New links between mathematical morphology and fuzzy property-oriented concept lattices

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Olomouc, Apr 3, 2014

What is Mathematical Morphology?

- *Mathematical Morphology* is a theory which provides tools for image analysis.
- It was originally proposed by *Jean Serra* and *Georges Matheron* in the early 60's.
- It covers a wide range of transformations: from *Filters* to *Edge detection*.
- Its foundations are based on *Set theory*, *Affine spaces* and *Lattice theory*.

What is Mathematical Morphology?

Different levels

There are different approaches to Mathematical Morphology, based mainly on the type of images.

- The original version of *Mathematical Morphology* relies on **binary (b/w) images**.
- Then, the theory can be extended to **Grey-scale Images**.
- Finally, a general **Algebraic Formulation** of Mathematical Morphology can be given.

Outline

- We begin by recalling some preliminary notions.
- Then, we will present the two most important transformations: *Erosions* and *Dilations*.
- Later, we introduce other morphologic transformations such as *Opening* and *Closing*.
- Finally we will describe some relationships between MM and FCA.

Binary Images

A (binary) image is either a subset of $\mathbb{Z} \times \mathbb{Z}$ or a subset of $\mathbb{R} \times \mathbb{R}$. Depending whether we are considering a discrete image or a continuous image.

Image in $\mathbb{Z} \times \mathbb{Z}$

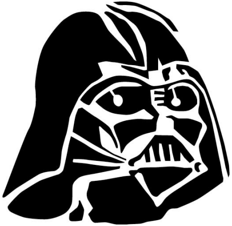


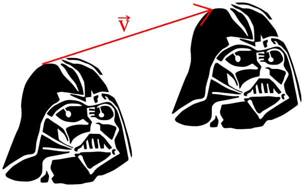
Image in $\mathbb{R} \times \mathbb{R}$

Set operations, translations and reflection

We assume that the basic operations between two sets A and B are well known; i.e *Intersection* $(A \cap B)$, *Union* $(A \cup B)$, *Complement* (A^c) and *substraction* $(A \setminus B)$.

We assume also that the notion of *translation* of a set A by a vector $\vec{v}$ and the notion of *reflexion* of a set with respect to *"the origin"* are known; i.e.

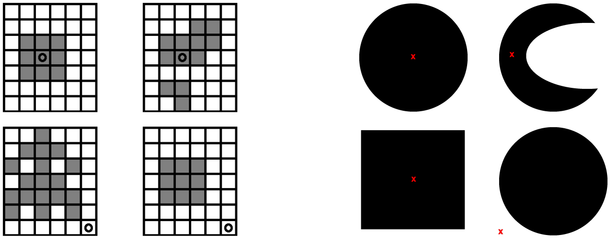
$$A_{\vec{v}} = \{a + \vec{v} \in \Omega \mid a \in A\} \qquad -A = \{-a \in \Omega \mid a \in A\}$$



Structuring elements

The definition

Every transformation in Mathematical Morphology is defined from *structuring elements*. Those elements are just images (i.e. either subsets of $\mathbb{Z} \times \mathbb{Z}$ or subsets of $\mathbb{R} \times \mathbb{R}$) with one origin.



The structuring element

Some remarks

- A priori, structuring elements do not have to satisfy any additional condition but being images with origins.
- But the choice of the structuring element is crucial to obtain searched results.
- Usually, structuring elements are considered small relative to the image; but there are exceptions.
- The number of structuring elements needed depends on the transformation. For instance, *dilations* only require one structuring element, for the *Hit-or-Miss* transformation are needed two and *thinning* is defined usually on eight.

Dilation

The definition

Given an image X, the *dilation by the structuring element* B is defined as:

$$\delta_B(X) = \{x + b \mid x \in X \text{ and } b \in B\} = \bigcup_{b \in B} X_b = \bigcup_{x \in X} B_x$$

The dilation of X by B coincides with the *Minkowski sum* of X and B. Thus, the notation $X \oplus B$ instead of $\delta_B(X)$ is usual in the Literature.

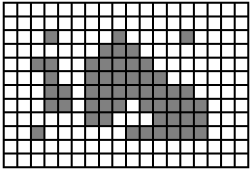
Remark

Note that for each structuring element B, we have a different dilation. So we have one dilation per each image we can define in the domain.

