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Geometric theory of two-qubit operations and its applications

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Motivation

Quantum bit(s)

Qubit

$$\mathcal{H}^2 \simeq \mathbb{C}^2 \qquad \mathcal{B} = \{|0\rangle, |1\rangle\}$$

$$|\phi\rangle = c_0|0\rangle + c_1|1\rangle$$

$$\langle m|n\rangle = \delta_{mn}$$

$$c_0, c_1 \in \mathbb{C}$$

$$|c_0|^2 + |c_1|^2 = 1$$

Example:

$$|\phi\rangle = c_H|H\rangle + c_V|V\rangle$$

Qubits

$$\bigotimes_{k=1}^{n} \mathcal{H}^2 = \mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \ldots \mathcal{H}^2 \quad \text{(n-times)} = \mathcal{H}^{2^n}$$

Examples:

product state

$$|\phi\rangle = c_{00}|00\rangle + c_{01}|01\rangle = |0\rangle \otimes (c_0|0\rangle + c_1|1\rangle)$$

entangled states

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Single-qubit operations in the standard basis

Bit flip

$$\hat{X} = \begin{pmatrix} \langle 0|\hat{X}|0\rangle & \langle 0|\hat{X}|1\rangle \\ \langle 1|\hat{X}|0\rangle & \langle 1|\hat{X}|1\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x$$

Phase flip

$$\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z$$

S-gate

$$\hat{S} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

T-gate

$$\hat{T} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

Hadamard gate

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$$

Two-qubit operations in the standard basis

$$\mathcal{B} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

Controlled-NOT

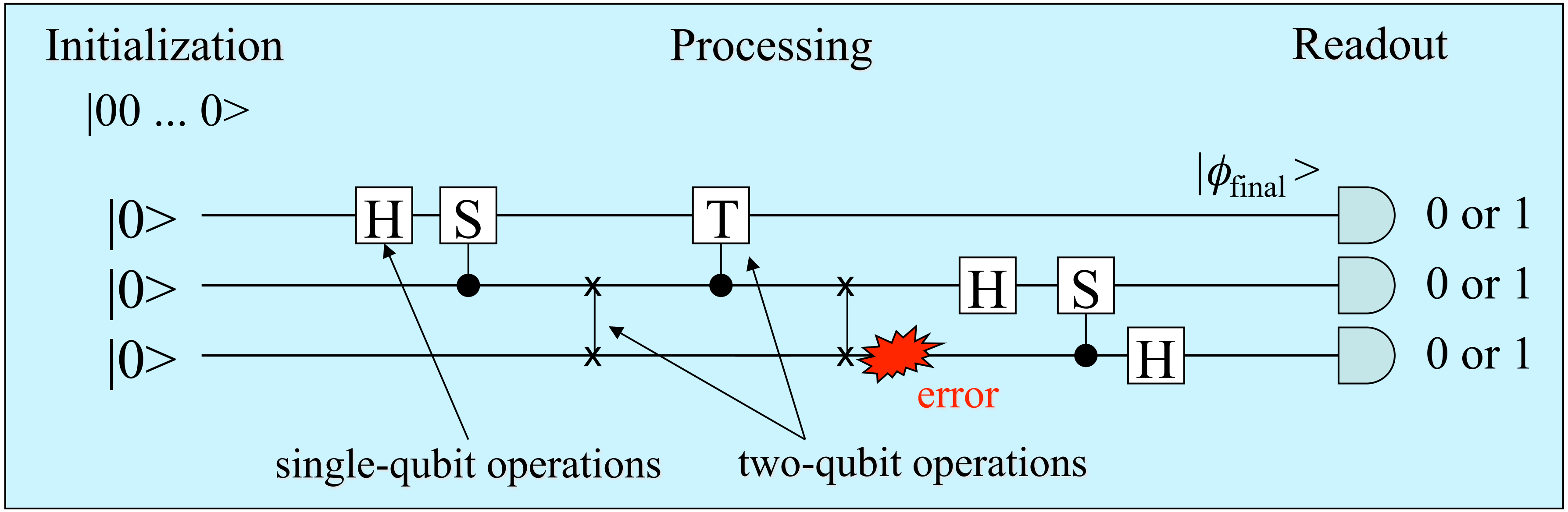
$$CNOT_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{X} = \hat{P}_0 \otimes \hat{I} + \hat{P}_1 \otimes \hat{X}$$

$$CNOT_{21} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \hat{I} \otimes |0\rangle\langle 0| + \hat{X} \otimes |1\rangle\langle 1| = \hat{I} \otimes \hat{P}_0 + \hat{X} \otimes \hat{P}_1$$

