The Lord of the Rings

Adventures and misadventures in OAM

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Outline

- **Motivation**
- **Phase space for continuous variables**
- **Phase space for angle-OAM**
- **Radial modes**
- **Quantum optics in terms of rings**

Motivation

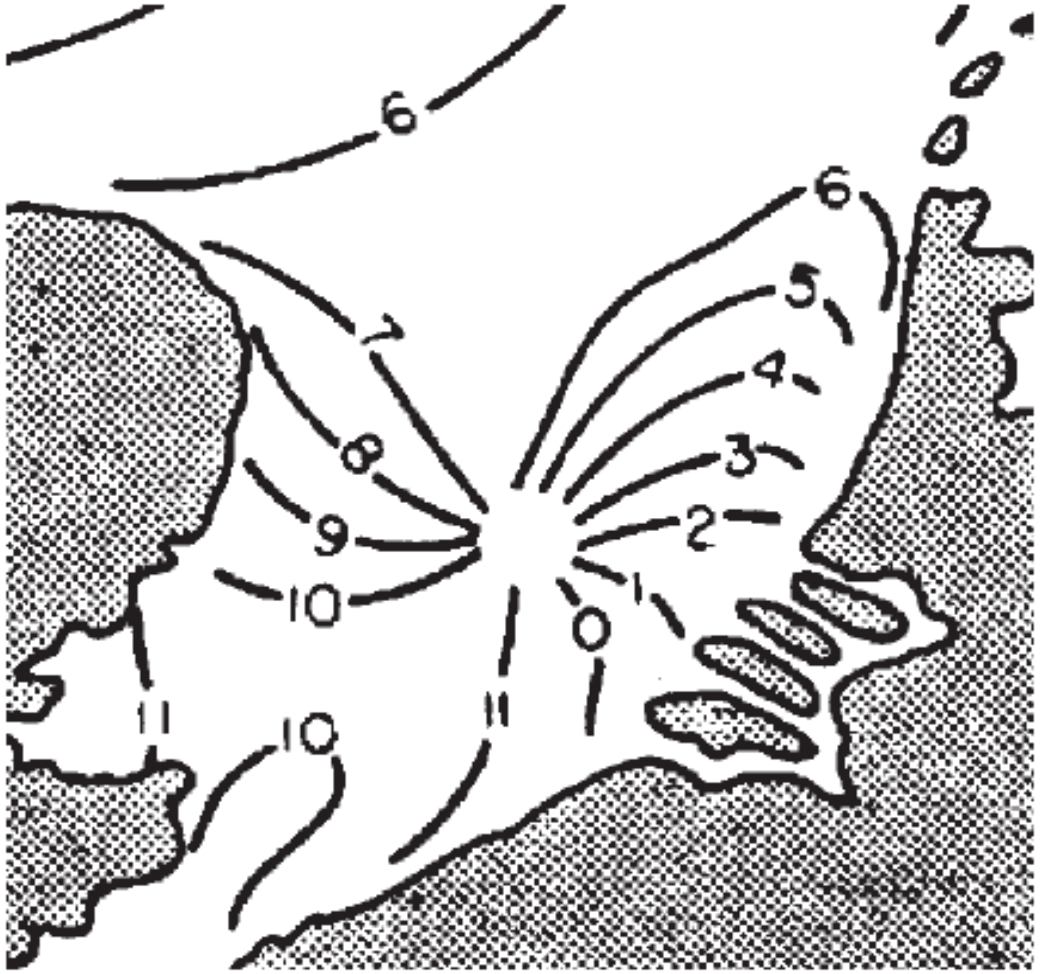
- **Phase-space methods**
  - ✓ Quantum mechanics appears as a statistical theory in phase space
  - ✓ Simple to understand (no abstract Hilbert-space concepts)
  - ✓ Simple to picture
  - ✓ Computationally efficient

- **Basic ingredients**
  - ✓ (Classical) phase space
  - ✓ (Quasi) distributions in phase space
  - ✓ Star product
  - ✓ Coherent states

- **Drawbacks**
  - ✓ The machinery works well for continuous variables (and symmetry)
  - ✓ For other systems we need a toolbox to deal with them

Phase singularities

Phase singularities



Whewell's amphidromy between England and Holland.

- William Whewell 1833
- Cotidal lines connecting places where tide is high at a given time (points of equal phase).
- 12 hour cycle.
- Extrapolating away from coasts revealed a point around which tidal waves revolve.
- Phase is undefined at this point, therefore amplitude must be zero

Phase dislocations in an optical field

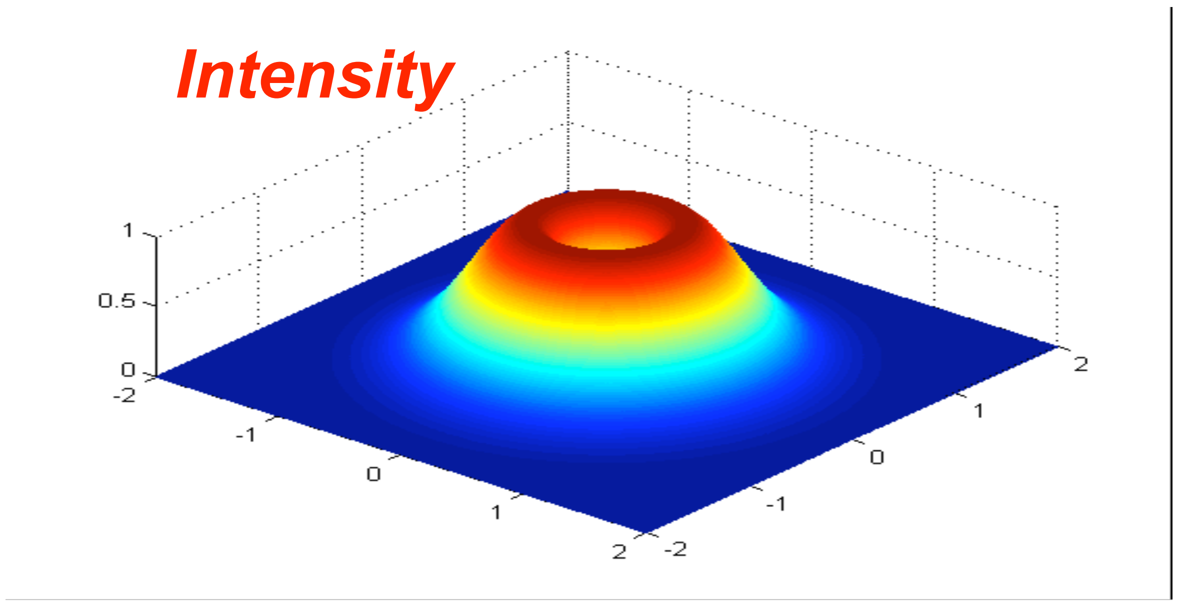
$$\mathbf{E}(r, \phi, z) = \mathbf{A}(r, z)\ \exp[i\Phi(r, \phi, z)]$$