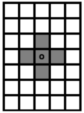
Dilation

An example in $\mathbb{Z} \times \mathbb{Z}$ (I of IV)

Let us compute the dilation of the image

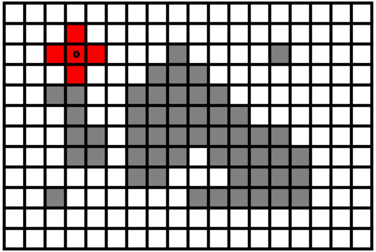


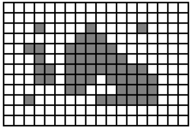
by the structuring element

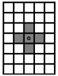


Dilation

An example in $\mathbb{Z} \times \mathbb{Z}$ (II of IV)



$X =$ 

$B =$ 

Dilation

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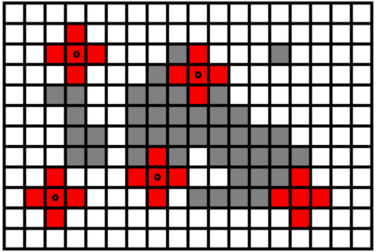
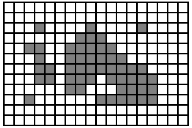
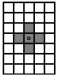
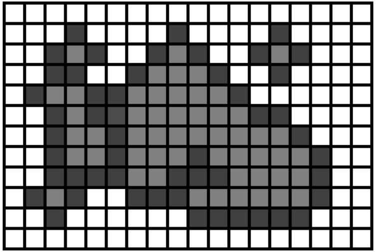
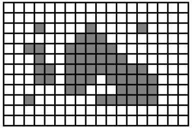


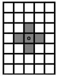
$X =$  $B =$ 

Dilation

An example in $\mathbb{Z} \times \mathbb{Z}$ (IV of IV)



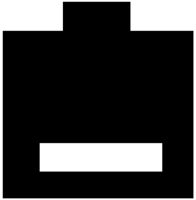
$X =$ 

$B =$ 

Dilation

An example in $\mathbb{R} \times \mathbb{R}$ (I of IV)

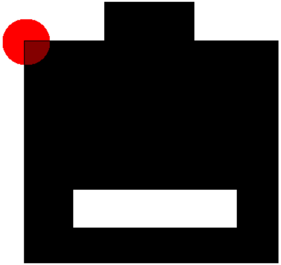
Let us compute the dilation of the image



by the structuring element $B =$ 

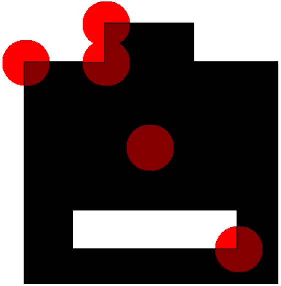
Dilation

An example in $\mathbb{R} \times \mathbb{R}$ (II of IV)



Dilation

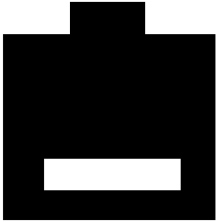
An example in $\mathbb{R} \times \mathbb{R}$ (III of IV)



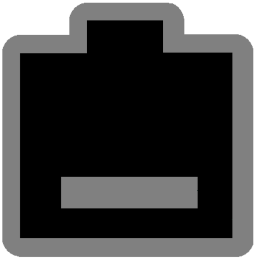
Dilation

An example in $\mathbb{R} \times \mathbb{R}$ (IV of IV)

X



$\delta_B(X)$



Dilation

Properties (I of II)

- **Monotonicity**:
  - If $X \subseteq Y$ then $\delta_B(X) \subseteq \delta_B(Y)$ for all structuring element B.
  - If $B_1 \subseteq B_2$ then $\delta_{B_1}(X) \subseteq \delta_{B_2}(X)$ for all image X.
- **Extensiveness**: *If the origin belongs to the structuring element* B, then $X \subseteq \delta_B(X)$.
- **Commutativity**: $\delta_B(X) = X \oplus B = B \oplus X = \delta_X(B)$
- **Translation invariance**: Let τ be a translation. Then

$$\delta_B(\tau(X)) = \tau(\delta_B(X)) = \delta_{\tau(B)}(X).$$

Dilation

Properties (II of II)