SWAP gate

$$SWAP = CNOT_{12} CNOT_{21} CNOT_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Standard model of quantum computation: quantum circuit



Universality ability to compute any computable function

Fault-tolerance ability to compute for arbitrary duration of time

Scalability ability to compute problems of arbitrary size

Universal set of quantum computing operations

Universal set of quantum computing gates is the set of operations that allows us to implement any computable function, i.e. any quantum computation algorithm or any unitary operation over n qubits, on a quantum computer.

Universality in quantum computation means the ability to generate an arbitrary element of the group of special unitary operations over n qubits, that is, an arbitrary element of the group $SU(2^n)$.

Solovay-Kitaev theorem

Given a set of gates that is dense in $SU(2^n)$ and closed under hermitian conjugation, any gate $U \in SU(2^n)$ can be approximated to an accuracy ϵ with a sequence of poly[log(1/ϵ)] gates from the set.

Example of a universal set:

any single qubit operation and one entangling gate

Overview

Geometric theory of two-qubit operations

- local invariants
- Cartan decomposition and three-torus
- Weyl chamber and local equivalence classes
- local equivalence classes of perfect entanglers

Geometric theory
Phys. Rev. A **67**, 042313 (2003)

Universality
Phys. Rev. Lett. **91**, 027903 (2003)
Phys. Rev. A **69**, 042309 (2004)
Phys. Rev. Lett. **93**, 020502 (2004)

Applications

- superconducting electronics
- trapped ion quantum computing

Phys. Rev. A **89**, 032301 (2014)

Optimal control applications

Optimal control applications
Phys. Rev. A **84**, 042315 (2011)
& a work in progress

Two-qubit gates as a metric space

- metric and invariant volume
- how large are control targets?
- what is the volume of the space of perfect entanglers?

Metric properties and applications
Entropy **15**, 1963 (2013)

I. Geometric theory of two-qubit operations

Jun Zhang
K. Birgitta Whaley
Shankar Sastry

Phys. Rev. A **67**, 042313 (2003)
Phys. Rev. Lett. **91**, 027903 (2003)
Phys. Rev. A **69**, 042309 (2004)
Phys. Rev. Lett. **93**, 020502 (2004)

Two-qubit gates

Unitary operators acting on the state of two quantum bits

$$U : \mathcal{H}^4 \to \mathcal{H}^4$$

form the group of four-by-four unitary matrices $U(4)$:

$$U(4) = U(1) \otimes SU(4)$$

where U(1) is a global phase and $SU(4)$ is the group of four-by-four unitary matrices with unit determinant.

Examples: in the standard computational basis: $\mathcal{B} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

$$U_I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad U_{H\otimes H} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad U_{CPHASE} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad U_Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & i & i & 0 \\ 0 & 1 & -1 & 0 \\ i & 0 & 0 & -i \end{pmatrix}$$

$$U_{DCNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad U_B = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 0 & i & 0 & 1 \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix} \qquad U_{\sqrt{SWAP}} = \frac{e^{-i\pi/8}}{\sqrt{2}}\begin{pmatrix} 1+i & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 1+i \end{pmatrix} \qquad U_{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$SU(4)$ group and $su(4)$ algebra

$SU(4)$ group $\quad e^{\sum_{ij}\theta_{ij}T_{ij}} \quad \longleftarrow \quad \sum_{ij}\theta_{ij}T_{ij}$ $\quad su(4)$ algebra

Generators:

$$T_{ij} = \frac{i}{2}\sigma_i^1 \otimes \sigma_j^2 = \frac{i}{2}\sigma_i^1\sigma_j^2$$

