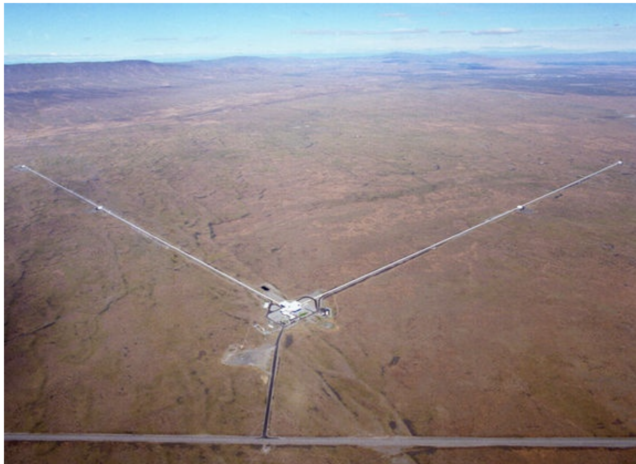
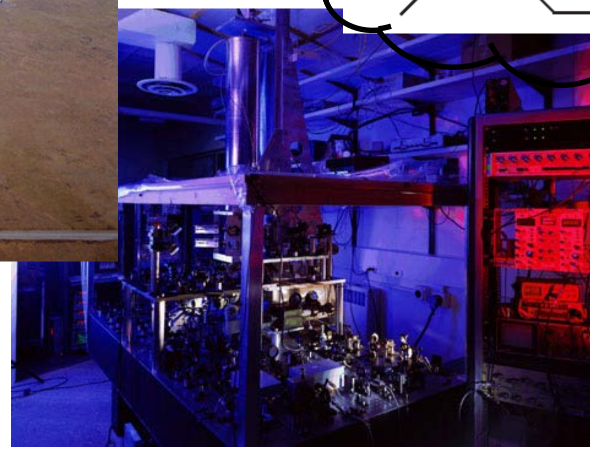
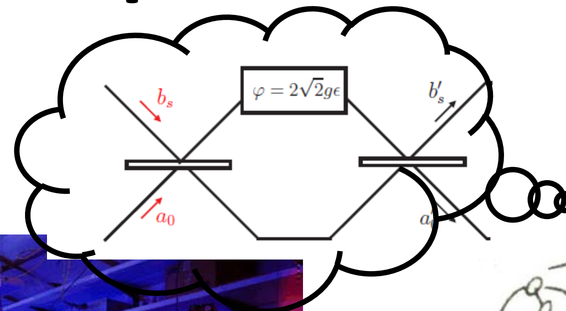
Gravitational wave detectors, atomic clocks – a naive theorist point of view







R. Demkowicz-Dobrzański[1], K. Banaszek[1], J. Kołodyński[1], M. Jarzyna[1], M. Guta[2], K. Macieszczak[1,2], R. Schnabel[3], M. Fraas[4]

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[2] School of Mathematical Sciences, University of Nottingham, United Kingdom

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[4] Theoretische Physik, ETH Zurich, 8093 Zurich, Switzerland

INNOVATIVE ECONOMY
NATIONAL COHESION STRATEGY

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SEVENTH FRAMEWORK PROGRAMME

UNIVERSITAS VARSOVIENSIS

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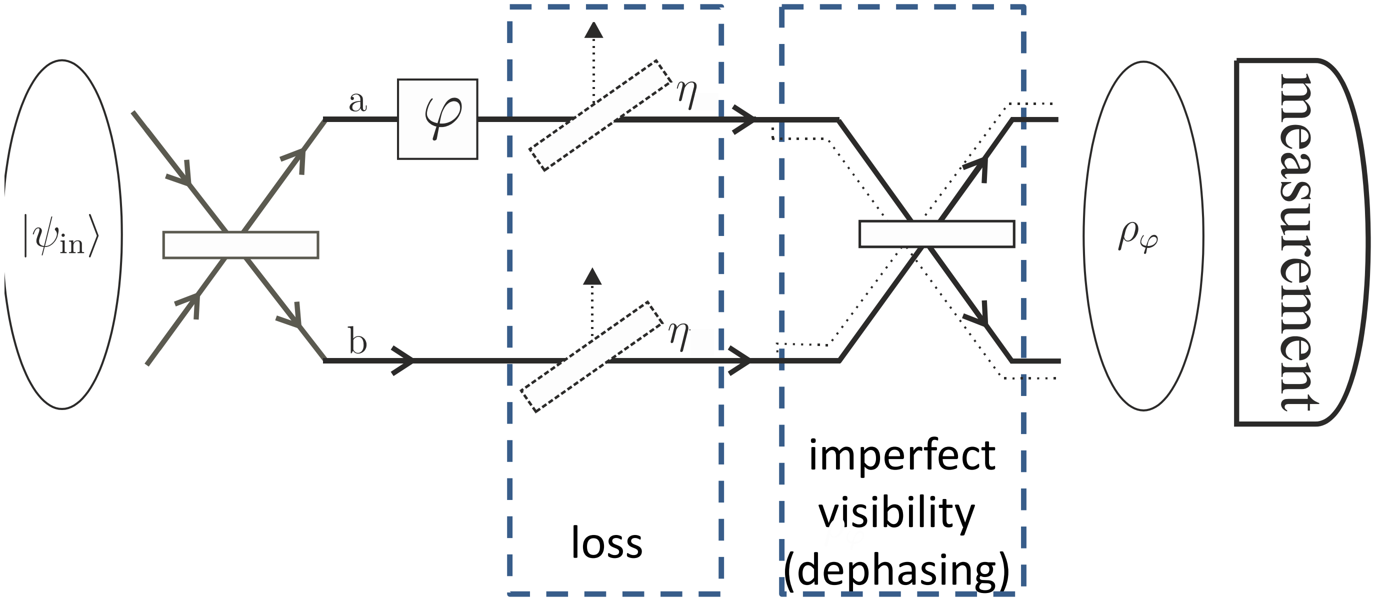
EUROPEAN REGIONAL DEVELOPMENT FUND

Quantum enhncement in an imperfect Mach-Zehnder interferometer



for classical light: $|\psi_{\rm in}\rangle = |\alpha\rangle_a |0\rangle_b$ $\quad \Delta\varphi = \frac{1}{\sqrt{\eta|\alpha|^2}} = \frac{1}{\sqrt{\eta\langle n\rangle}}$ shot noise

What is the maximal quantum enhanced precision we can get using non-classical states of light with fixed total energy at the input?

$$\Delta\varphi \geq \frac{1}{\sqrt{F_Q(\rho_\varphi)}}$$

Quantum Cramer-Rao bound

$$F_Q(\rho_\varphi) = \mathrm{Tr}(\rho_\varphi L_\varphi^2)$$

Quantum Fisher Information

$$\frac{\mathrm{d}\rho_\varphi}{\mathrm{d}\rho_\varphi} = \frac{1}{2}(\rho_\varphi L_\varphi + L_\varphi \rho_\varphi)$$

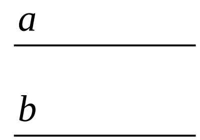
Symmetric logarithmic derrivative

Maximize F_Q over input states

Mode vs particle description of light

A general *N* photon two mode state:

$$|\psi_N\rangle = \sum_{n=0}^{N} \alpha_n |n\rangle_a |N-n\rangle_b$$



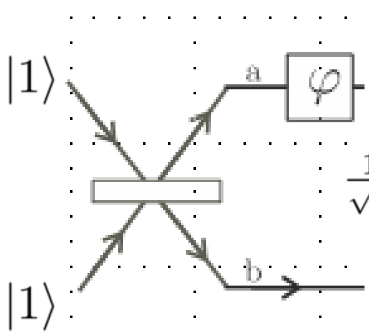
Written in the language of *N* formally distinguishable particles:

$$|n\rangle_a |N-n\rangle_b = \mathcal{S}(\underbrace{|a\rangle \otimes \cdots \otimes |a\rangle}_{n} \otimes \underbrace{|b\rangle \otimes \cdots \otimes |b\rangle}_{N-n})$$

↑ symetrization

Mode vs particle entanglement

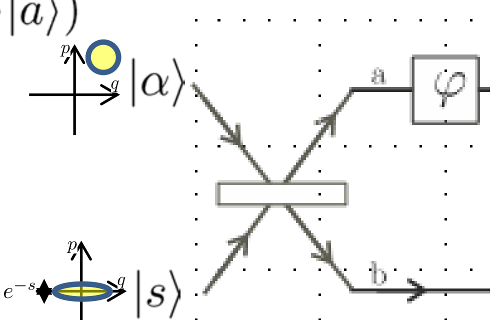
$$|\psi\rangle = |1\rangle_a |1\rangle_b = \frac{1}{\sqrt{2}}(|a\rangle|b\rangle + |b\rangle|a\rangle)$$



$$\frac{1}{\sqrt{2}}(e^{2i\varphi}|2\rangle_a|0\rangle_b + |0\rangle_a|2\rangle_b)$$

enhanced sensitivity

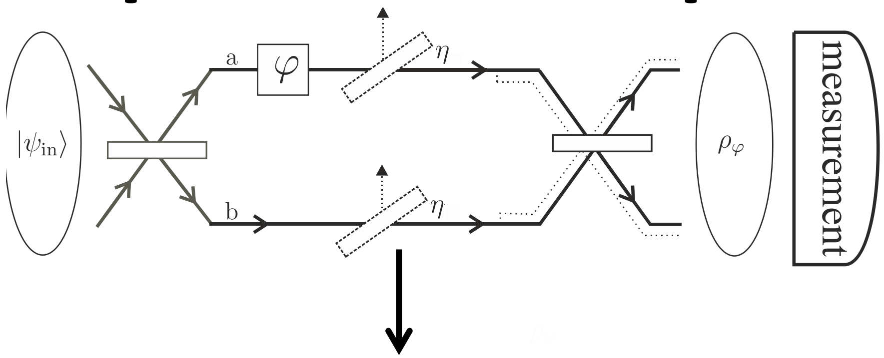
Hong-Ou-Mandel interference

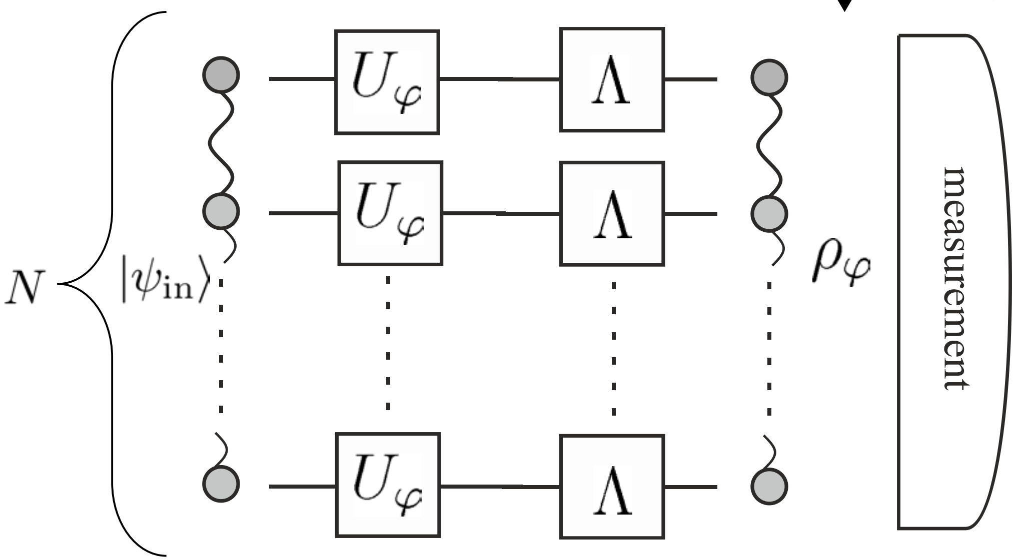


when projected on a fixed photon number sector yields a particle entangled states

It is the particle entanglement that is the fundamental source for quantum precision enhancement