- **Distributivity with respect to Union**:
  - $\delta_B(X \cup Y) = \delta_B(X) \cup \delta_B(Y)$
  - $\delta_{B_1 \cup B_2}(X) = \delta_{B_1}(X) \cup \delta_{B_2}(X)$
- **Relationship with the intersection**:
  - $\delta_B(X \cap Y) \subseteq \delta_B(X) \cap \delta_B(Y)$
  - $\delta_{B_1 \cap B_2}(X) \subseteq \delta_{B_1}(X) \cap \delta_{B_2}(X)$
- **Associativity:**

$$X \oplus (Y \oplus Z) = (X \oplus Y) \oplus Z$$

$$\text{(in terms of } \delta_B) \quad \delta_{B_1}(\delta_{B_2}(Z)) = \delta_{\delta_{B_1}(B_2)}(X) = \delta_{B_1 \oplus B_2}(X)$$

Erosion

The definition

Given an image X, the *erosion by the structuring element* B is defined as:

$$\epsilon_B(X) = \{z \mid B_z \subseteq X\} = \bigcap_{b \in B} X_{-b}$$

The erosion of X by B coincides with the *Minkowski substraction* of X by B. Thus, the notation $X \ominus B$ instead of $\epsilon_B(X)$ is usual in the Literature.

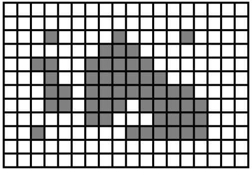
Remark

As in the case of dilations, for each structuring element B, we have a different erosion. So we have one erosion per each image we can define in the domain.

Erosion

An example in $\mathbb{Z} \times \mathbb{Z}$ (I of IV)

Let us compute the erosion of the image



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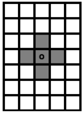
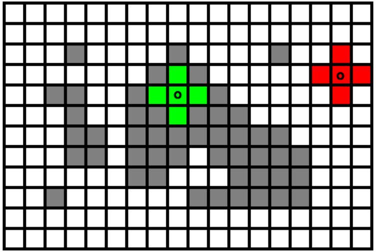
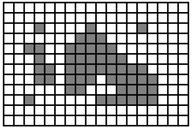
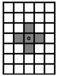


Erosion

An example in $\mathbb{Z} \times \mathbb{Z}$ (II of IV)



$X =$ 

$B =$ 

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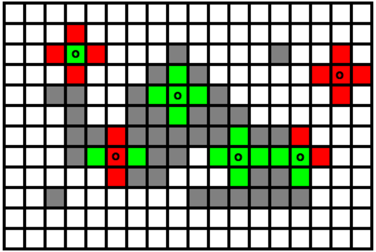
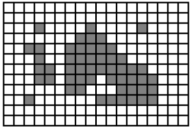
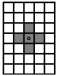


$X =$  $B =$ 

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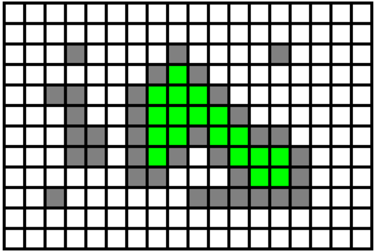
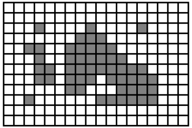
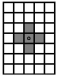


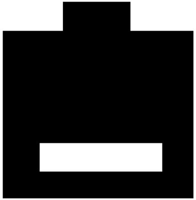
$X =$ 

$B =$ 

Erosion

An example in $\mathbb{R} \times \mathbb{R}$ (I of IV)

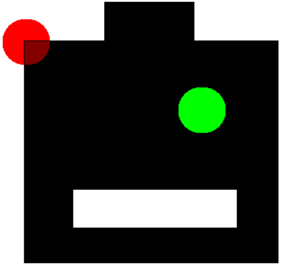
Let us compute the erosion of the image



by the structuring element $B =$ 

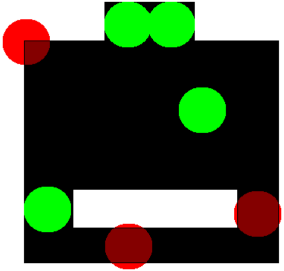
Erosion

An example in $\mathbb{R} \times \mathbb{R}$ (II of IV)



Erosion

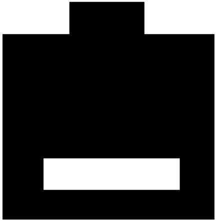
An example in $\mathbb{R} \times \mathbb{R}$ (III of IV)



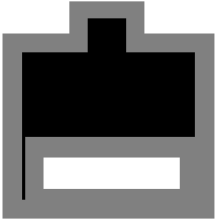
Erosion

An example in $\mathbb{R} \times \mathbb{R}$ (IV of IV)

X



$\epsilon_B(X)$



Erosion

Properties (I of II)

- **Monotonicity with respect to the Image**:

  If $X \subseteq Y$ then $\epsilon_B(X) \subseteq \epsilon_B(Y)$ for all structuring element B.