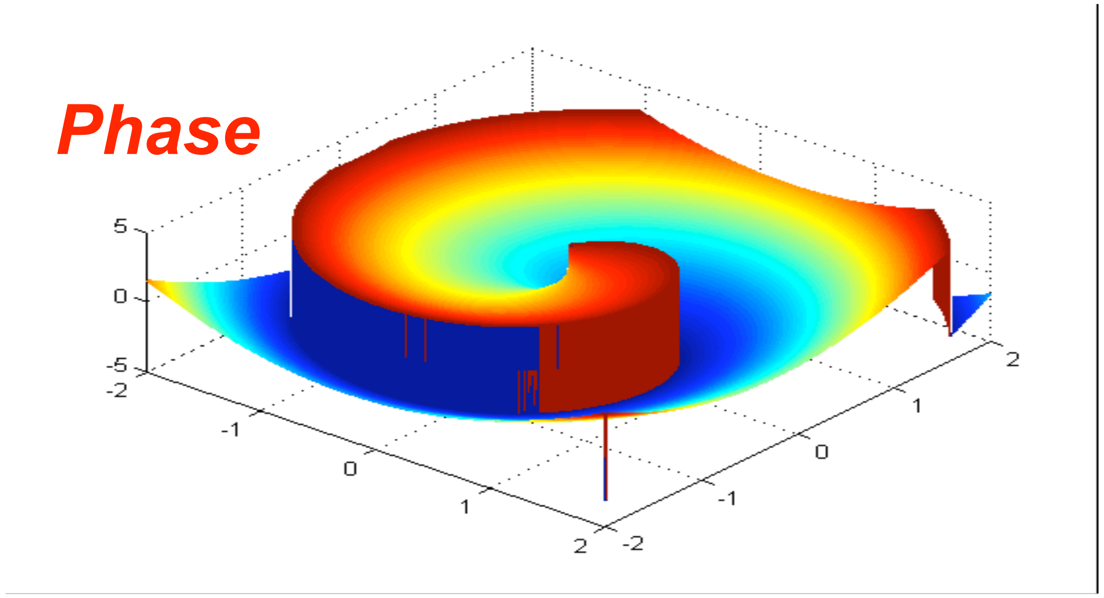
- Dislocations

$$\mathrm{Re}\left[\mathbf{E}(r, \phi, z)\right] = \mathrm{Im}\left[\mathbf{E}(r, \phi, z)\right] = 0$$

- Screw dislocation (vortex)

$$\mathrm{Re}\left[\mathbf{E}(r_0)\right] = \mathrm{Im}\left[\mathbf{E}(r_0)\right] = 0$$