Quantum enhanced interferometry using the particle description





uncorrelated noise models
commute with the phase encoding

phase encoding $U_\varphi = e^{i\sigma_z \varphi/2}$

decoherence $\Lambda(\cdot) = \sum_i K_i \cdot K_i^\dagger$

imperfect viisbility – loss of coherence between the modes (local qubit dephasing)

$$K_0 = \sqrt{\tfrac{1+\eta}{2}}\mathbb{1} \qquad K_1 = \sqrt{\tfrac{1-\eta}{2}}\sigma_z$$

loss – we use three dimensional output space

$$\{|a\rangle, |b\rangle, |\mathrm{vac}\rangle\}$$

$$K_0 = \begin{pmatrix} \sqrt{\eta} & 0 \\ 0 & \sqrt{\eta} \\ 0 & 0 \end{pmatrix} \quad K_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \sqrt{1-\eta} & 0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \sqrt{1-\eta} \end{pmatrix}$$

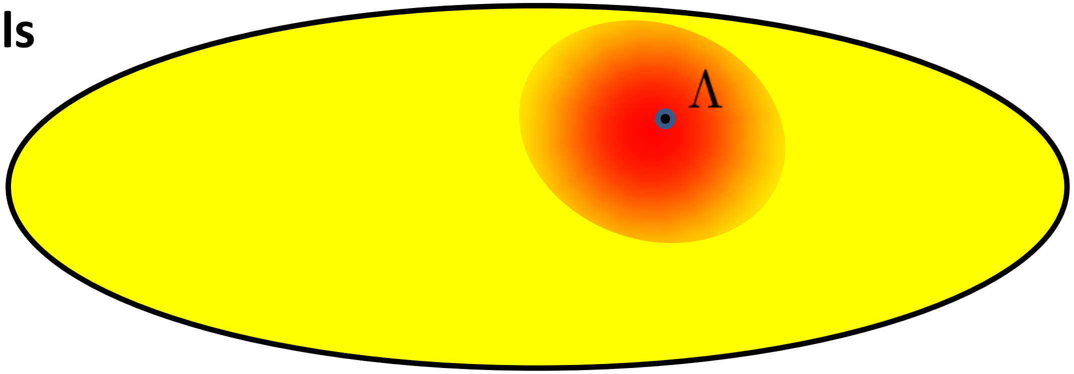
photon survives   lost in mode a   lost in mode b

Find the bounds on the quantum Fisher information as a function of *N*

Classical simulation of a quantum channel

Convex set of quantum channels

$$\Lambda = \int \mathrm{d}X \, p(X) \Lambda_X$$



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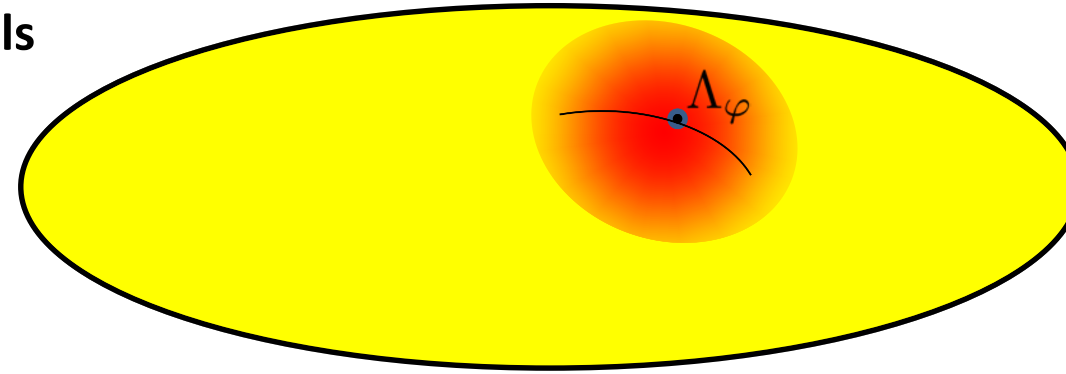


Parameter dependence moved to mixing probabilities

Before:

$$\varphi \to \Lambda_\varphi \left[\rho\right] \to \tilde{\varphi}$$

After:

$$\varphi \to p_\varphi \to X \to \Lambda_X \left[\rho\right] \to \tilde{\varphi}$$

By Markov property….

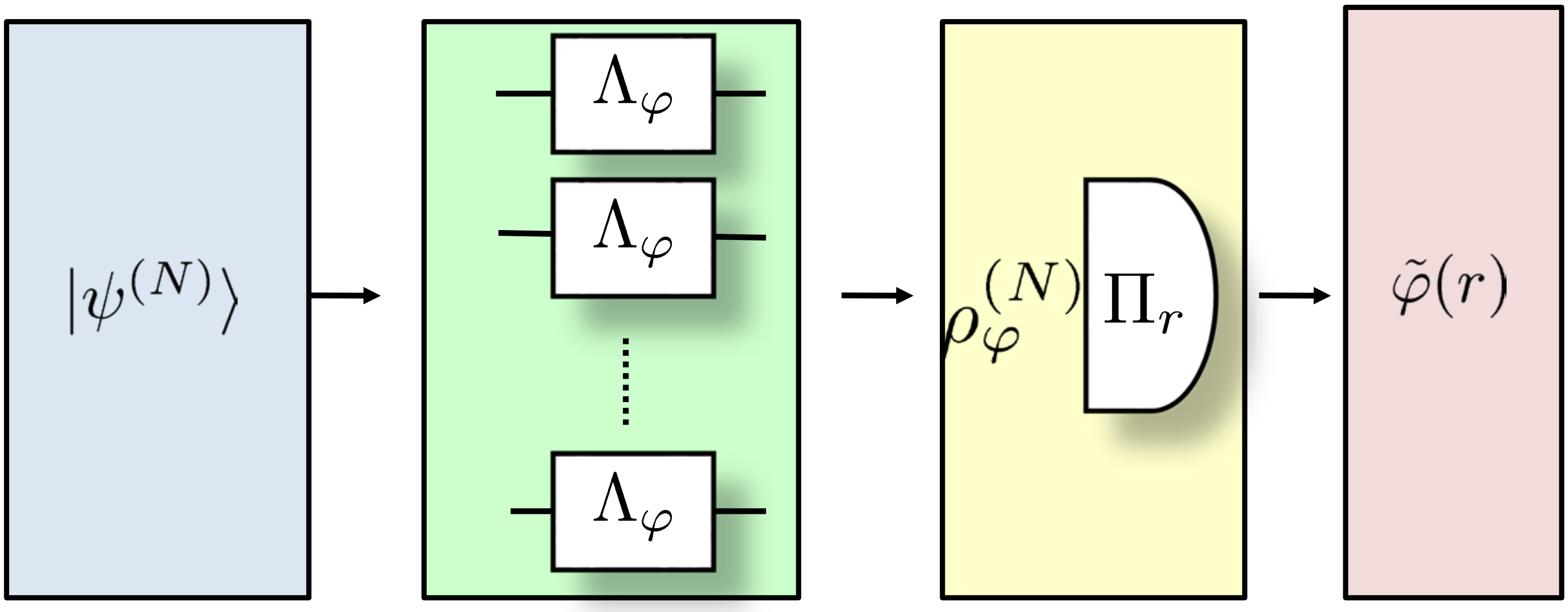
Estimating φ directly from X is no worse than from measurement on $\Lambda_\varphi[\rho]$

$$F_Q\left[\Lambda_\varphi(\rho)\right] \leq F_{\mathrm{cl}}\left[p_\varphi(X)\right]$$

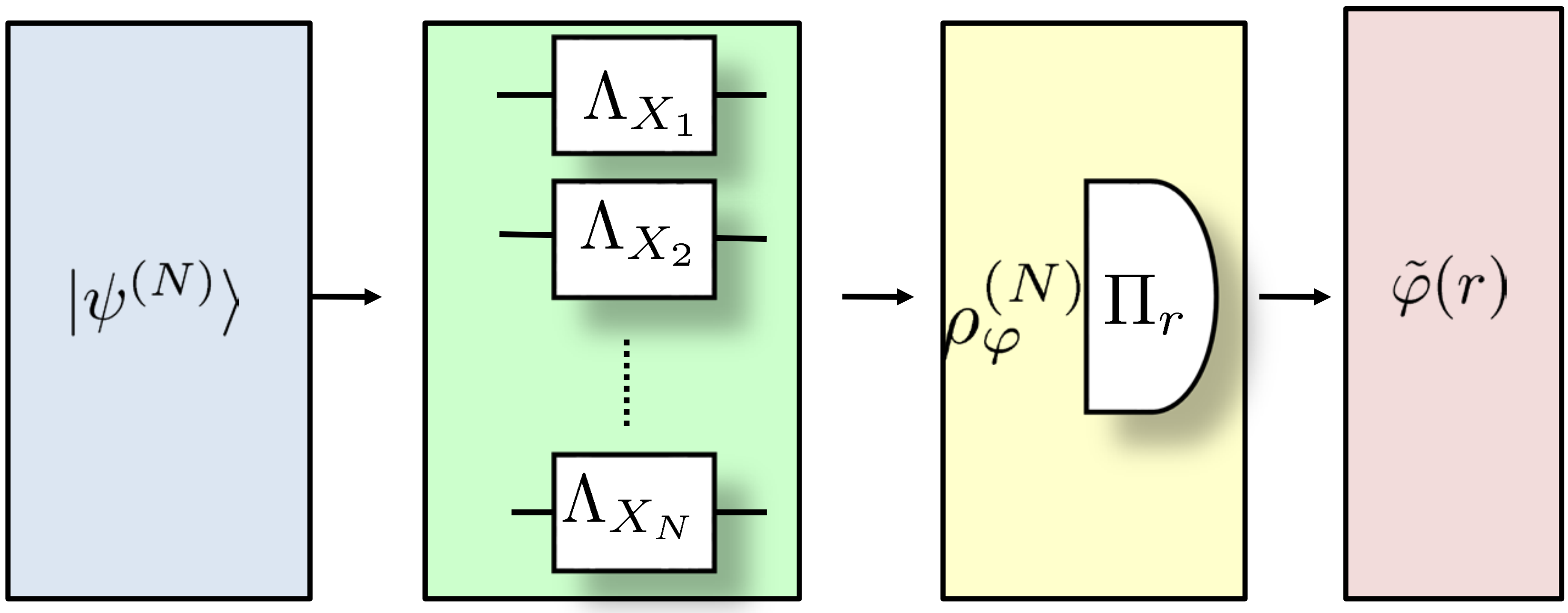
$$F_{\mathrm{cl}}[p_\varphi(X)] = \int dX \, \frac{[\partial_\varphi p_\varphi(X)]^2}{p_\varphi(X)}$$

K. Matsumoto, arXiv:1006.0300 (2010)

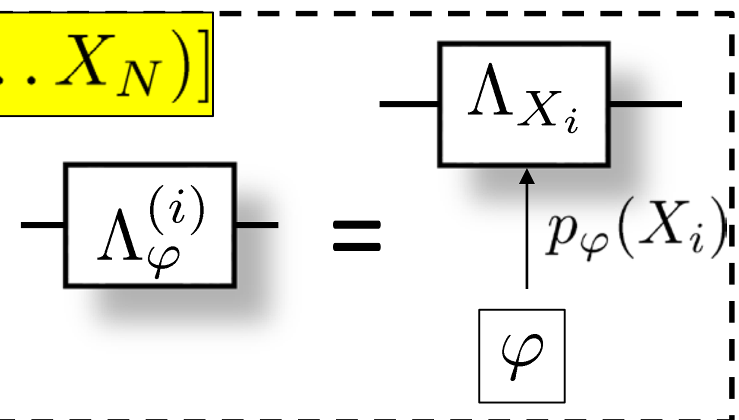
Classical simulation of *N* channels used in parallel