Example:

$$T_{x0} = \frac{i}{2}\sigma_x^1 \otimes I$$

$[T_{ij}, T_{kl}]$	T_{x0}	T_{y0}	T_{z0}	T_{0x}	T_{0y}	T_{0z}	T_{xx}	T_{xy}	T_{xz}	T_{yx}	T_{yy}	T_{yz}	T_{zx}	T_{zy}	T_{zz}
T_{x0}	0	$-T_{z0}$	T_{y0}	0	0	0	0	0	0	$-T_{zx}$	$-T_{zy}$	$-T_{zz}$	T_{yx}	T_{yy}	T_{yz}
T_{y0}	T_{z0}	0	$-T_{x0}$	0	0	0	T_{zx}	T_{zy}	T_{zz}	0	0	0	$-T_{xx}$	$-T_{xy}$	$-T_{xz}$
T_{z0}	$-T_{y0}$	T_{x0}	0	0	0	0	$-T_{yx}$	$-T_{yy}$	$-T_{yz}$	T_{xx}	T_{xy}	T_{xz}	0	0	0
T_{0x}	0	0	0	0	$-T_{0z}$	T_{0y}	0	$-T_{xz}$	T_{xy}	0	$-T_{yz}$	T_{yy}	0	$-T_{zz}$	T_{zy}
T_{0y}	0	0	0	T_{0z}	0	$-T_{0x}$	T_{xz}	0	$-T_{xx}$	T_{yz}	0	$-T_{yx}$	T_{zz}	0	$-T_{zx}$
T_{0y}	0	0	0	$-T_{0y}$	T_{0x}	0	$-T_{xy}$	T_{xx}	0	$-T_{yy}$	T_{yx}	0	$-T_{zy}$	T_{zx}	0
T_{xx}	0	$-T_{zx}$	T_{yx}	0	$-T_{xz}$	T_{xy}	0	$-T_{0z}$	T_{0y}	$-T_{z0}$	0	0	T_{y0}	0	0
T_{xy}	0	$-T_{zy}$	T_{yy}	T_{xz}	0	$-T_{xx}$	T_{0z}	0	$-T_{0x}$	0	$-T_{z0}$	0	0	T_{y0}	0
T_{xz}	0	$-T_{zz}$	T_{yz}	$-T_{xy}$	T_{xx}	0	$-T_{0y}$	T_{0x}	0	0	0	$-T_{z0}$	0	0	T_{y0}
T_{yx}	T_{zx}	0	$-T_{xx}$	0	$-T_{yz}$	T_{yy}	T_{z0}	0	0	0	$-T_{0z}$	T_{0y}	$-T_{x0}$	0	0
T_{yy}	T_{zy}	0	$-T_{xy}$	T_{yz}	0	$-T_{yx}$	0	T_{z0}	0	T_{0z}	0	$-T_{0x}$	0	$-T_{x0}$	0
T_{yz}	T_{zz}	0	$-T_{xz}$	$-T_{yy}$	T_{yx}	0	0	0	T_{z0}	$-T_{0y}$	T_{0x}	0	0	0	$-T_{x0}$
T_{zx}	$-T_{yx}$	T_{xx}	0	0	$-T_{zz}$	T_{zy}	$-T_{y0}$	0	0	T_{x0}	0	0	0	$-T_{0z}$	T_{0y}
T_{zy}	$-T_{yy}$	T_{xy}	0	T_{zz}	0	$-T_{zx}$	0	$-T_{y0}$	0	0	T_{x0}	0	T_{0z}	0	$-T_{0x}$
T_{zz}	$-T_{yz}$	T_{xz}	0	$-T_{zy}$	T_{zx}	0	0	0	$-T_{y0}$	0	0	T_{x0}	$-T_{0y}$	T_{0x}	0

Cartan decomposition of su(4)

$$su(4) = k \oplus p$$

$$[k, k] \subset k$$
$$[p, k] \subset p$$
$$[p, p] \subset k$$

$$k = span\{T_{x0}, T_{y0}, T_{z0}, T_{0x}, T_{0y}, T_{0z}\}$$

$$p = span\{T_{xx}, T_{xy}, T_{xz}, T_{yx}, T_{yy}, T_{yz}, T_{zx}, T_{zy}, T_{zz}\}$$