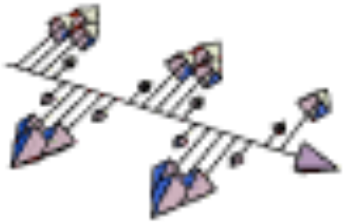




Angular momentum of light

Spin angular momentum

Linear $\sigma = 0$



Right circular $\sigma = 1$

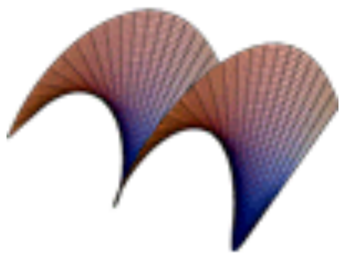
Left circular $\sigma = -1$

Orbital angular momentum

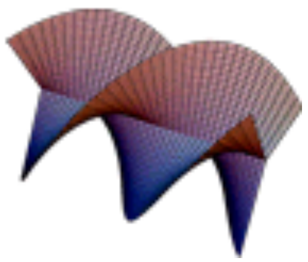
m = 0

m = 1

m = 2



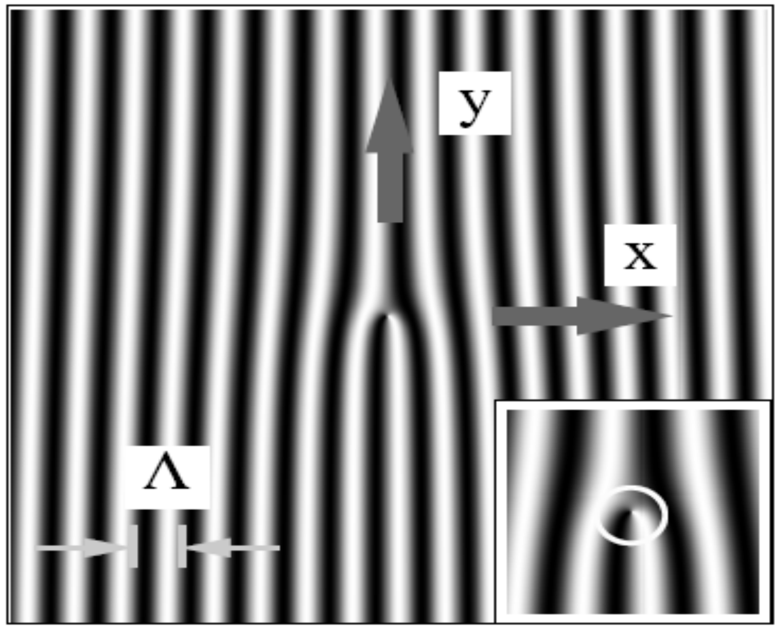
m = 3

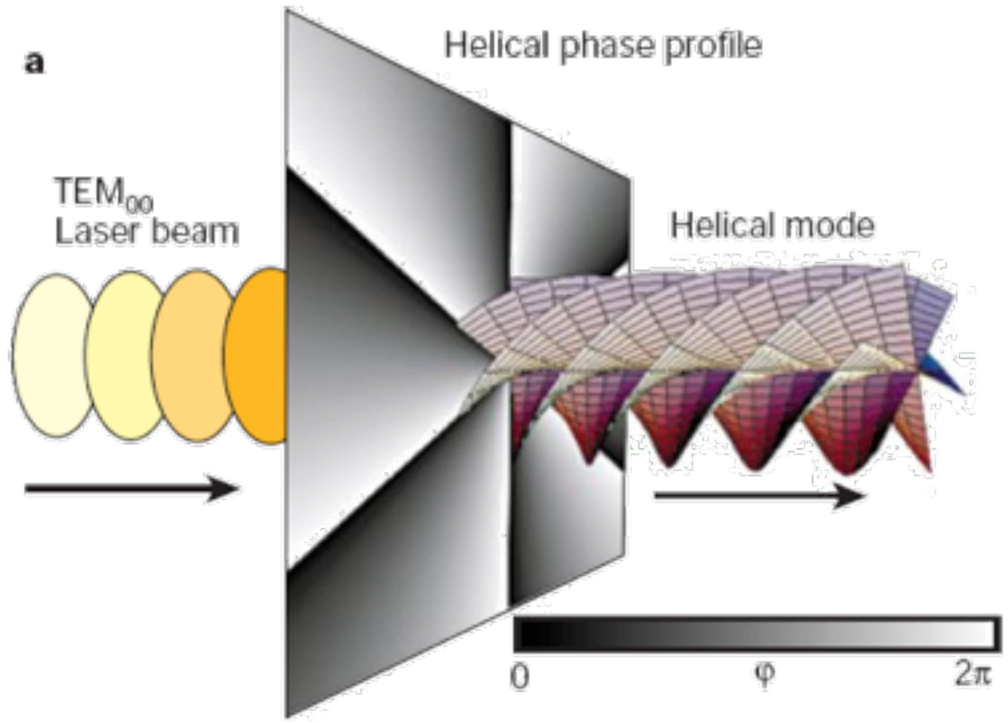


http://www.physics.gla.ac.uk/Optics/projects/singlePhotonOAM/

Generation of vortex beams

Holographic mask





Topological charge

Importance far beyond quantum optics: Vortices in other physical sytems, such as

Quantum fluids, BECs, molecular rotations, ferromagnets, accoustical waves...

OAM of vortex beams

Total energy

$$W = \frac{1}{2}\varepsilon_0 \int d^3r\, \mathbf{E}^* \cdot \mathbf{E}$$

Total angular momentum

Orbital   Spin

$$\mathbf{J} = \varepsilon_0 \int d^3r\, (\mathbf{r} \times \mathbf{E} \times \mathbf{B}) = \mathbf{L} + \mathbf{S}$$

Laguerre-Gauss modes

$$\frac{J_z}{W} = \frac{1}{\omega}(\ell + \sigma)$$

$\sigma = 0$, *linear polarization*
$\sigma = \pm 1$, *circular polarization*

OAM per photon

$$\frac{L_z}{\text{per photon}} = \ell\hbar$$

Phase space for continuous variables

Dynamical symmetry group

✓ Hermitian coordinate and momentum operators

$$[\hat{q}, \hat{p}] = i$$

Heisenberg-Weyl

✓ Phase space: coadjoint orbit associated with an irreducible representation of the dynamical symmetry group: $\mathbb{R}^2$

✓ Generators of translations in position and momentum

$$\hat{U}(q) = \exp(-iq\hat{p}) \qquad \hat{V}(p) = \exp(ip\hat{q})$$

✓ Unitary Weyl form

$$\hat{V}(p)\hat{U}(q) = e^{iqp}\,\hat{U}(q)\hat{V}(p)$$

Phase space for continuous variables

- ✓ Displacement operators

$$\hat{D}(q,p) = e^{-iqp/2}\,\hat{U}(p)\hat{V}(q)\,, = \exp[i(p\hat{q} - q\hat{p})]$$

Complete orthonormal set in the space of operators acting on $\mathcal{H}$

- ✓ Coherent states

$$|q,p\rangle = \hat{D}(q,p)\,|\psi_0\rangle$$

- ✓ Fiducial state $|\psi_0\rangle$
  - Fundamental state of the harmonic oscillator (Gaussian!!)
  - Minimum uncertainty state
  - Eigenstate of the Fourier transform

Phase space for continuous variables

Wigner function

- ✓ Map the density matrix into a classical function on phase space

$$W(q,p) = \mathrm{Tr}[\hat{\varrho}\,\hat{w}(q,p)]$$

$$\hat{\varrho} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{w}(q,p) W(q,p)\, dq dp$$

- ✓ Wigner kernel (Stratonovich-Weyl quantizer)

$$\hat{w}(q,p) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \exp[-i(pq' - qp')]\, \hat{D}(q',p')\, dq' dp'$$

Double
Fourier transform

Phase space for OAM

- Hermitian angular coordinate and orbital angular momentum operators

$$[\hat{\phi}, \hat{L}] \stackrel{!}{=} i$$

Periodicity!!