Classical simulation of *N* channels used in parallel

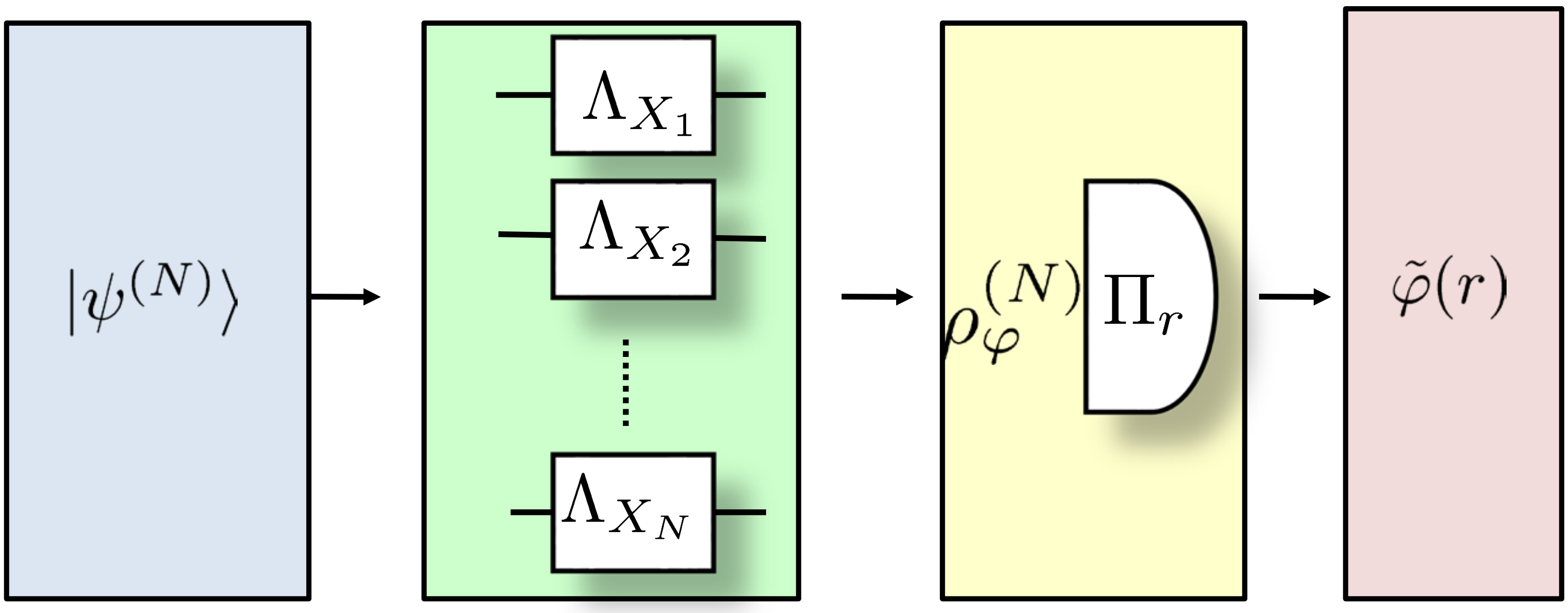


$$F_Q\left[\Lambda_\varphi^{\otimes N}(|\psi^{(N)}\rangle)\right] \leq F_{\mathrm{cl}}\left[p_\varphi(X_1, \dots X_N)\right]$$



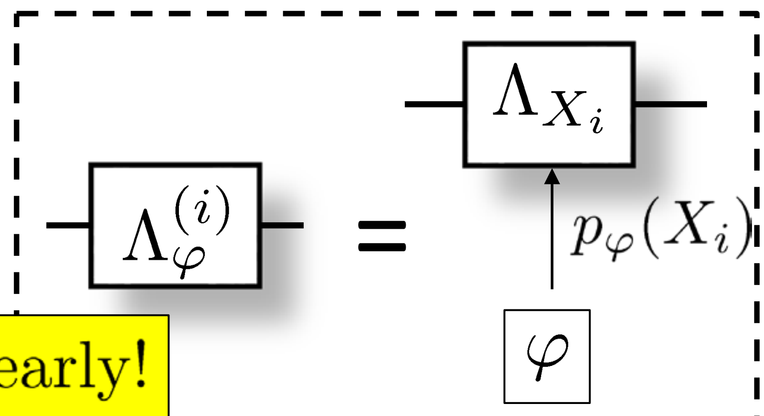
X_i - are independent variables!

Classical simulation of *N* channels used in parallel



$$F_Q\left[\Lambda_\varphi^{\otimes N}(\rho^N)\right] \leq N F_{\mathrm{cl}}\left[p_\varphi(X)\right]$$

X_i - are independent variables!

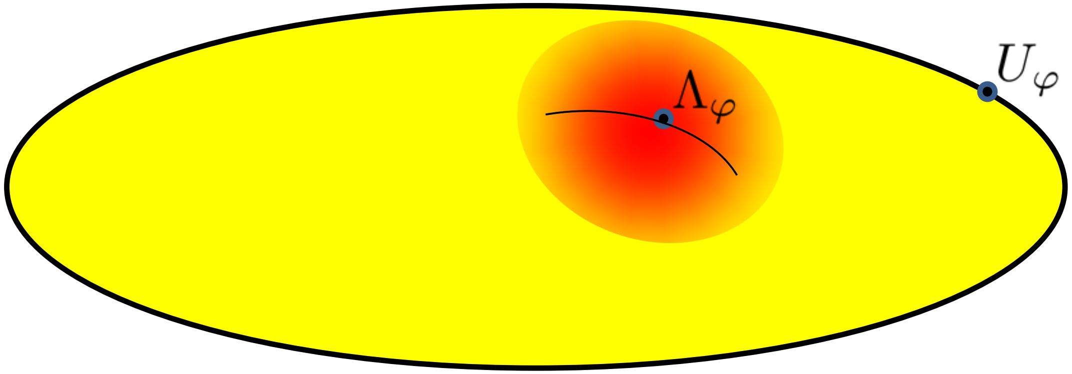


If F_{cl} is finite F_Q scales at most linearly!

Precision bounds thanks to classical simulation

$$\Lambda_\varphi = \int \mathrm{d}X \, p_\varphi(X) \Lambda_X$$

$$\Delta\varphi \geq \frac{1}{\sqrt{F_{\mathrm{cl}}(p_\varphi) N}}$$



- For unitary channels $F_{\mathrm{cl}} = \infty$ Heisenberg scaling possible
- Generic decoherence model will manifest shot noise scaling
- To get the tighest bound we need to find the classical simulation with lowest F_{cl}

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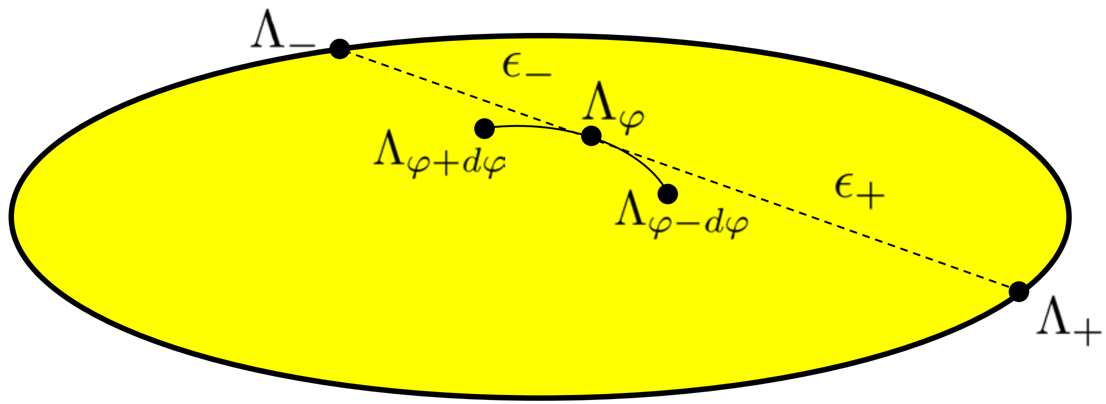
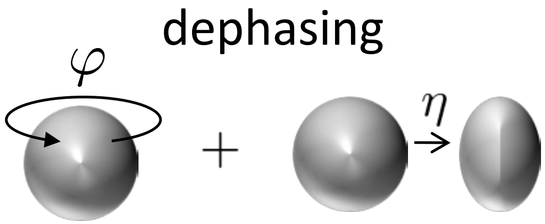


- For unitary channels $F_{\mathrm{cl}} = \infty$ Heisenberg scaling possible
- Generic decoherence model will manifest shot noise scaling
- To get the tighest bound we need to find the classical simulation with lowest F_{cl}

$$\Delta\varphi \geq \sqrt{\frac{\epsilon_+ \epsilon_-}{N}} \qquad \Lambda_\pm = \Lambda_\varphi \pm \frac{d\Lambda_\varphi}{d\varphi}\epsilon_\pm$$

RDD, J. Kolodynski, M. Guta, Nature Communications **3**, 1063 (2012)

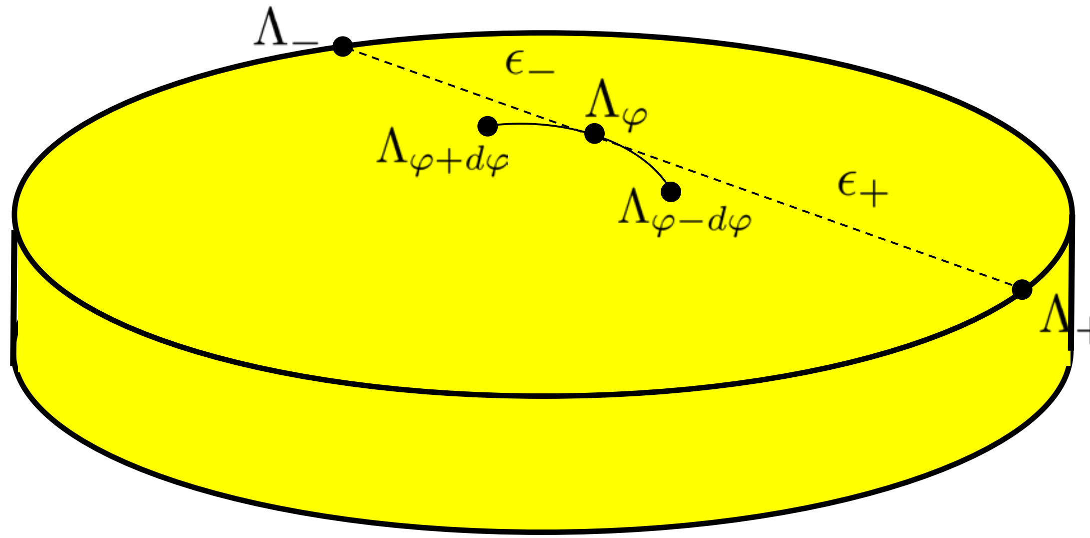
Example: dephasing



$$\Lambda_\varphi(\rho) = U_\varphi \left(\sum_i K_i \rho K_i^\dagger \right) U_\varphi^\dagger$$

$$K_1 = \sqrt{\frac{1+\eta}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$K_2 = \sqrt{\frac{1-\eta}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



$$\Lambda_\pm = \Lambda_\varphi \pm \frac{d\Lambda_\varphi}{d\varphi}\epsilon_\pm$$

$$\varepsilon_\pm = \frac{\sqrt{1-\eta^2}}{\eta}$$

$$\Delta\tilde{\varphi} \geq \sqrt{\frac{\varepsilon_+\varepsilon_-}{N}} = \frac{\sqrt{1-\eta^2}}{\eta}\frac{1}{\sqrt{N}}$$

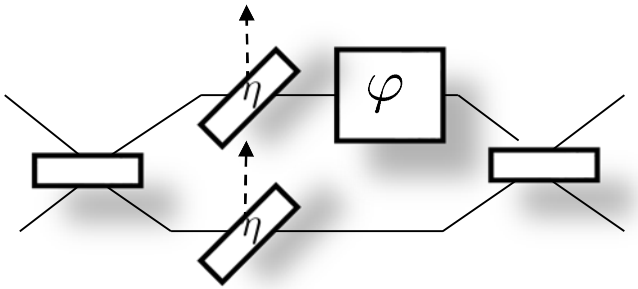
For „classical strategies” $\Delta\tilde{\varphi} = \frac{1}{\eta}\frac{1}{\sqrt{N}}$

Maximal quantum enhancment $= \frac{1}{\sqrt{1-\eta^2}}$

RDD, J. Kolodynski, M. Guta, Nature Communications **3**, 1063 (2012)