- **Antitonicity with respect to the Structuring element**:

  If $B_1 \subseteq B_2$ then $\epsilon_{B_1}(X) \supseteq \epsilon_{B_2}(X)$ for all image X.

- **Aniti-extensiveness**: *If the origin belongs to the structuring element B*, then $\epsilon_B(X) \subseteq X$.

- **Translation invariance**: Let τ be a translation. Then

$$\epsilon_B(\tau(X)) = \tau(\epsilon_B(X)) = \epsilon_{-\tau(B)}(X).$$

Erosion

Properties (II of II)

- **Relationships with the union of images**: $\epsilon_B(X \cup Y) \supseteq \epsilon_B(X) \cup \epsilon_B(Y)$.
- **Distributivity with respect to intersection of images**: $\epsilon_B(X \cap Y) = \epsilon_B(X) \cap \epsilon_B(Y)$.
- **On union and intersection of structuring elements**:
  - $\epsilon_{B_1 \cup B_2}(X) = \epsilon_{B_1}(X) \cap \epsilon_{B_2}(X)$
  - $\epsilon_{B_1 \cap B_2}(X) \supseteq \epsilon_{B_1}(X) \cup \epsilon_{B_2}(X)$
- **Serial Composition** (Instead of Associative property):

$$\epsilon_{B_1}(\epsilon_{B_2}(Z)) = \epsilon_{\delta_{B_1}(B_2)}(X) = \epsilon_{B_1 \oplus B_2}(X)$$

Note that **erosions are not commutative**. So for erosions is important to distinguish between image and structuring element.

Relationship between Erosion and Dilation

Duality

Theorem (Duality)

Let X be an image and let B be a structuring element. Then:

$$(\delta_B(X))^c = \epsilon_{-B}(X^c)$$

where $-B = \{-b \mid b \in B\}$ is the reflection of B with respect to the origin.

Relationship between Erosion and Dilation

Adjoint pairs

Theorem (Adjointness)

Let X and Y be two images and let B be a structuring element. Then:

$$\delta_B(X) \subseteq Y \iff X \subseteq \epsilon_B(Y)$$

Remark

The above result is equivalent to state that (δ_B, ϵ_B) forms an adjoint pair for all structuring element B.

Fuzzy approach

Fuzzy image

Definition

A fuzzy image F is a fuzzy set on the Euclidean plane $\mathbb{R}^2$; i.e. is a mapping $F\colon \mathbb{R}^2 \to [0,1]$.

Structuring elements are defined as usual:

Definition

Fuzzy structuring elements are fuzzy images with one origin in $\mathbb{R}^2$

Dilation

Let us recall the classical definition of dilation. Given an image X (i.e. a subset of $\mathbb{R}^2$), the *dilation of X by the structuring element B* is defined as:

$$\delta_B^*(X) = \bigcup_{x \in X} B_x$$

Definition

Let $\otimes$ be a conjunction. Given one fuzzy image F and one structuring element B, the *fuzzy dilation of F by B* is the fuzzy image defined by:

$$\delta_B^{\otimes}(F)(x) = \bigvee_{y \in \mathbb{R}^2} B(x-y) \otimes F(y)$$

Erosion

Let us recall the classical definition of erosion. Given an image X, the *erosion of* X *by the structuring element* B is defined by:

$$\epsilon_B^*(X) = \{z \in \mathbb{R}^2 \mid B_z \subseteq X\}$$

Definition

Let $\rightarrow$ be a implication. Given one fuzzy image F and one structuring element B, the *fuzzy erosion of* F *by* B is defined as:

$$\epsilon_B^{\rightarrow}(F)(x) = \bigwedge_{y \in \mathbb{R}^2} B(y - x) \rightarrow F(y)$$