- Phase space: discrete cylinder: $\mathcal{S}_1 \times \mathbb{Z}$

- Generators of translations in position and momentum

$$\hat{U}(\phi) = \exp(-i\phi\,\hat{L}) \qquad \hat{V}(\ell) = \exp(i\ell\,\hat{\phi})$$

- Unitary Weyl form

$$\hat{V}(\ell)\hat{U}(\phi) = e^{i\ell\phi}\,\hat{U}(\phi)\hat{V}(\ell)$$

Phase space for OAM

- ✓ Displacement operators

$$\hat{D}(\ell, \phi) = \exp[i\alpha(\ell, \phi)]\, \hat{V}(\ell)\hat{U}(\phi)$$

Complete orthonormal set in the space of operators acting on $\mathcal{H}$

- ✓ Coherent states

$$|\ell, \phi\rangle = \hat{D}(\ell, \phi)\, |\Psi_0\rangle$$

- ✓ Fiducial state $|\psi_0\rangle$
  - Gaussian

$$\Psi_0(\phi) \propto \vartheta_3 \left(\frac{\phi}{2} \Big| \frac{1}{e^{1/2}} \right)$$

Phase space for OAM

Wigner function

- ✓ Map the density matrix into a classical function on the cylinder

$$W(\ell, \phi) = \mathrm{Tr}[\hat{\varrho}\, \hat{w}(\ell, \phi)] \, ,$$

$$\hat{\varrho} = 2\pi \sum_{\ell \in \mathbb{Z}} \int_{-\pi}^{\pi} \hat{w}(\ell, \phi) W(\ell, \phi) \, d\phi \, .$$

- ✓ Wigner kernel (Stratonovich-Weyl quantizer)

$$\hat{w}(\ell, \phi) = \frac{1}{(2\pi)^2} \sum_{\ell' \in \mathbb{Z}} \int_{-\pi}^{\pi} \exp[-i(\ell' \phi - \ell \phi')] \hat{D}(\ell', \phi') \, d\phi'$$

Double
Fourier transform

Some examples

Angular momentum eigenstate $|\ell_0\rangle$

$$W_{|\ell_0\rangle}(\ell,\phi) = \frac{1}{2\pi}\delta_{\ell\ell_0}$$

Angle eigenstate $|\phi_0\rangle$

$$W_{|\phi_0\rangle}(\ell,\phi) = \frac{1}{2\pi}\delta_{2\pi}(\phi-\phi_0)$$

Coherent state $|\ell_0,\phi_0\rangle$ with $\langle\ell|\ell_0,\phi_0\rangle = e^{-(\ell-\ell_0)/2+i\ell\phi}$

$$W(\ell,\phi) \propto e^{-(\ell-\ell_0)}\theta_3(\phi-\phi_0|1/e)$$

discrete and circular Gaussian

$$+ \sim \theta_3(\phi-\phi_0+i/2|1/e)\sum_{\ell'\in\mathbb{Z}}\frac{(-1)^{\ell_0-\ell+\ell'}\, e^{-\ell'^2-\ell'}}{\ell_0-\ell+\ell'+1/2}$$

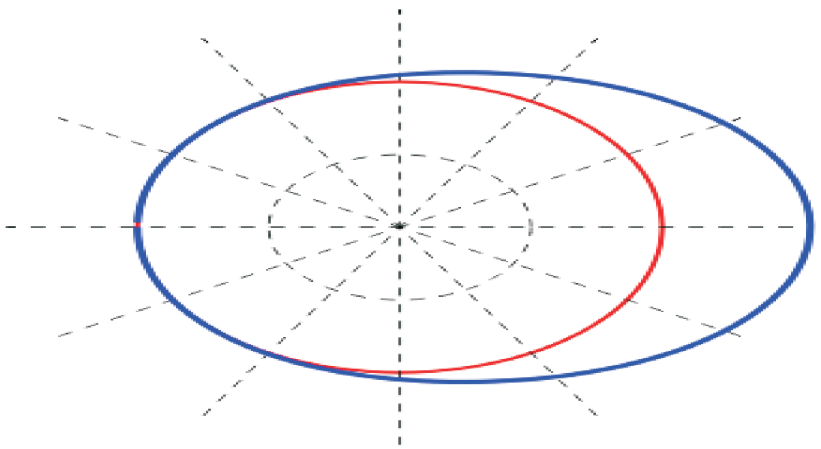
Superposition state $|\psi\rangle = |\ell_1\rangle + |\ell_2\rangle$

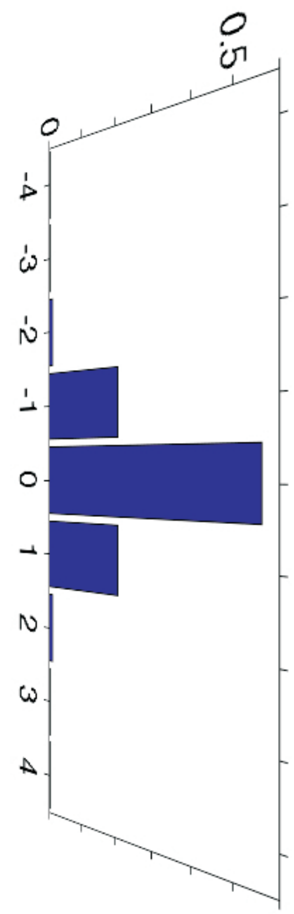
$$W_{|\psi\rangle}(\ell,\phi) \propto \delta_{\ell\ell_1} + \delta_{\ell\ell_2} + \ldots$$