Example: loss

Lossy interferometer

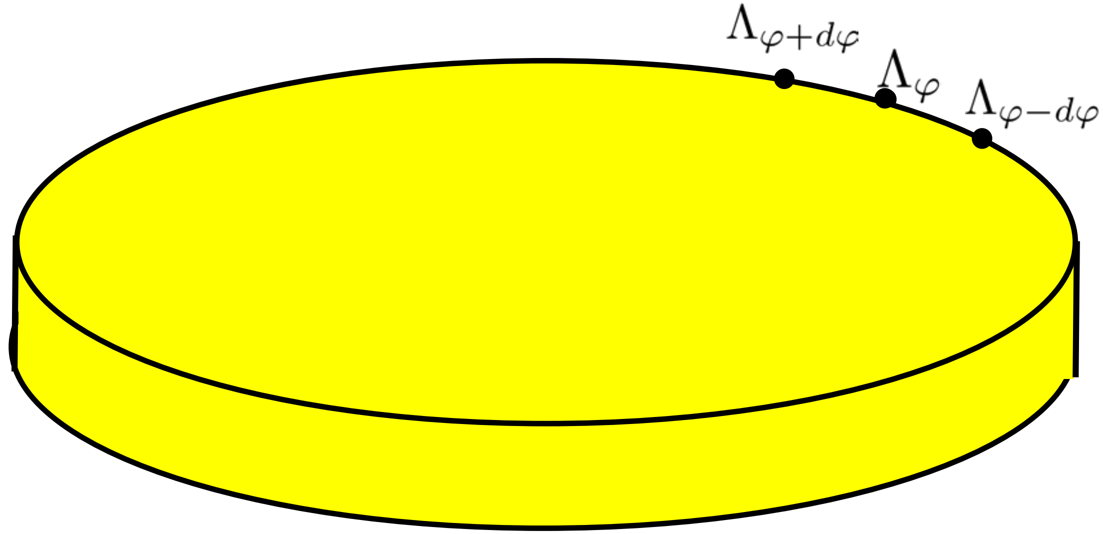


$$\Lambda_\varphi(\rho) = U_\varphi \left(\sum_i K_i \rho K_i^\dagger \right) U_\varphi^\dagger$$

$$K_0 = \begin{pmatrix} \sqrt{\eta} & 0 \\ 0 & \sqrt{\eta} \\ 0 & 0 \end{pmatrix}$$ photon transmitted

$$K_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \sqrt{1-\eta} & 0 \end{pmatrix}$$ photon lost from the upper arm

$$K_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \sqrt{1-\eta} \end{pmatrix}$$ photon lost from the lower arm

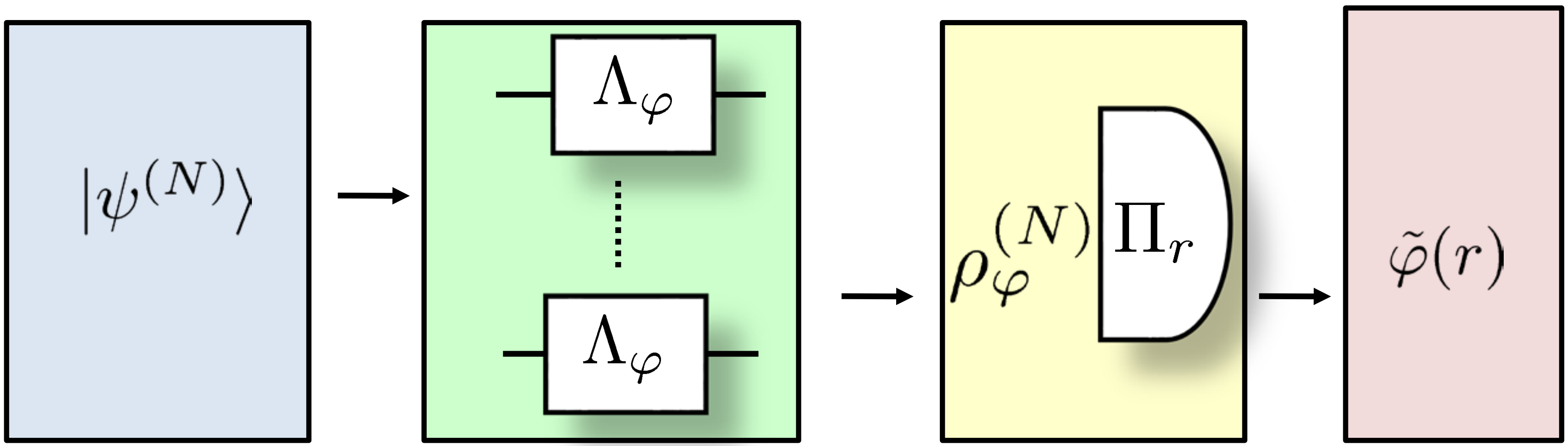


$$\Delta\tilde{\varphi} \geq \sqrt{\frac{\epsilon_+\epsilon_-}{N}} = 0$$

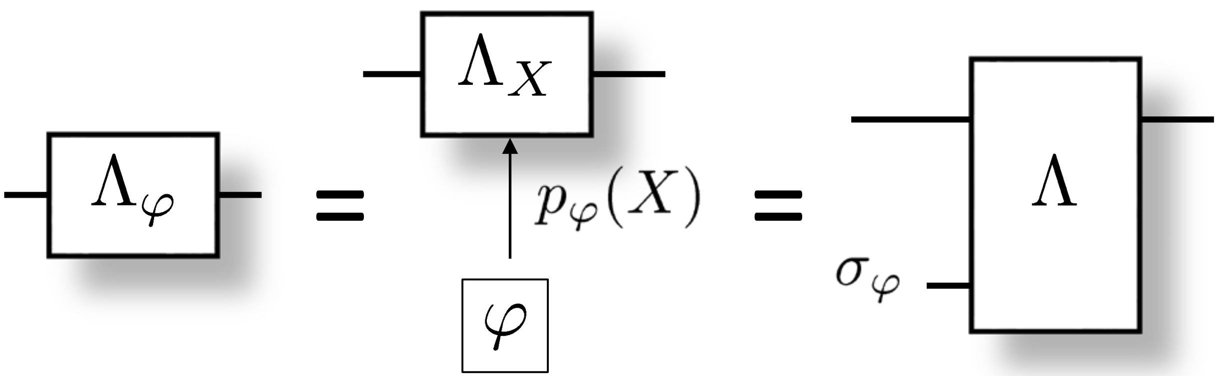
Bound useless

Need to generalize the idea of classical simmulation

Quantum simulation

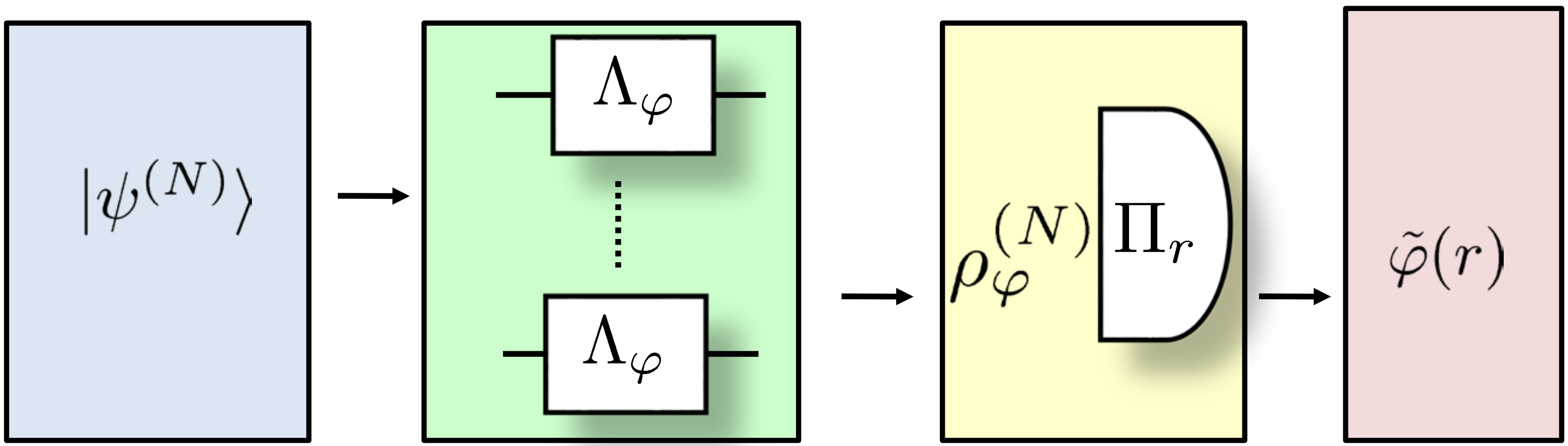


Classical simulation

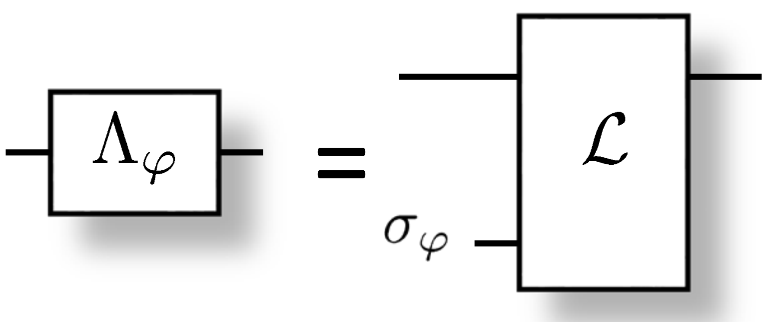


$\sigma_\varphi = \sum_X p_\varphi(X)|X\rangle\langle X|$ $\Lambda(\rho \otimes \sigma_\varphi) = \sum_X p_\varphi(X)\Lambda_X(\rho)$

Quantum simulation

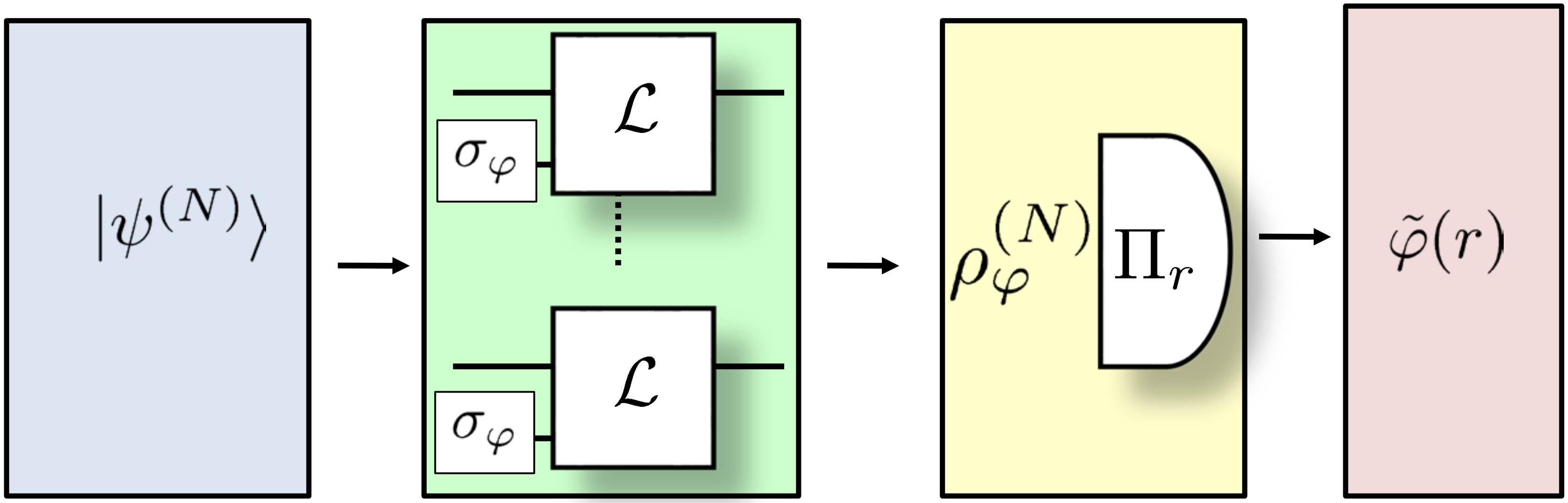


Quantum simulation



σ_φ arbitrary state $\mathcal{L}$ arbitrary map

Quantum simulation



$$\rho_\varphi^{(N)} = \mathcal{L}^{\otimes N}\left(|\psi^{(N)}\rangle\langle\psi^{(N)}| \otimes \sigma_\varphi^{\otimes N}\right)$$