Dilation and Erosion

Properties

- Dilations and Erosion are translation invariant, that is, for all translation τ

$$\tau(\delta_B^{\otimes}(F)) = \delta_B^{\otimes}(\tau(F)) \qquad \text{and} \qquad \tau(\epsilon_B^{\rightarrow}(F)) = \epsilon_B^{\rightarrow}(\tau(F))$$

- Dilation commutes with supremum: $\delta_B^{\otimes}(\bigvee_{i\in\mathbb{I}} F_i) = \bigvee_{i\in\mathbb{I}} \delta_B^{\otimes}(F_i)$
- Erosion commutes with infimum: $\epsilon_B^{\rightarrow}(\bigwedge_{i\in\mathbb{I}} F_i) = \bigwedge_{i\in\mathbb{I}} \epsilon_B^{\rightarrow}(F_i)$

Proposition

Let $\rightarrow$ be an implication and $\otimes$ be a conjunction on $[0,1]$. Then:

$$(\otimes, \rightarrow) \text{ is an adjoint pair} \iff (\delta_B^{\otimes}, \epsilon_B^{\rightarrow}) \text{ is an adjoint pair}$$

for all structuring element B.

Fuzzy property-oriented concept lattices

Recalling the basics

Definition

Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ be mappings, then $(\&, \swarrow, \nwarrow)$ is an *adjoint triple* with respect to P_1, P_2, P_3 if:

1. $\&$ is order-preserving in both arguments.
2. $\swarrow$ and $\nwarrow$ are order-preserving on the first argument and order-reversing on the second.
3. $x \leq_1 z \swarrow y$ iff $x \,\&\, y \leq_3 z$ iff $y \leq_2 z \nwarrow x$, where $x \in P_1$, $y \in P_2$ and $z \in P_3$.

Gödel, product and Łukasiewicz t-norms, together with their residuated implications, can be seen as examples of adjoint triples.

Fuzzy property-oriented concept lattices

Frames and contexts

Definition

Given two complete lattices $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$, a poset $(P, \leq)$ and one adjoint triple with respect to P, L_2, L_1, $(\&, \nwarrow)$, a *fuzzy property-oriented frame* is the tuple

$$(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&, \nwarrow)$$

Definition

Let $(L_1, L_2, P, \&, \nwarrow)$ be a fuzzy property-oriented frame. A *context* is a tuple (A, B, R) such that A and B are non-empty sets (usually interpreted as attributes and objects, respectively), R is a P-fuzzy relation $R\colon A \times B \to P$.

From now on, we will consider a fixed fuzzy property-oriented frame $(L_1, L_2, P, \&, \nwarrow)$ and a fixed context (A, B, R).

Fuzzy property-oriented concept lattices

Necessity and possibility operators

Definition

The mappings ${}^{\uparrow_\Pi}\colon L_2^B \to L_1^A$ and ${}^{\downarrow^N}\colon L_1^A \to L_2^B$ are defined as

$$
\begin{aligned}
g^{\uparrow_\Pi}(a) &= \bigvee\{R(a,b) \mathbin{\&} g(b) \mid b \in B\} \\
f^{\downarrow^N}(b) &= \bigwedge\{f(a) \nwarrow R(a,b) \mid a \in A\}
\end{aligned}
$$

These definitions generalize the classical possibility and necessity operators. Moreover, $({}^{\uparrow_\Pi}, {}^{\downarrow^N})$ is an isotone Galois connection (also known as adjunction) and, therefore, ${}^{\uparrow_\Pi\downarrow^N}\colon L_2^B \to L_2^B$ is a closure operator and ${}^{\downarrow^N\uparrow_\Pi}\colon L_1^A \to L_1^A$ is an interior operator.

Fuzzy property-oriented concept lattices

The definition

A *concept*, in this environment, is a pair of mappings $\langle g, f\rangle$, with $g \in L^B, f \in L^A$, such that $g^{\uparrow_\Pi} = f$ and $f^{\downarrow^N} = g$, which will be called *fuzzy property-oriented concept*. In that case, g is called the *extent* and f, the *intent* of the concept.

Definition

The associated *fuzzy property-oriented concept lattice* to the fixed frame and context (or, the *concept lattice of* (A, B, R) *based on rough set theory*) is defined as the set

$$\mathcal{F}_{\Pi N} = \{\langle g, f\rangle \in L_2^B \times L_1^A \mid g^{\uparrow_\Pi} = f \text{ and } f^{\downarrow^N} = g\}$$

in which the ordering is defined by $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ iff $g_1 \preceq_2 g_2$ (or equivalently $f_1 \preceq_1 f_2$).