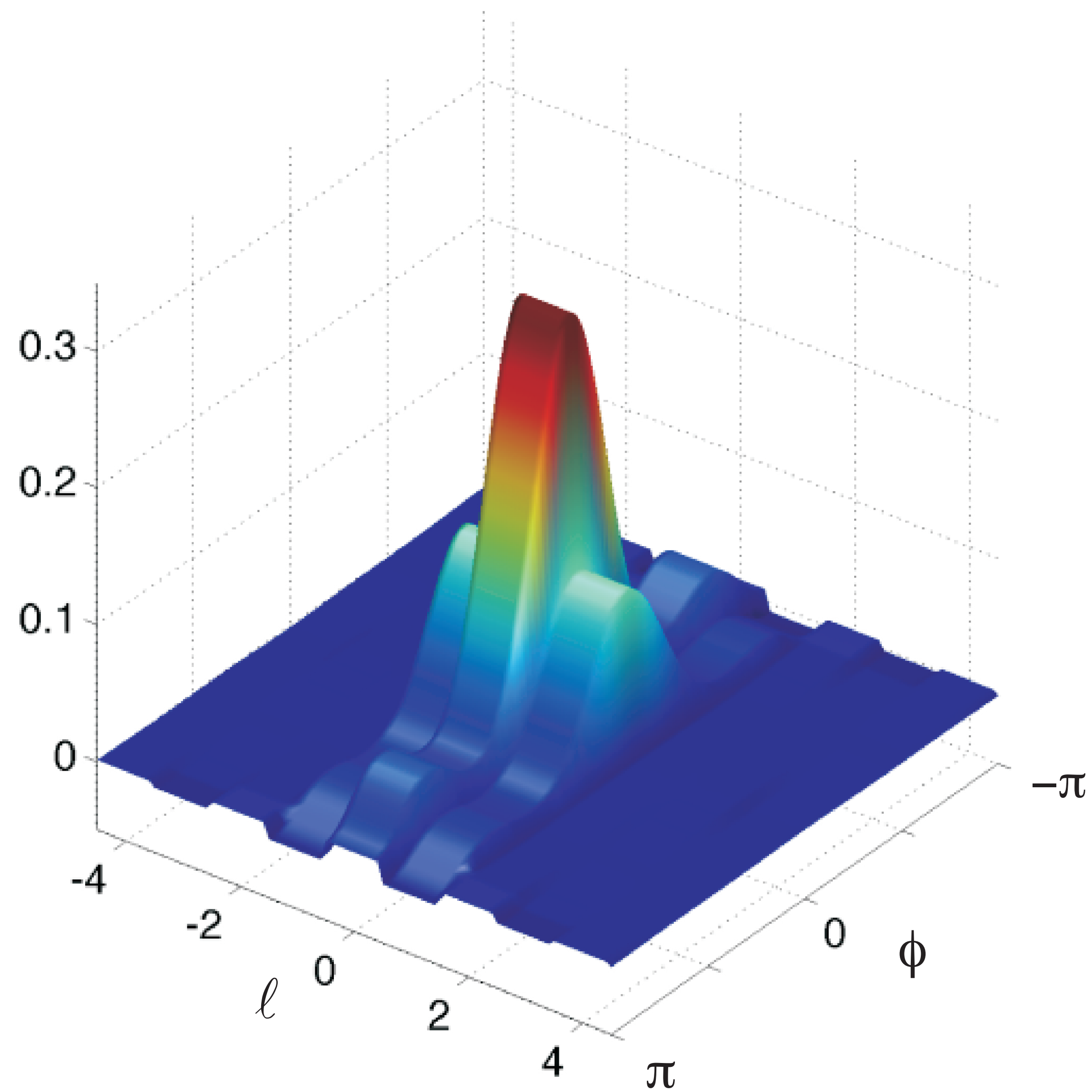
$2\cos[\phi(\ell_1+\ell_2)/2]$ (even)

$\cos[\phi(\ell_1+\ell_2)/2]\dfrac{(-1)^{\ell+(\ell_1+\ell_2-1)/2}}{\ell_1+\ell_2-2\ell}$ (odd)