Fisher information cannot increase under parameter independent CP map

$$F_Q\left[\rho_\varphi^{(N)}\right] \leq F_Q\left[\sigma_\varphi^{\otimes N}\right] = N F_Q[\sigma_\varphi]$$

$$\Delta\varphi \geq \frac{1}{\sqrt{N F_Q[\sigma_\varphi]}}$$

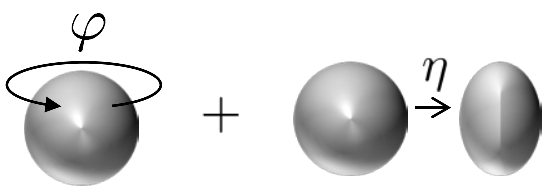
We should look for the ,,worst'' quantum simulation to get the tightest bounds

Search for the „worst" Quantum simulation

A semi-definite programm $F_Q^{\text{worst}}(\sigma_\varphi) = 4 \min_h \lambda$

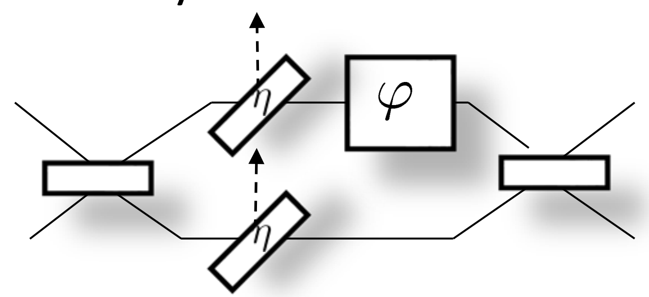
$\sum_i \dot{\tilde{K}}_i^\dagger \dot{\tilde{K}}_i \leq \lambda \mathbb{1}$ $\quad \sum_i \dot{\tilde{K}}_i^\dagger K_i = 0$ $\quad \dot{\tilde{K}}_i = \dot{K}_i + \sum_j \mathrm{i} h_{ij} K_j$

dephasing



$$\Delta\tilde{\varphi} \geq \frac{\sqrt{1-\eta^2}}{\eta} \frac{1}{\sqrt{N}}$$

Lossy interferometer



$$\Delta\tilde{\varphi} \geq \sqrt{\frac{1-\eta}{\eta}} \frac{1}{\sqrt{N}}$$

the same as from classical simulation

lossy interferometer -> dephasing $\eta \to \eta^2$

Heisenberg 1/*N* scaling lost!

RDD, J. Kolodynski, M. Guta, Nature Communications **3**, 1063 (2012)
J. Kolodynski, RDD, New J. Phys. 15, 073043 (2013)

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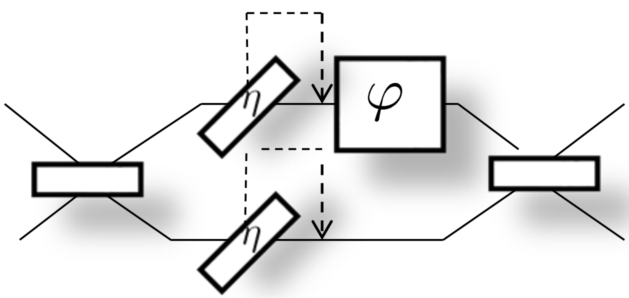
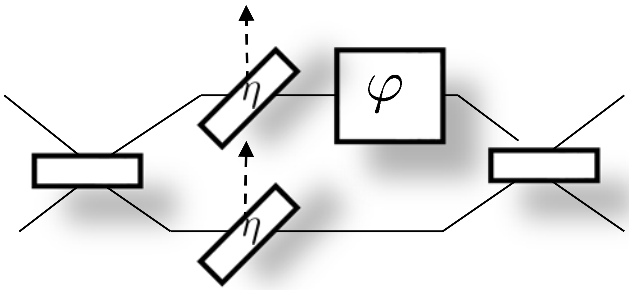
$\sum_i \dot{\tilde{K}}_i^\dagger \dot{\tilde{K}}_i \leq \lambda \mathbb{1}$ $\sum_i \dot{\tilde{K}}_i^\dagger K_i = 0$ $\dot{\tilde{K}}_i = \dot{K}_i + \sum_j \mathrm{i} h_{ij} K_j$

dephasing



$$\Delta\tilde{\varphi} \geq \frac{\sqrt{1-\eta^2}}{\eta}\frac{1}{\sqrt{N}}$$

Lossy interferometer



$$\Delta\tilde{\varphi} \geq \sqrt{\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}}$$

dephasing = losses + sending back decohered photons $\eta \to \eta^2$

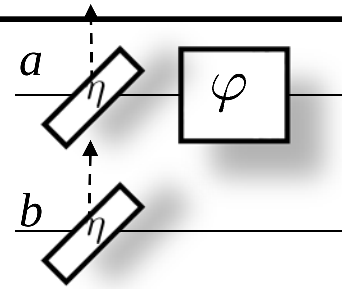
RDD, J. Kolodynski, M. Guta, Nature Communications **3**, 1063 (2012)
J. Kolodynski, RDD, New J. Phys. 15, 073043 (2013)

Explicit example of a quantum simulation

lossy interferometr:

$$\Delta\tilde{\varphi} \geq \sqrt{\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}}$$

$$\Lambda_\varphi =$$



we will prove this bound for

$$\eta = 1/2 \qquad \Delta\varphi \geq \frac{1}{\sqrt{N}}$$

$$|\psi\rangle \to \boxed{\Lambda_\varphi} \to \frac{1}{2}|\psi_\varphi\rangle\langle\psi_\varphi| + \frac{1}{2}|\mathrm{vac}\rangle\langle\mathrm{vac}|$$

photon lost with probability 1/2

$$\eta = 1/2, \quad F_Q[\Lambda_\varphi^{\otimes N}(\rho_N)] \leq F_Q(|\varphi\rangle\langle\varphi|^{\otimes N}) \leq N$$

quantum simulation:

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(e^{i\varphi}|0\rangle + |1\rangle) \to$$

$$|\psi\rangle = \alpha|a\rangle + \beta|b\rangle \to$$

$$\boxed{\mathcal{L}} \to \frac{1}{2}|\psi_\varphi\rangle\langle\psi_\varphi| + \frac{1}{2}|\mathrm{vac}\rangle\langle\mathrm{vac}|$$

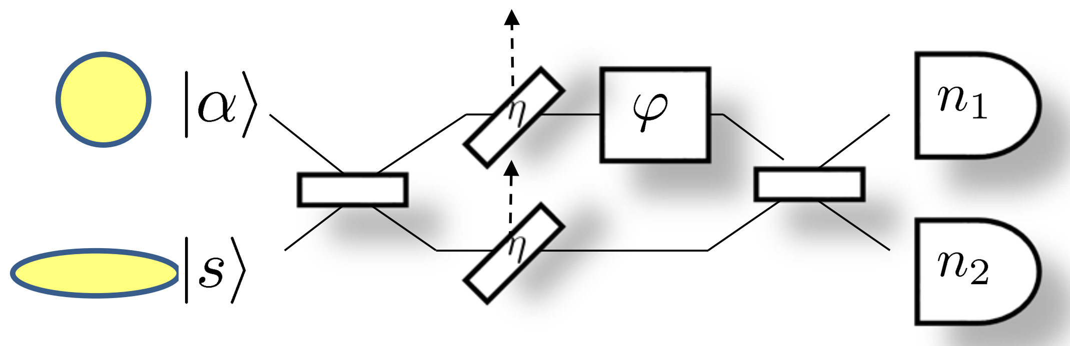
$$|\varphi\rangle|\psi\rangle = \frac{1}{\sqrt{2}}\underbrace{(\alpha e^{i\varphi}|0\rangle|a\rangle + \beta|1\rangle|b\rangle)} + \frac{1}{\sqrt{2}}(\beta e^{i\varphi}|0\rangle|b\rangle + \alpha|1\rangle|a\rangle)$$

$\mathcal{L}$ projects on this subspace (probability 1/2), if unsuccesful returns $|\mathrm{vac}\rangle$

Saturating the fundamental bounds is simple!

$$\Delta\tilde{\varphi} \geq \sqrt{\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}}$$

Fundamental bound



For strong beams:

$$\Delta\tilde{\varphi} \approx \sqrt{\frac{1-\eta+\eta e^{-2s}}{\eta|\alpha|^2}}$$

Simple estimator based on n_1- n_2 measurement

C. Caves, Phys. Rev D **23**, 1693 (1981)

Weak squezing + simple measurement + simple estimator = optimal strategy!

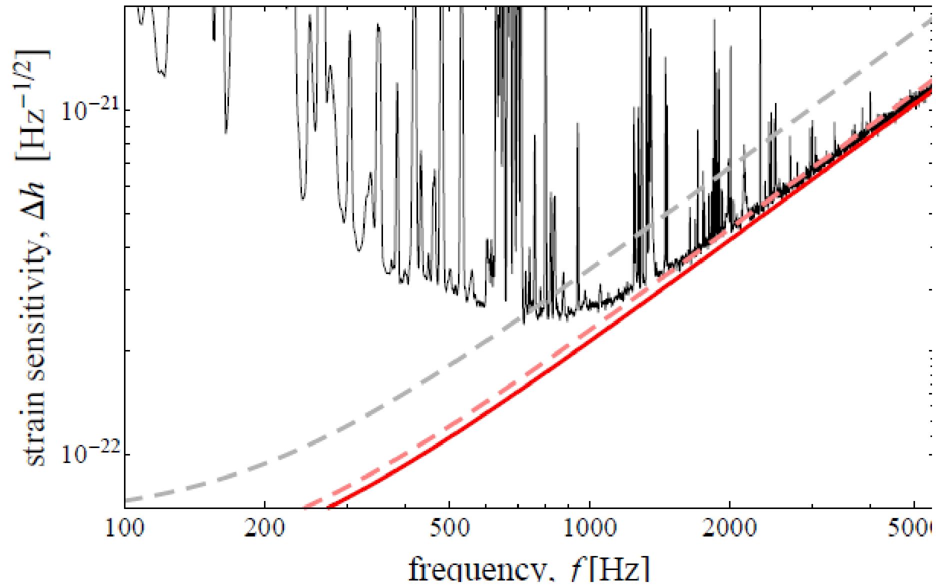
True also for dephasing (also atomic dephasing – spin squeezed states are optimal)

S. Huelga, et al. Phys. Rev. Lett **79,** 3865 (1997), B. M. Escher, R. L. de Matos Filho, L. Davidovich *Nature Phys.* 7, 406–411 (2011), D. Ulam-Orgikh and M. Kitagawa, Phys. Rev. A **64, 052106** (2001).