An abstract approach to mathematical morphology

Algebraic formulations

Definition

Let $(L_1, \leq_1)$ and $(L_2, \leq)$ be two complete lattices:

- A mapping $\varepsilon\colon L_1 \to L_2$ is called an *erosion* if for all $X \subseteq L_1$ we have: $\varepsilon(\bigwedge X) = \bigwedge_{x \in X} \varepsilon(x)$
- A mapping $\delta\colon L_2 \to L_1$ is called a *dilation* if for all $Y \subseteq L_2$ we have: $\delta(\bigvee Y) = \bigvee_{y \in Y} \delta(y)$

Theorem

If (ε, δ) is an adjoint pair, then ε is an erosion and δ is a dilation.

On the other hand, the converse can be written in the following sense:

Adjoints, dilations and erosions

From an algebraic standpoint

Theorem

Let $\varepsilon\colon L_1 \to L_2$ *be an erosion. Then there exists exactly one dilation* $\delta_\varepsilon\colon L_2 \to L_1$ *such that* $(\varepsilon, \delta_\varepsilon)$ *forms an adjoint pair. Specifically, such a dilation can be determined by the expression for every* $Y \in L_2$*:*

$$\delta_\varepsilon(Y) = \bigwedge\{Z \in L_2 \mid Y \leq \varepsilon(Z)\}$$

Similarly, for every dilation $\delta\colon L_2 \to L_1$ *there exists exactly one erosion* $\varepsilon_\delta\colon L_1 \to L_2$ *such that* $(\varepsilon_\delta, \delta)$ *forms an adjoint pair. Moreover, such an erosion is determined by the expression for every* $X \in L_1$*:*

$$\varepsilon_\delta(X) = \bigvee\{Z \in L_1 \mid \delta(Z) \leq X\}$$

The first explicit link

In some sense, a first link between both theories can be straightforwardly obtained from the previous Theorem, using the fact that the necessity and possibility operators form an adjunction (or isotone Galois connection). Therefore, we have:

Proposition

Every necessity (respectively, possibility) operator of a property-oriented concept lattice is an erosion (resp., dilation).

The converse is also true in some sense. That is, any erosion operator ε can be identified with the necessity operator associated to some fuzzy property-oriented concept lattice.

Deeper results

Building derivation operators associated to erosions/dilations

Theorem

Consider two complete lattices, $\overline{L}_1$ *and* $\overline{L}_2$*, and an erosion operator* $\varepsilon\colon \overline{L}_1 \to \overline{L}_2$*, then there exists a frame* $(L_1, L_2, P, \&, \nwarrow)$*, a context* (A, B, R) *and two isomorphisms* $\phi_1\colon \overline{L}_1 \to L_1^A$*,* $\phi_2\colon \overline{L}_2 \to L_2^B$*, such that* $\varepsilon = \phi_2^{-1} \circ {\downarrow^N} \circ \phi_1$*, where* ${\downarrow^N}$ *is the necessity operator associated with the frame and context.*

Similarly, we have

Theorem

Given two complete lattices, $\overline{L}_1$ *and* $\overline{L}_2$*, and a dilation operator* $\varepsilon\colon \overline{L}_2 \to \overline{L}_2$*, there exists a frame* $(L_1, L_2, P, \&, \nwarrow)$*, a context* (A, B, R) *and two isomorphisms* $\phi_1\colon \overline{L}_1 \to L_1^A$*,* $\phi_2\colon \overline{L}_2 \to L_2^B$*, such that* $\delta = \phi_1^{-1} \circ {\uparrow^\Pi} \circ \phi_2$*, where* ${\uparrow^\Pi}$ *is the necessity operator associated with the frame and context.*

Conclusions

- The existence of these first links allows for foreseeing future developments in which both frameworks can be merged, so that algorithms given for fuzzy concept lattices could be applied to mathematical morphology and vice versa.
- As future work, we will further extend the link by considering extended frameworks in both topics.
  - On the one hand, using the new theoretical results recently obtained on the constructions of Galois connections within multi-adjoint concept lattices.
  - On the other hand, considering recent advances in fuzzy mathematical morphology, for instance, *lattice morphological image processing* which uses L-fuzzy sets as images and structuring elements.