Coherent state

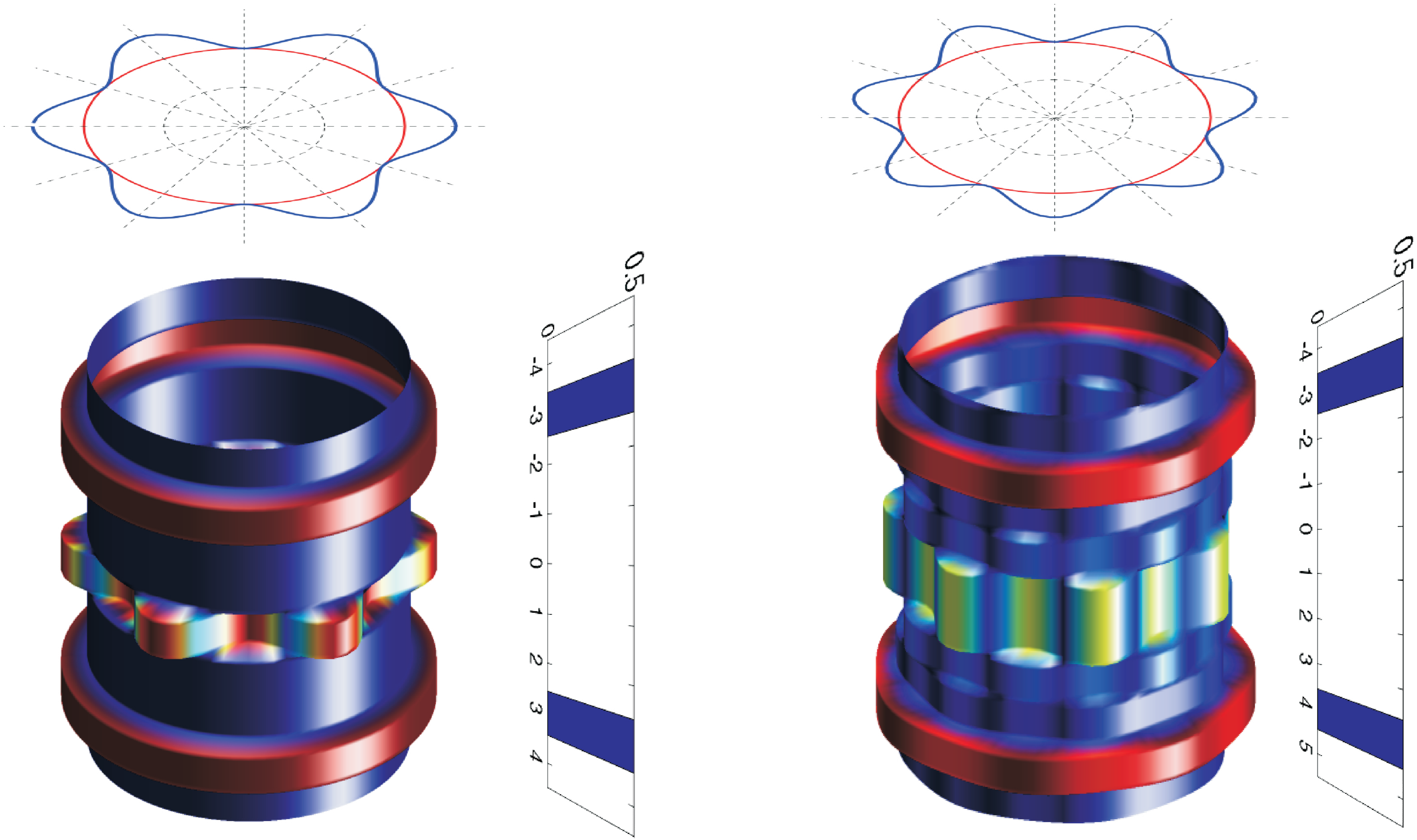






Note small negativities for $\ell = \pm 1,\ \phi \sim \pm\pi$

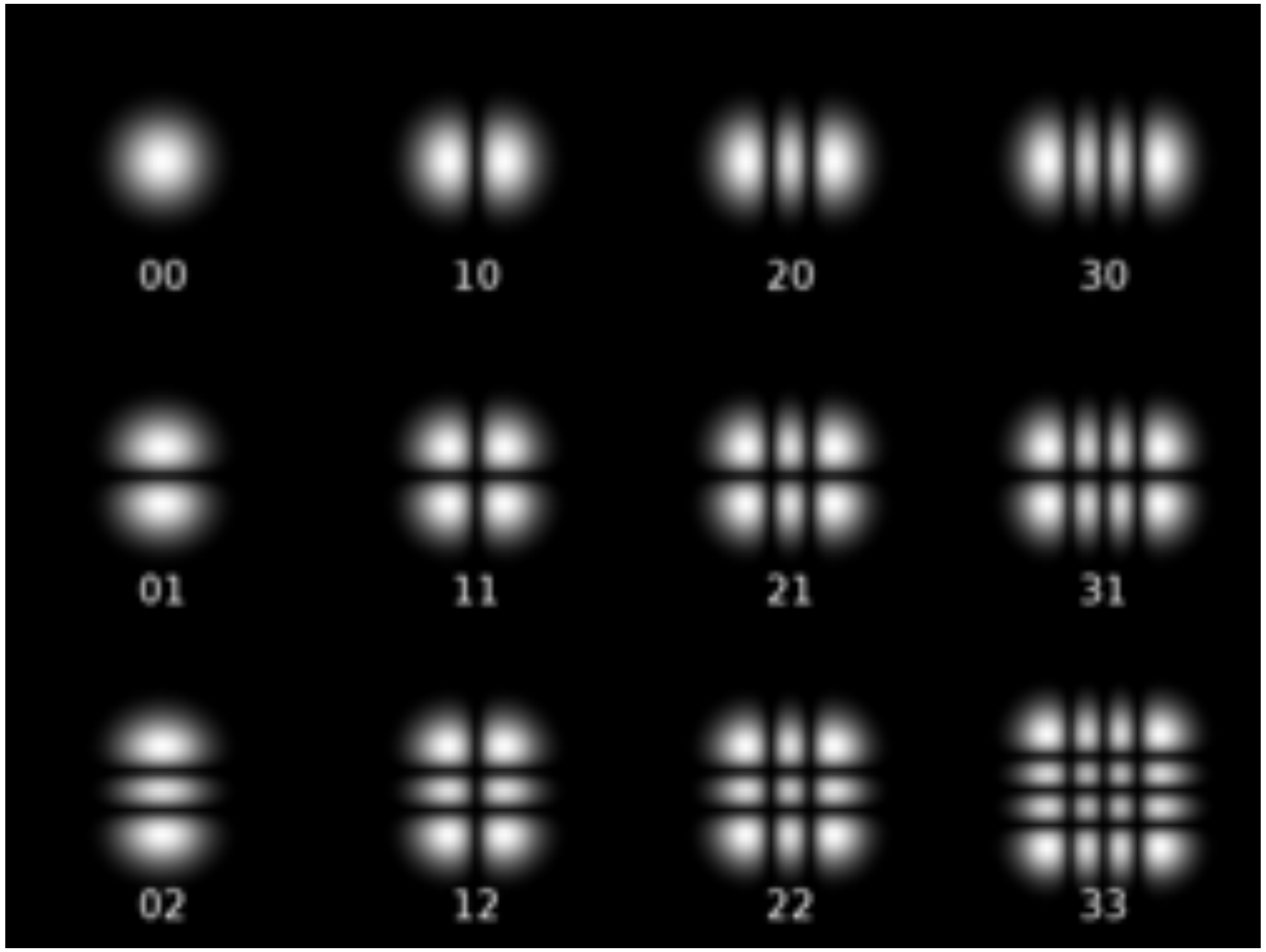
Superposition states



OAM from a 2D oscillator

- ✓ Stationary states (Cartesian coordinates)

$$\Psi_{n_x n_y}(x,y) = \sqrt{\frac{\alpha^2}{\pi\, 2^{n_x+n_y} n_x! n_y!}} H_{n_x}(\alpha x) H_{n_y}(\alpha y) \exp[-\alpha^2(x^2+y^2)/2]$$

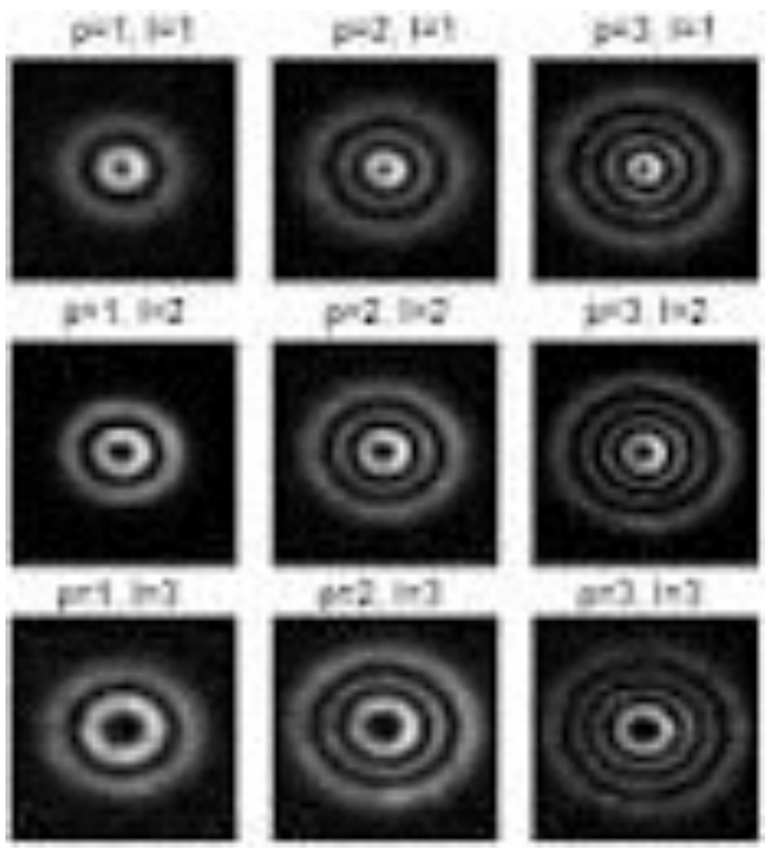


OAM from a 2D oscillator

- ✓ Stationary states (cyndrical symmetry)