The bound may also be saturated with low rank matrix product states

M. Jarzyna, RDD, Phys. Rev. Lett. 110, 240405 (2013)

GEO600 interferometer at the fundamental quantum bound



coherent light
+10dB squeezed
fundamental bound

LETTERS
PUBLISHED ONLINE: 11 SEPTEMBER 2011 | DOI: 10.1038/NPHYS2083
nature physics

A gravitational wave observatory operating beyond the quantum shot-noise limit

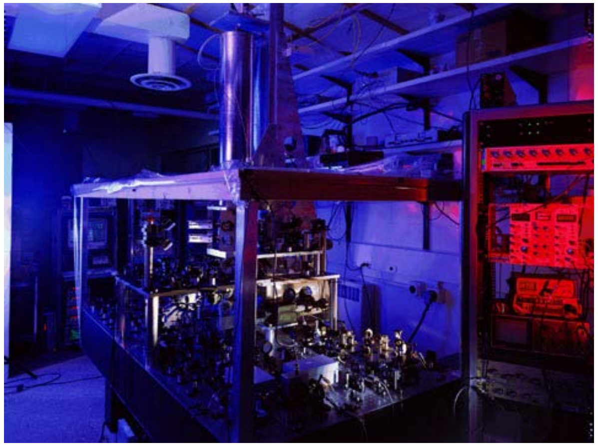
The LIGO Scientific Collaboration †*

$$\frac{\Delta\varphi_{\mathrm{squeezed}}}{\Delta\varphi_{\mathrm{standard}}} \approx 0.66$$

The most general quantum strategies could improve the precision by at most 8%

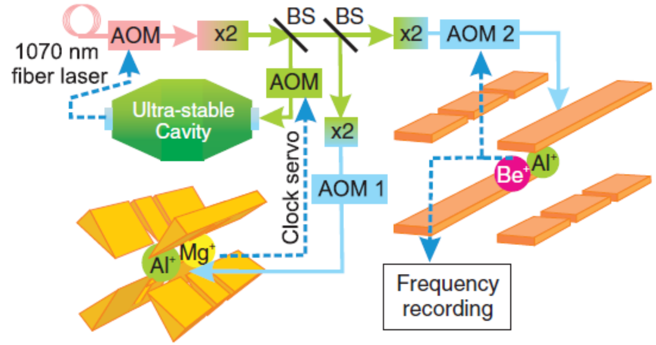
RDD, K. Banaszek, R. Schnabel, Phys. Rev. A, 041802(R) (2013)

Atomic clocks



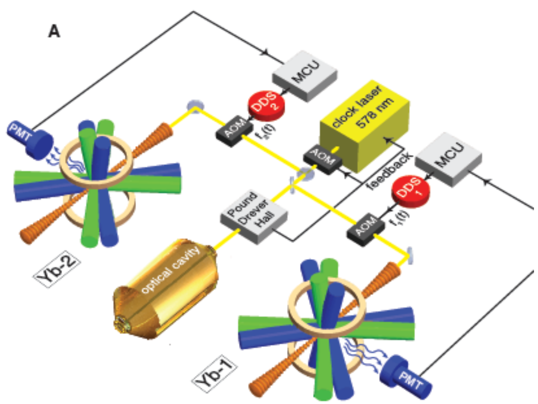
Cs atomic fountain clock (NIST)

$$\Delta t/t \approx 10^{-16}$$



Al^{+} ion atomic clock (2010, NIST)

$$\Delta t/t \approx 10^{-17}$$



Optical lattice clocks
(NIST, SYRTE, Tokyo), approaching:

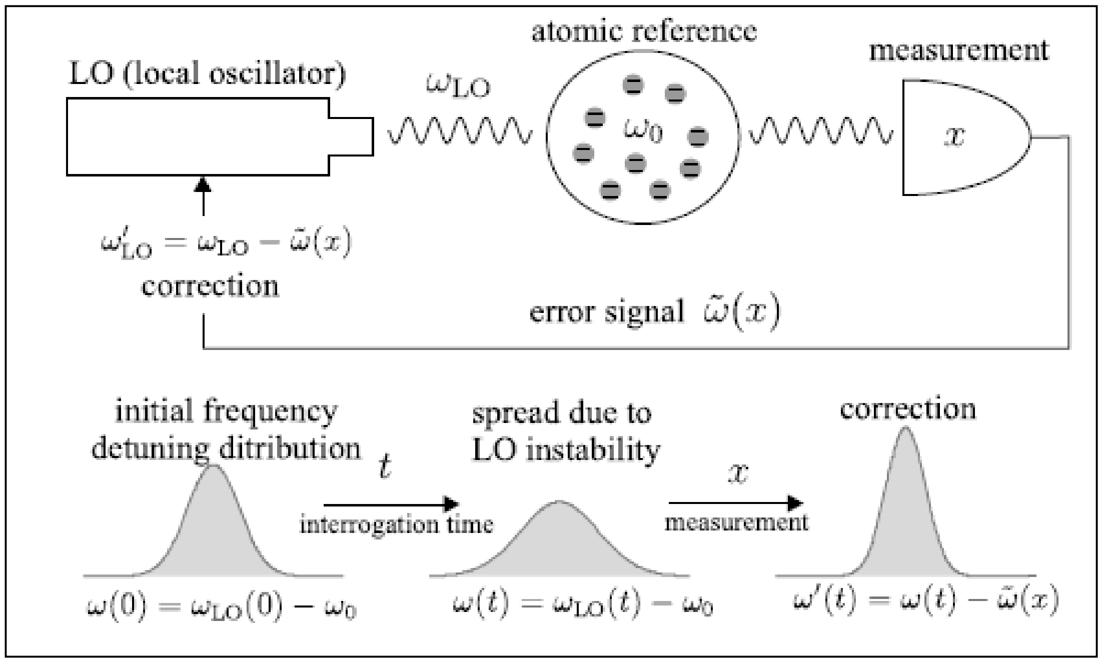
$$\Delta t/t \approx 10^{-18}$$

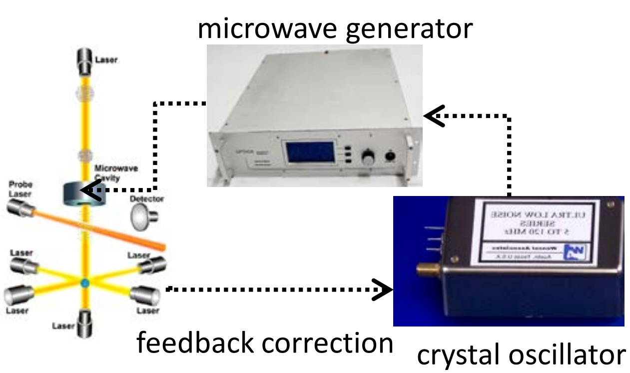
Measuring the age of the universe with 1s precision!

Sensing elevation changes due to gravitational redshift with 1cm precision! (relativistic geodesy)

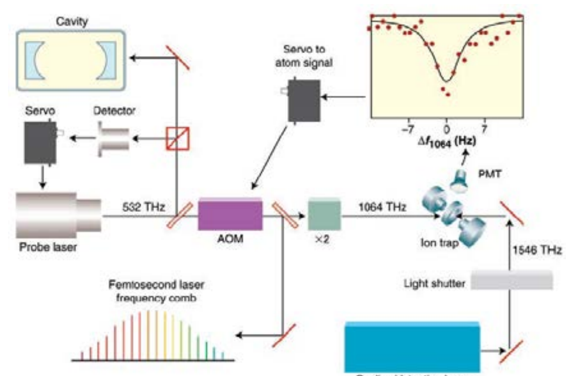
Can quantum correlations (entanglement) improve the precision even further?

Basic scheme of atomic clock operation





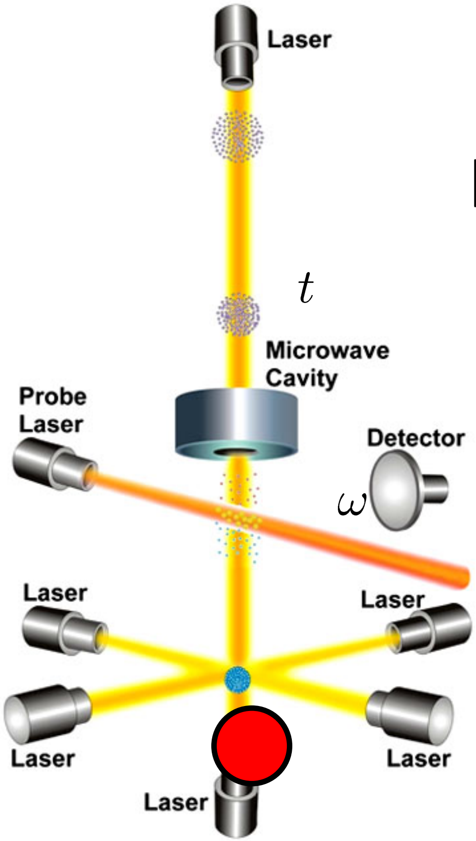
microwave atomic clock



optical atomic clock

Frequency estimation

Ramsey interferometry



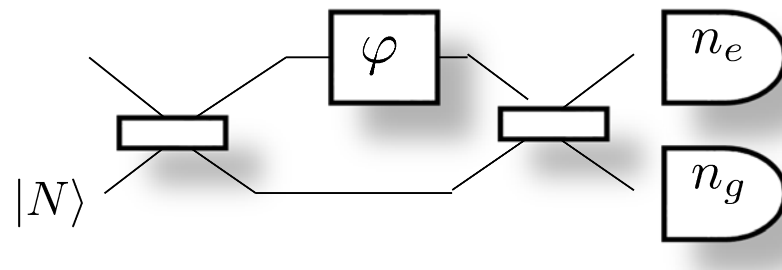
$$|\psi\rangle = |g\rangle^{\otimes N} \longrightarrow |\psi\rangle = \left(\frac{|g\rangle + |e\rangle}{\sqrt{2}}\right)^{\otimes N} \longrightarrow |\psi\rangle = \left(\frac{|g\rangle + e^{-i\omega_0 t}|e\rangle}{\sqrt{2}}\right)^{\otimes N}$$

$$\longrightarrow \quad |\psi\rangle = \left[\sin\left(\frac{(\omega_0-\omega)t}{2}\right)|g\rangle + \cos\left(\frac{(\omega_0-\omega)t}{2}\right)|e\rangle\right]^{\otimes N}$$

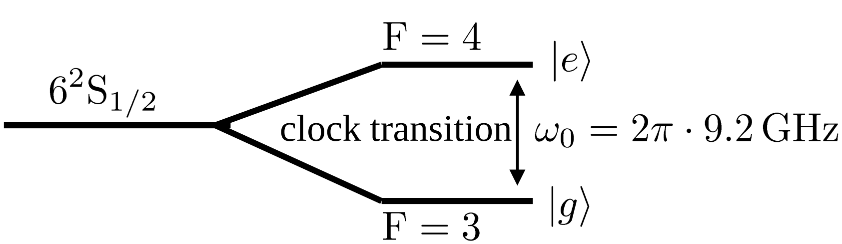
$$\langle n_g\rangle = N\sin^2\left(\frac{(\omega_0-\omega)t}{2}\right) \qquad \langle n_e\rangle = N\cos^2\left(\frac{(\omega_0-\omega)t}{2}\right)$$

Exactly the same as in the Mach-Zehnder

$$\varphi = (\omega_0 - \omega)t$$



Cs atomic fountain clock (NIST)