$[T_{ij}, T_{kl}]$	T_{x0}	T_{y0}	T_{z0}	T_{0x}	T_{0y}	T_{0z}	T_{xx}	T_{xy}	T_{xz}	T_{yx}	T_{yy}	T_{yz}	T_{zx}	T_{zy}	T_{zz}
T_{x0}	0	$-T_{z0}$	T_{y0}	0	0	0	0	0	0	$-T_{zx}$	$-T_{zy}$	$-T_{zz}$	T_{yx}	T_{yy}	T_{yz}
T_{y0}	T_{z0}	0	$-T_{x0}$	0	0	0	T_{zx}	T_{zy}	T_{zz}	0	0	0	$-T_{xx}$	$-T_{xy}$	$-T_{xz}$
T_{z0}	$-T_{y0}$	T_{x0}	0	0	0	0	$-T_{yx}$	$-T_{yy}$	$-T_{yz}$	T_{xx}	T_{xy}	T_{xz}	0	0	0
T_{0x}	0	0	0	0	$-T_{0z}$	T_{0y}	0	$-T_{xz}$	T_{xy}	0	$-T_{yz}$	T_{yy}	0	$-T_{zz}$	T_{zy}
T_{0y}	0	0	0	T_{0z}	0	$-T_{0x}$	T_{xz}	0	$-T_{xx}$	T_{yz}	0	$-T_{yx}$	T_{zz}	0	$-T_{zx}$
T_{0y}	0	0	0	$-T_{0y}$	T_{0x}	0	$-T_{xy}$	T_{xx}	0	$-T_{yy}$	T_{yx}	0	$-T_{zy}$	T_{zx}	0
T_{xx}	0	$-T_{zx}$	T_{yx}	0	$-T_{xz}$	T_{xy}	0	$-T_{0z}$	T_{0y}	$-T_{z0}$	0	0	T_{y0}	0	0
T_{xy}	0	$-T_{zy}$	T_{yy}	T_{xz}	0	$-T_{xx}$	T_{0z}	0	$-T_{0x}$	0	$-T_{z0}$	0	0	T_{y0}	0
T_{xz}	0	$-T_{zz}$	T_{yz}	$-T_{xy}$	T_{xx}	0	$-T_{0y}$	T_{0x}	0	0	0	$-T_{z0}$	0	0	T_{y0}
T_{yx}	T_{zx}	0	$-T_{xx}$	0	$-T_{yz}$	T_{yy}	T_{z0}	0	0	0	$-T_{0z}$	T_{0y}	$-T_{x0}$	0	0
T_{yy}	T_{zy}	0	$-T_{xy}$	T_{yz}	0	$-T_{yx}$	0	T_{z0}	0	T_{0z}	0	$-T_{0x}$	0	$-T_{x0}$	0
T_{yz}	T_{zz}	0	$-T_{xz}$	$-T_{yy}$	T_{yx}	0	0	0	T_{z0}	$-T_{0y}$	T_{0x}	0	0	0	$-T_{x0}$
T_{zx}	$-T_{yx}$	T_{xx}	0	0	$-T_{zz}$	T_{zy}	$-T_{y0}$	0	0	T_{x0}	0	0	0	$-T_{0z}$	T_{0y}
T_{zy}	$-T_{yy}$	T_{xy}	0	T_{zz}	0	$-T_{zx}$	0	$-T_{y0}$	0	0	T_{x0}	0	T_{0z}	0	$-T_{0x}$
T_{zz}	$-T_{yz}$	T_{xz}	0	$-T_{zy}$	T_{zx}	0	0	0	$-T_{y0}$	0	0	T_{x0}	$-T_{0y}$	T_{0x}	0

