\odot = \{\circ_{b,a} : (b, a) \in B \times A\}$ is a system of isotone and left-continuous mappings $\circ_{b,a} : U_b \times V_a \longrightarrow P_{b,a}$.

Translation:

- B and A will be the index sets I and J,
- complete lattices U_i or V_j for any $(i,j) \in B \times A = I \times J$ will be the complete lattices $\langle \mathrm{Ext}(\mathcal{C}_i), \leq \rangle$ and $\langle \mathrm{Int}(\mathcal{D}_j), \leq \rangle$,
- $P_{i,j}$ will be a complete lattice of all fuzzy relations from $L^{B_i \times A_j}$,
- any value of relation r will be a binary relation $r(i,j) = r_{i,j} \in L^{B_i \times A_j}$,
- operation $\circ_{i,j} : \mathrm{Ext}(\mathcal{C}_i) \times \mathrm{Int}(\mathcal{D}_j) \longrightarrow L^{B_i \times A_j}$ is defined as

$$(f \circ_{i,j} g)(b,a) = f(b) \otimes g(a)$$

for any $f \in \mathrm{Ext}(\mathcal{C}_i)$ and $g \in \mathrm{Int}(\mathcal{D}_j)$ and any $(b,a) \in B_i \times A_j$. The mapping $\circ_{i,j}$ is isotone due to isotonicity of $\otimes$.

$$\nwarrow (\Phi)^j = \bigvee \{g \in \mathrm{Int}(\mathcal{D}_j) : (\forall i \in I) \Phi^i \circ_{i,j} g \leq r_{i,j}\}$$

$$\searrow (\Psi)^i = \bigvee \{f \in \mathrm{Ext}(\mathcal{C}_i) : (\forall j \in J) f \circ_{i,j} \Psi^j \leq r_{i,j}\}$$

$$\uparrow_{\widehat{\mathcal{K}}} (\Phi) \leq \nwarrow (\Phi) \text{ and } \downarrow_{\widehat{\mathcal{K}}} (\Psi) \leq \searrow (\Psi)$$

for any $\Phi \in \prod_{i \in I} \mathrm{Ext}(\mathcal{C}_i)$ and $\Psi \in \prod_{j \in J} \mathrm{Int}(\mathcal{D}_j)$

Thank you!