$$\Psi_{n\ell}(r,\varphi) = A_{n\ell}(r)\exp(i\ell\varphi)$$

$$A_{n\ell}(r) = \frac{\sqrt{2\alpha^2 p!}}{\sqrt{(p+|\ell|)!}}\, e^{-\alpha^2 r^2/2}(\alpha r)^{|\ell|} L_p^{|\ell|}(\alpha^2 r^2)$$

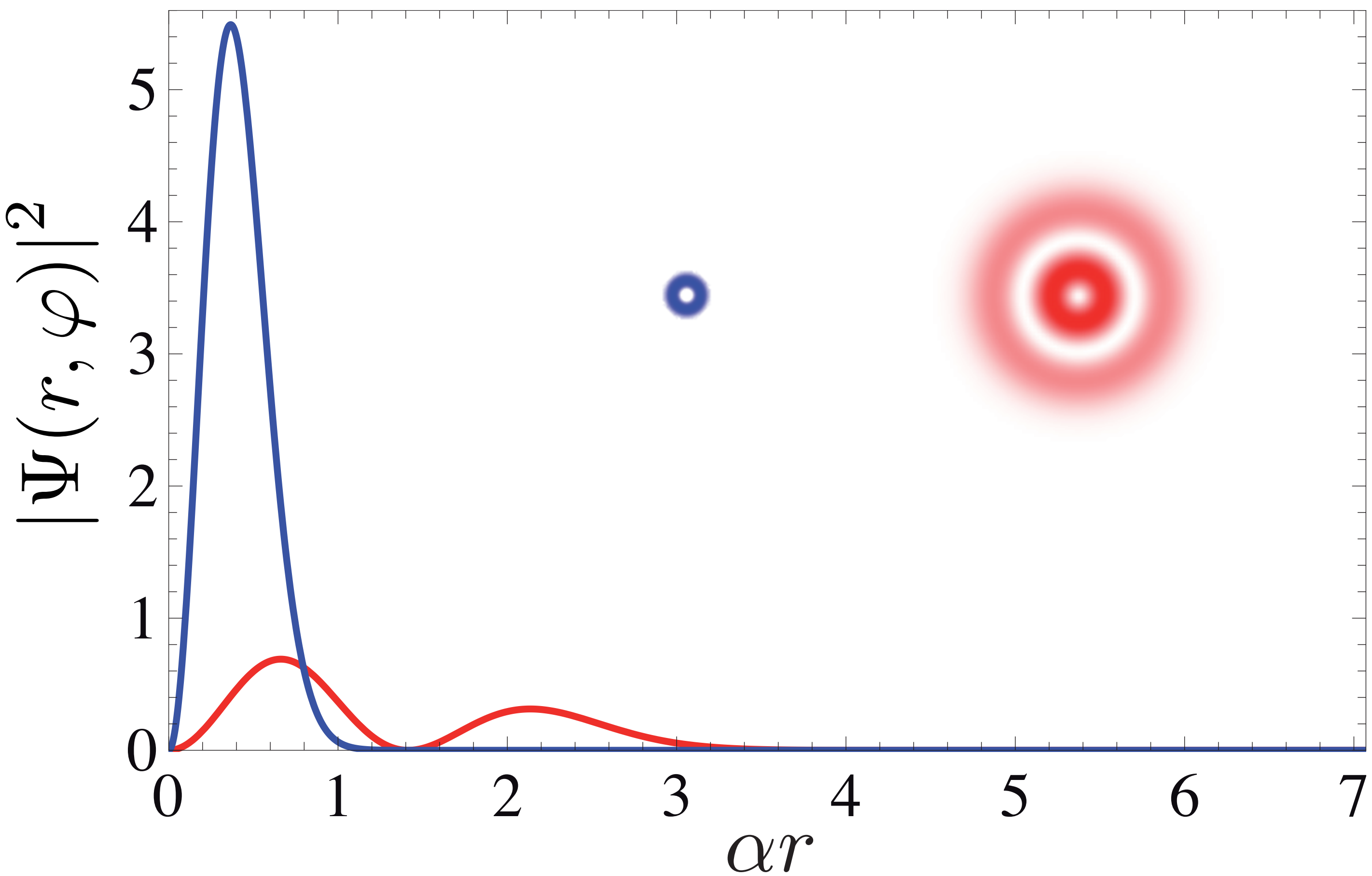


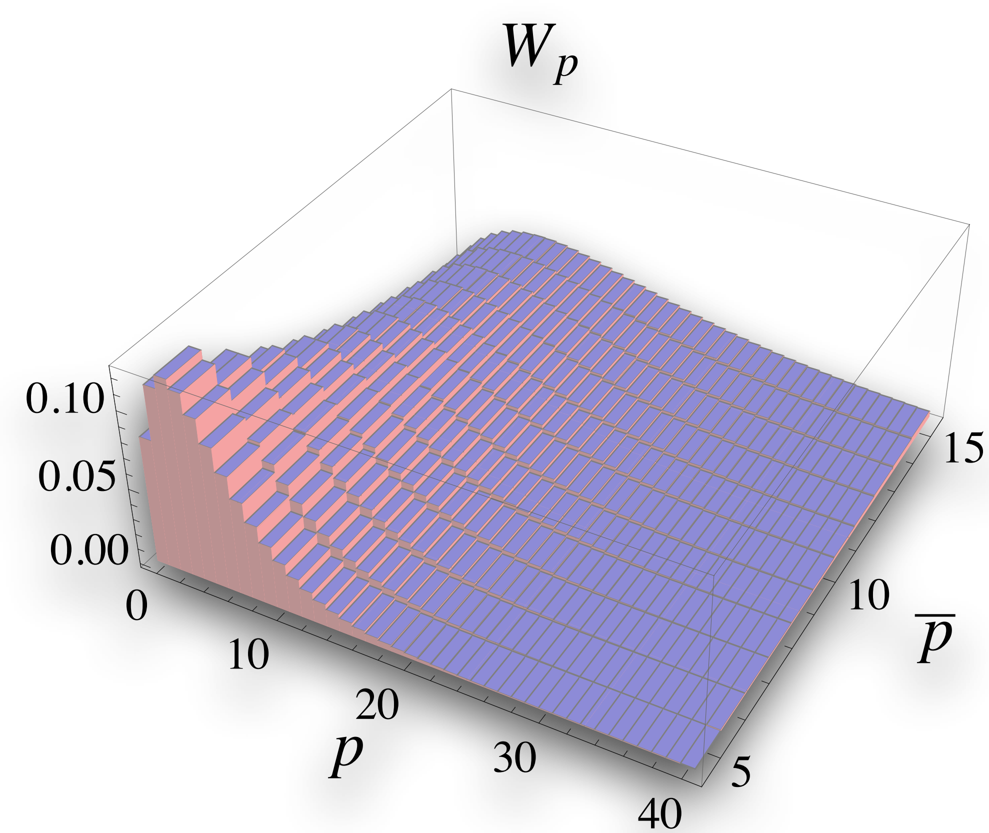
$$p = \frac{1}{2}(n - |\ell|)$$

- ✓ Number of rings

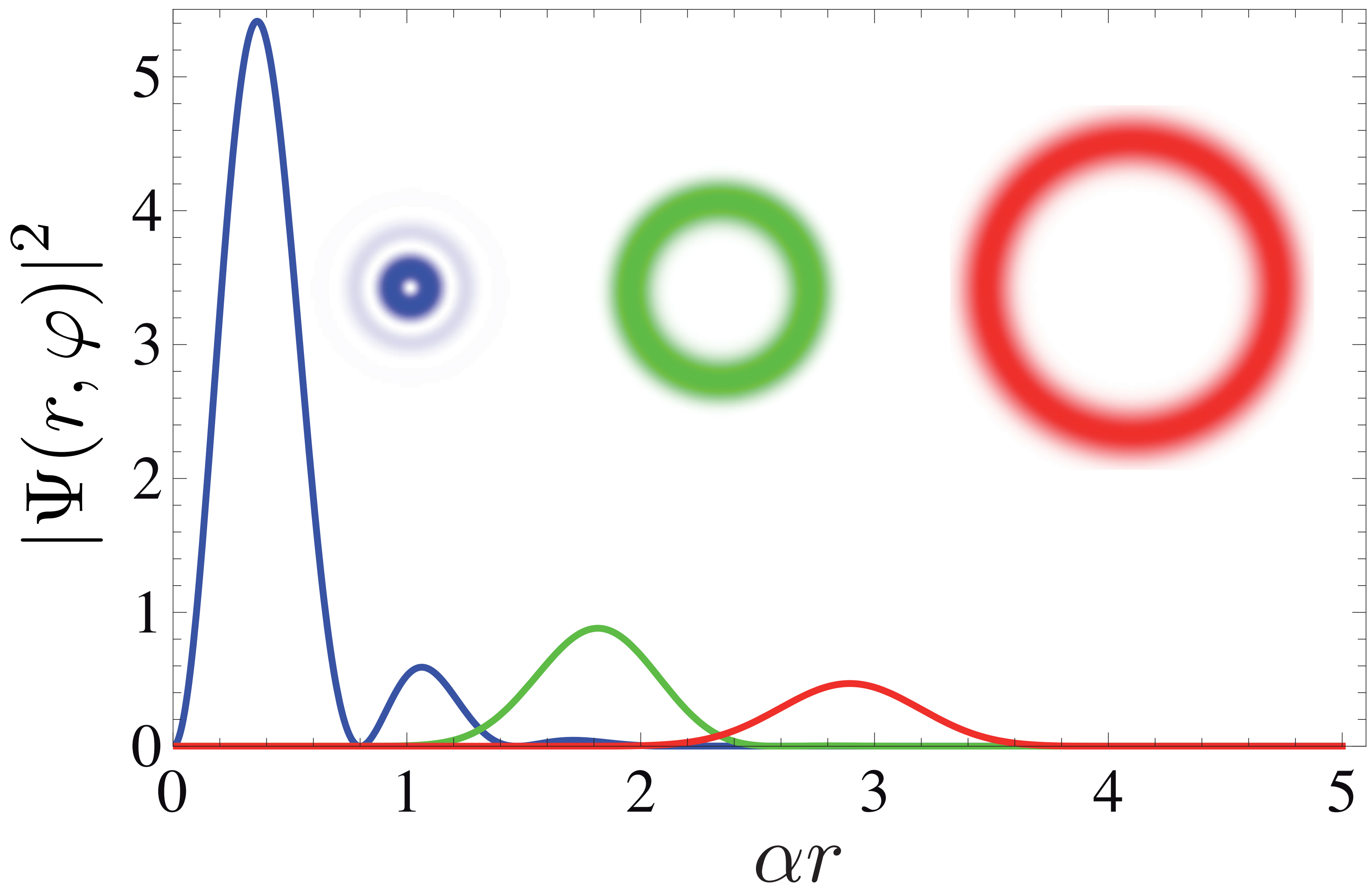
Is p a number operator?

Quantum optics in terms or rings?





Intelligent states



Conclusions

- ✓ A phase space description of the canonical pair angle-angular momentum is possible by extending known techniques from the position-momentum phase space in the proper consistent manner

- ✓ We have provided a handy toolbox to deal with the radia index the Laguerre-Gauss modes and how it can be used to construct a consistent quantum theory of this variable