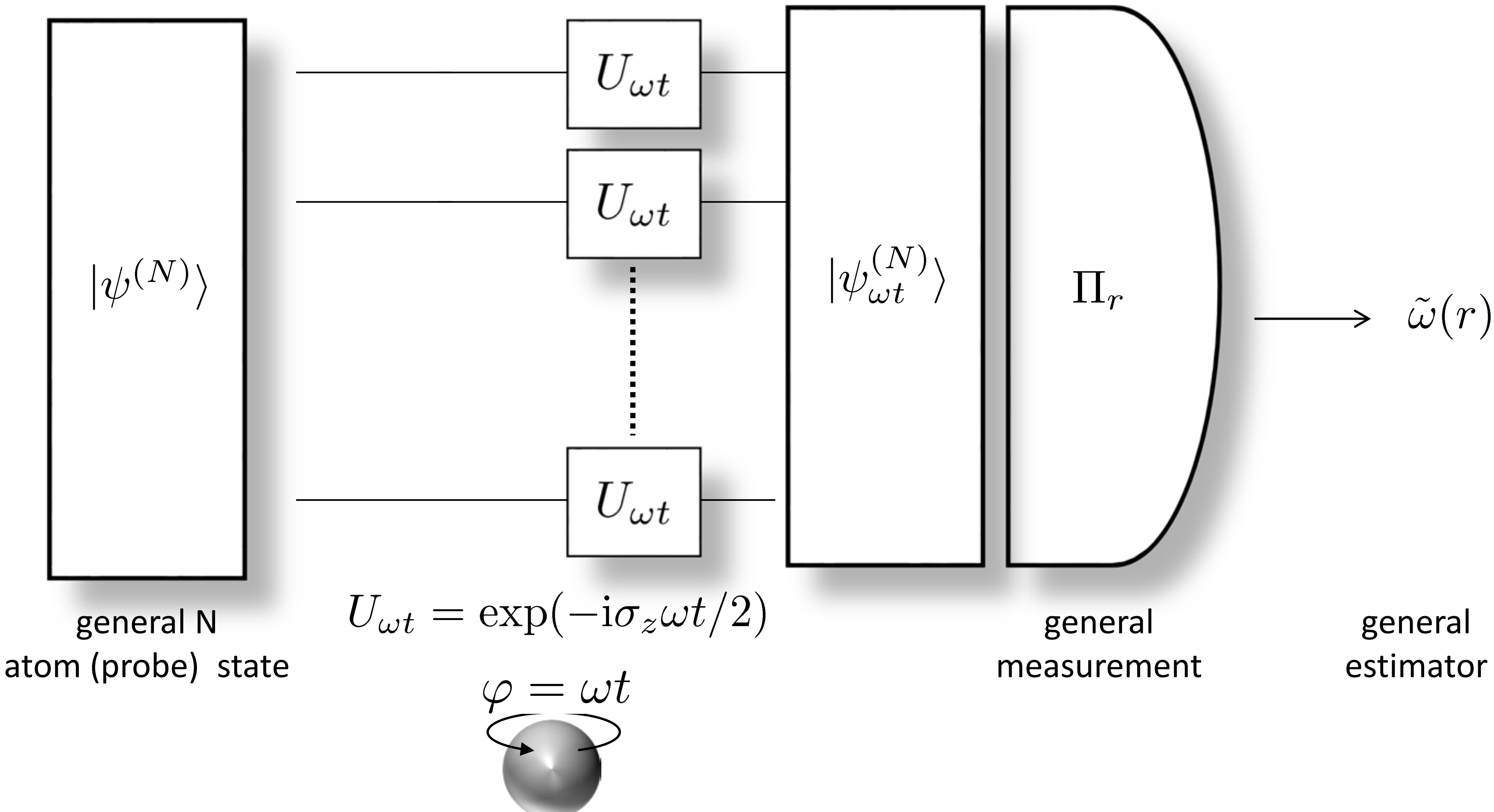


Frequency difference estimation precision

$$\Delta\varphi \approx \frac{1}{\sqrt{N}} \longrightarrow \Delta\omega \approx \frac{1}{\sqrt{N}}\frac{1}{t}$$

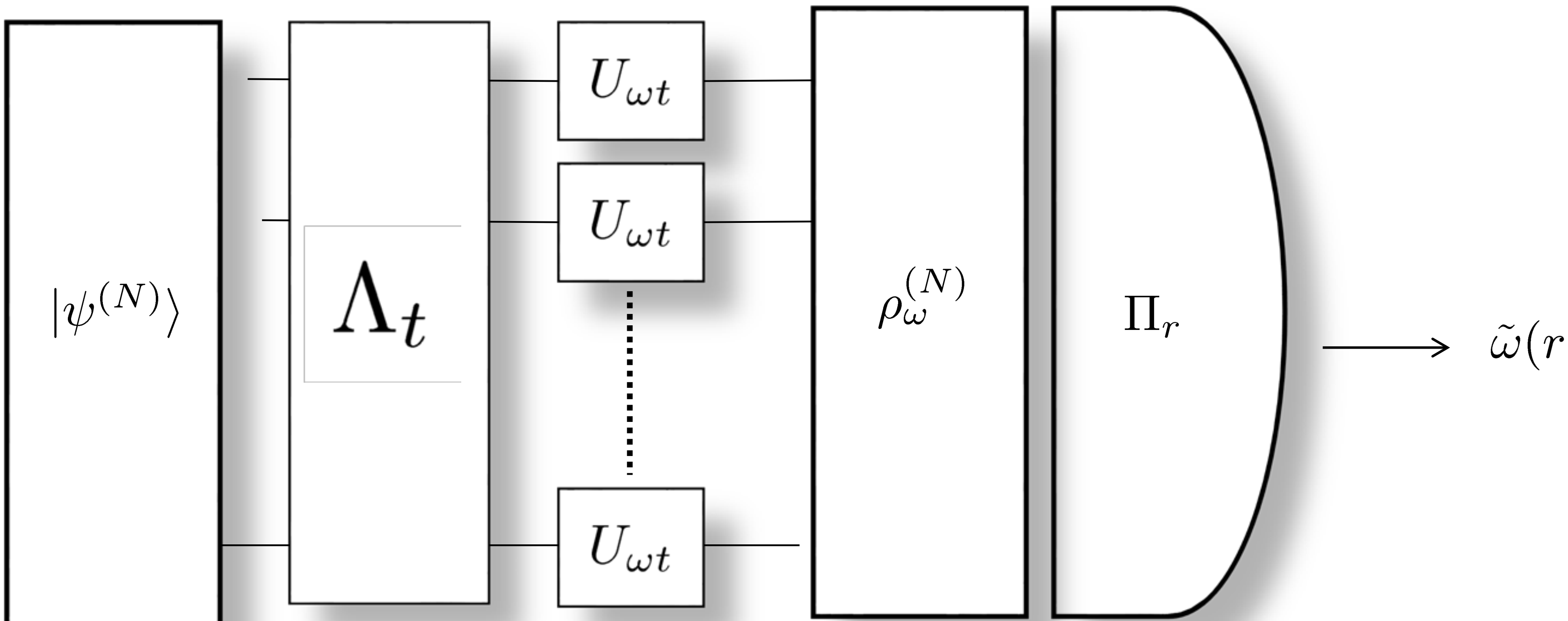
quantum projection noise

Frequency estimation



Minimize $\Delta^2\tilde{\omega} = \langle(\tilde{\omega} - \omega)^2\rangle$ over $|\psi^{(N)}\rangle$, t, Π_r and $\tilde{\omega}$

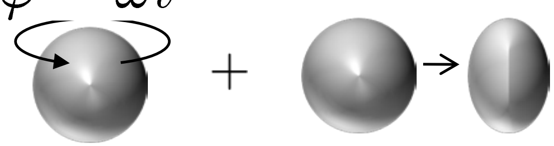
Frequency estimation
with decoherence (collective dephasing)



general N atom (probe) state

$U_{\omega t} = \exp(-\mathrm{i}\sigma_z \omega t/2)$

$\varphi = \omega t$



general measurement

general estimator

$$\text{Minimize } \Delta^2\tilde{\omega} = \langle(\tilde{\omega}-\omega)^2\rangle \text{ over } |\psi^{(N)}\rangle,\ t,\ \Pi_r \text{ and } \tilde{\omega}$$

Quantum Fisher Information approach

In the frequency estimation context:

$$\rho_\omega = e^{-\mathrm{i}H\omega t}\Lambda_t(\rho)e^{\mathrm{i}H\omega t} \qquad e^{-\mathrm{i}H\omega t} = U_{\omega t}^{\otimes N} \qquad H = \frac{1}{2}\sum_{i=1}^{N}\sigma_z^{(i)}$$

$$F_Q(\rho_\omega) = 2t^2\sum_{ij}\frac{|\langle i|H|j\rangle|^2(\lambda_i-\lambda_j)^2}{\lambda_i+\lambda_j} =: F_Q(\Lambda_t(\rho), Ht),$$

$\lambda_i, |i\rangle$ - eigenvalues, eigenvectors of $\Lambda_t(\rho)$, $\rho = |\psi^{(N)}\rangle\langle\psi^{(N)}|$

Phase -> Frequency $\varphi = \omega t$

$$F_Q(\rho_\omega) = t^2 F_Q(\rho_\varphi), \quad \Lambda(\rho) = \Lambda_t(\rho)$$

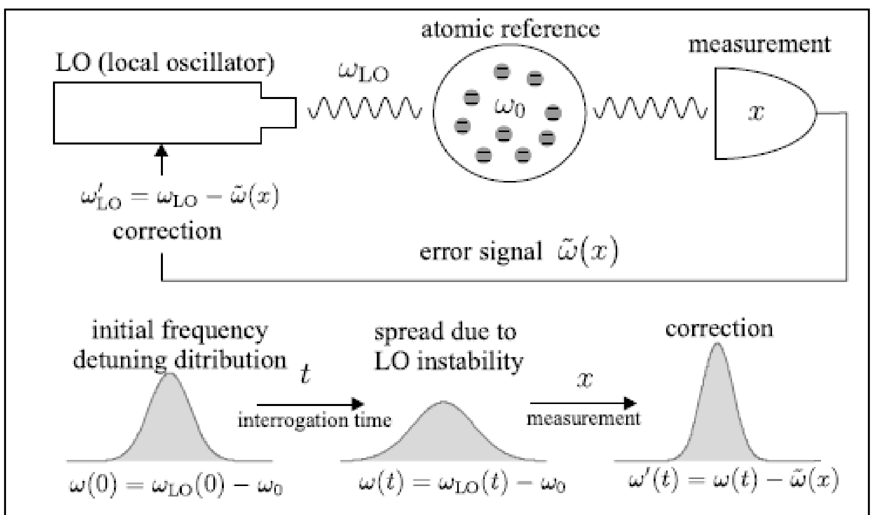
Time is an additional control parameter, but appart from that simmilar problem as phase estimation

Why Quantum Fisher Information approach is not enough?

Quantum Fisher Information approach well justified for „local sensing" (narrow priors)

Freedom in choosing time of evolution makes „local sensing" regime not well defined apriori

Saturation of Cramer-Rao bound not always guranteed in a single shot



In atomic clocks we are interested in Stationary clock operation:

LO spread compensated by the estimation

$$\Delta^2\tilde{\omega}(t) = \Delta^2\omega(0)$$

Bayesian approach more justified

Bayesian approach to frequency estimation

$$\rho_\omega = e^{-\mathrm{i}H\omega t}\Lambda_t(\rho)e^{\mathrm{i}H\omega t}$$

$$\Delta^2\tilde{\omega} = \int \mathrm{d}\omega p(\omega) \int \mathrm{d}r \mathrm{Tr}(\rho_\omega \Pi_r)[\omega - \tilde{\omega}(r)]^2$$

prior distribution — $p(\omega)$; measurement — Π_r; estimator — $\tilde{\omega}(r)$

For given input states and fixed time *t,* the optimal estimation strategy yields:

$$\Delta^2\tilde{\omega} = \Delta^2\omega - \mathrm{Tr}(\bar{\rho}L^2) \qquad \bar{\rho}' = \frac{1}{2}\left(\bar{\rho}L + L\bar{\rho}\right)$$

$$\bar{\rho} = \int \mathrm{d}\omega p(\omega)\rho_\omega \qquad \bar{\rho}' = \int \mathrm{d}\omega p(\omega)\omega\ \rho_\omega \qquad L = \sum_i \tilde{\omega}_i|\tilde{\omega}_i\rangle\langle\tilde{\omega}_i|$$

For gaussian prior $p(\omega)$: $\bar{\rho}' = \frac{\mathrm{d}\bar{\rho}}{\mathrm{d}\omega}\Delta^2\omega$

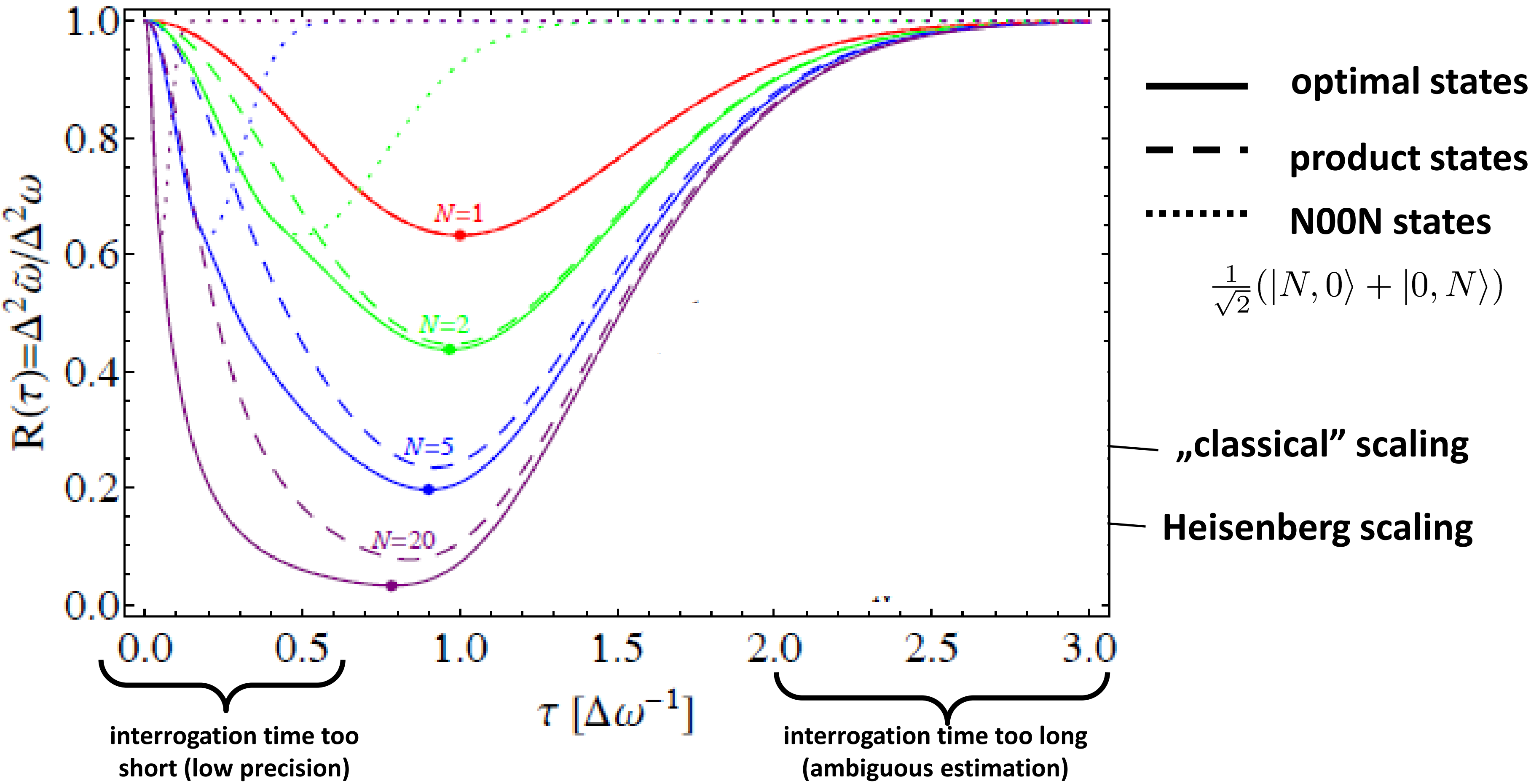
$$\Delta^2\tilde{\omega} = \Delta^2\omega[1 - \Delta^2\omega F_Q(\bar{\rho}, Ht)]$$

Optimal states in Fisher approach -> Optimal states in Bayesian approach

Helstrom, Quantum detection and estimation theory (1976)
K. Macieszczak, R. Demkowicz-Dobrzanski, M. Fraas, arXiv:1311.5576 (New simple proof)

Optimal reduction of frequency variance

numerical results, no decoherene



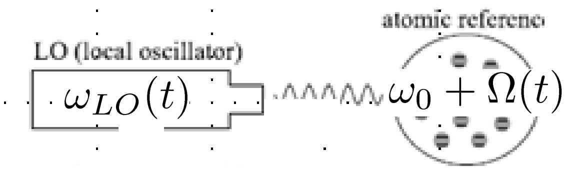
Very fast iterative method (requires only one eigendecomposition at each step):
random state -> corresponding optimal measurement -> corresponding optimal state etc.

K. Macieszczak, R. Demkowicz-Dobrzanski, M. Fraas, arXiv:1311.5576

Decoherence

Decoherence model:

Atomic frequency fluctuations, $\Omega(t)$
LO frequency fluctuations, $\omega(t) = \omega_{LO}(t) - \omega_0$



Deriving the optmal estimation strategy:

Output state at time *t* for a given realization of stochastic processes $\omega(t), \Omega(t)$

$$\rho_t = U_t \rho U_t^\dagger \qquad U_t = e^{-\mathrm{i}H\int_0^t \omega(s)+\Omega(s)\mathrm{d}s}$$

Measurement is performed at time *t*, to estimate $\omega(t)$ -> feedback is applied to the LO

The resulting variance of the LO frequency detuning:

$$\Delta^2\tilde{\omega}(t) = \langle \int \mathrm{d}x\, \mathrm{Tr}(\rho_t \Pi_x)[\omega(t) - \tilde{\omega}(x)]^2 \rangle,$$

averaging with respect to stochastic processes $\omega(t), \Omega(t)$

Optimization analogous as in the decoherence free case

Optimal Bayesian estimation strategy

with collective dephasing

For Gaussian stochastic processes $\omega(t), \Omega(t)$, LO frequency variance:

$$\Delta^2\tilde{\omega}(t) = \Delta^2\omega(t) - K_1^{\omega}(t)^2 F(\bar{\rho}, Ht).$$

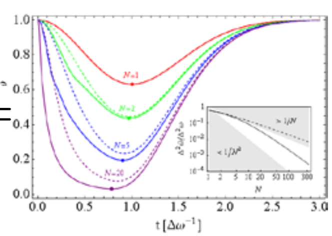
spread due to LO instability — feedback correction

$K_1^{\omega}(t) = \frac{1}{t}\int_0^t \mathrm{d}t_1 \langle \omega(t_1)\omega(t)\rangle$

$\bar{\rho} = \langle U_t \rho U_t^{\dagger}\rangle$ averaged as if with prior distribution: $\Delta^2\omega_K(t) = K_2^{\omega}(t) + K_2^{\Omega}(t).$

$K_2^{\omega}(t) = \frac{1}{t^2}\int_0^t\int_0^t \mathrm{d}t_1\mathrm{d}t_2\langle\omega(t_1)\omega(t_2)\rangle$ $\quad K_2^{\Omega}(t) = \frac{1}{t^2}\int_0^t\int_0^t \mathrm{d}t_1\mathrm{d}t_2\langle\Omega(t_1)\Omega(t_2)\rangle$

Solution to the problem **with decoherence** requires no addtional numerical optimization compared with the **decoherence free** case!

$R(t) =$ 

$$\Delta^2\tilde{\omega}(t) = \Delta^2\omega(t) - \frac{K_1^{\omega}(t)^2}{\Delta^2\omega_K(t)}\left[1 - R\left(t\sqrt{\Delta^2\omega_K(t)}\right)\right]$$

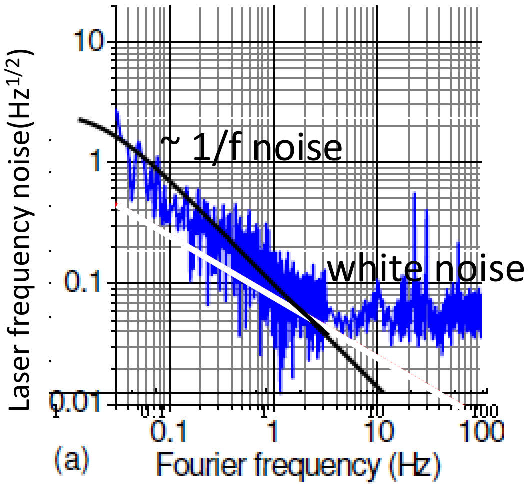
just need to rescale the prior to find the optimal states

Example

motivated by NIST optical clock system: Nature Photonics 5, 158 (2011)

$\Omega(t)$ atomic dephasing Markovian

$\omega(t)$ LO dephasing : Ornstein-Uhlenbeck process - diffusive spread of LO frequency distribution for small times



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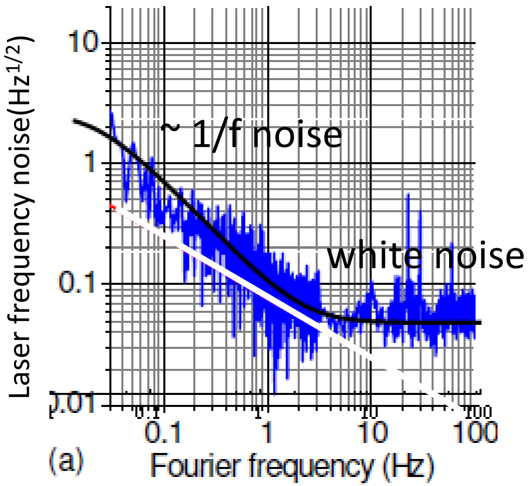
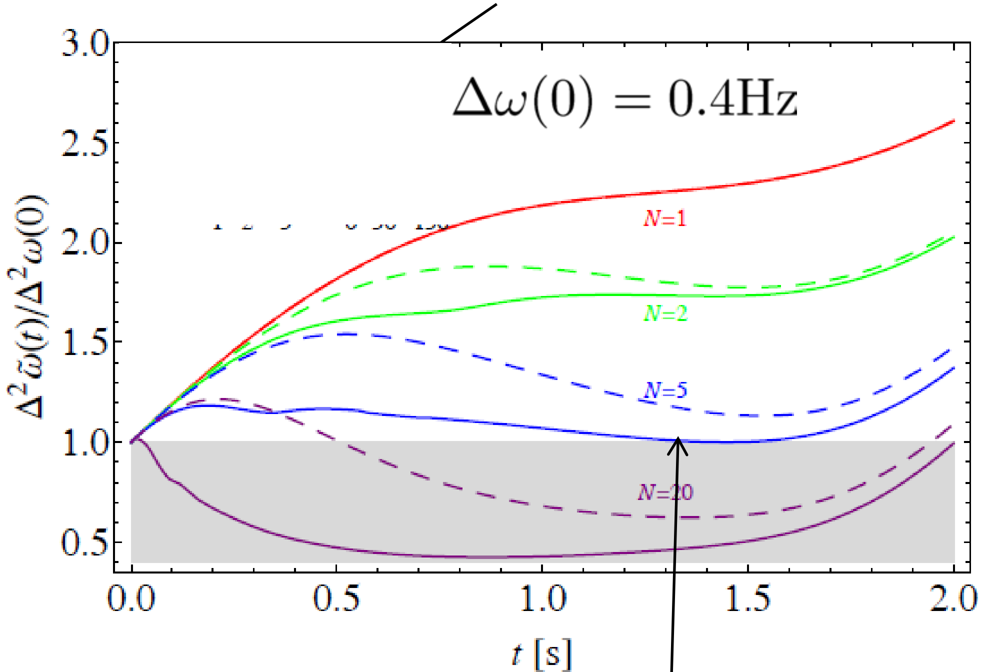


Achievable stationary variance as a function of number of atoms prepared in the optimal (solid) and product (dashed) states



$$\Delta^2\tilde{\omega}(t) = \Delta^2\omega(t) - \frac{K_1^\omega(t)^2}{\Delta^2\omega_K(t)}[1 - R(t\sqrt{\Delta^2\omega_K(t)})]$$

Stationary operation at 0.4Hz uncertainty possible with the optimal 5 atom state!

Stationary clock operation: $\Delta^2\tilde{\omega}(t) = \Delta^2\omega(0)$

K. Macieszczak, R. Demkowicz-Dobrzanski, M. Fraas, arXiv:1311.5576

Summary…

Fundamental precision bounds for optical intereferometry derrived and shown to be relatively easy to saturate

RDD, J. Kolodynski, M. Guta, , Nature Communications **3**, 1063 (2012)
RDD, K. Banaszek, R. Schnabel, Phys. Rev. A, 041802(R) (2013)
J. Kolodynski, RDD, New J. Phys. 15, 073043 (2013)
M. Jarzyna, RDD, Phys. Rev. Lett. 110, 240405 (2013)

Optimal Bayesian strategies for frequency estimation under collective dephasing derrived

K. Macieszczak, RDD, M. Fraas, arXiv:1311.5576

To do: Consider a mutistep estimation-feedback process and find fundamental bounds on the Allan variance