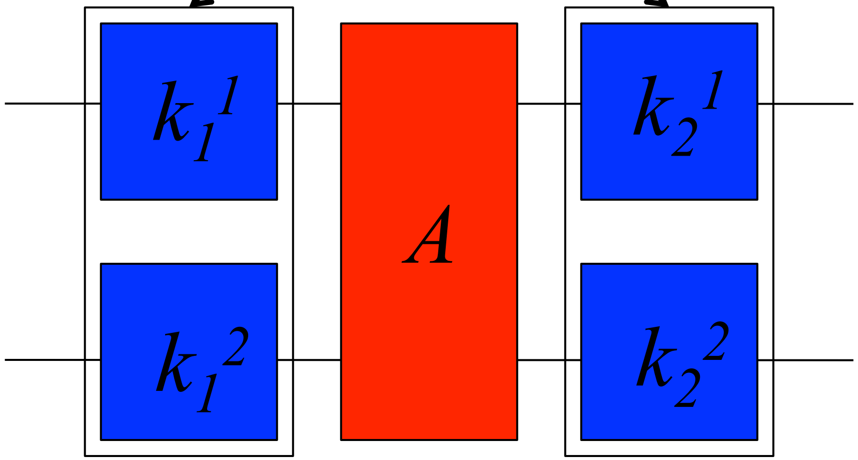
Cartan, maximal Abelian, subalgebra:

$$a = span\{T_{xx}, T_{yy}, T_{zz}\} = span\frac{i}{2}\{\sigma_x^1\sigma_x^2, \sigma_y^1\sigma_y^2, \sigma_z^1\sigma_z^2\} \subset p$$

Cartan decomposition of SU(4)

$U \in SU(4)$

$$U = k_1 \, A \, k_2 = k_1 \, e^{\frac{i}{2}\left(c_1 \sigma_x^1 \sigma_x^2 + c_2 \sigma_y^1 \sigma_y^2 + c_3 \sigma_z^1 \sigma_z^2 \right)} k_2$$



$k_1, k_2 \in SU(2) \otimes SU(2)$

Parameter counting:

$6 + 3 + 6 = 15 = 4^2 - 1$

If two gates have the same *A* in the Cartan decomposition, they are locally equivalent:

$$U_1 = k_1 U_2 k_2$$

Local equivalence and construction of local invariants

Two gates are locally equivalent if they differ only by local operations

$$U_1 = k_1 U_2 k_2 \qquad k_1,\ k_2 \in SU(2) \otimes SU(2)$$

Construction:

1) Cartan decomposition (fix: the standard computational basis)

$$U = k_1\, A\, k_2 = k_1\, e^{\frac{i}{2}\left(c_1\, \sigma_x^1 \sigma_x^2 + c_2\, \sigma_y^1 \sigma_y^2 + c_3\, \sigma_z^1 \sigma_z^2 \right)} k_2$$

2) transformation into the Bell basis

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & i & i & 0 \\ 0 & 1 & -1 & 0 \\ i & 0 & 0 & -i \end{pmatrix}$$

$$|00> \to \frac{1}{\sqrt{2}}(|00> + |11>)$$
$$|01> \to \frac{i}{\sqrt{2}}(|01> + |10>)$$
$$|10> \to \frac{1}{\sqrt{2}}(|01> - |10>)$$
$$|11> \to \frac{i}{\sqrt{2}}(|00> - |11>)$$

$$U_B = Q^\dagger U Q = Q^\dagger k_1 Q\; Q^\dagger A Q\; Q^\dagger k_2 Q = O_1 F O_2$$

$$O_1, O_2 \in SO(4) \qquad O_k^T O_k = I$$

$$F = Q^\dagger A Q = diag\left\{ e^{i\frac{c_1 - c_2 + c_3}{2}}, e^{i\frac{c_1 + c_2 - c_3}{2}}, e^{i\frac{c_1 + c_2 + c_3}{2}}, e^{i\frac{-c_1 + c_2 + c_3}{2}} \right\}$$

$$i/2\{\sigma_x^1\sigma_x^2, \sigma_y^1\sigma_y^2, \sigma_z^1\sigma_z^2\} \to i/2\{\sigma_z^1, -\sigma_z^2, \sigma_z^1\sigma_z^2\}$$

J. Makhlin, QIP, 1, 243 (2003)

J. Zhang, J. Vala, S. Sastry, K.B. Whaley Phys. Rev. A 67, 042313 (2003)

Local equivalence and construction of local invariants

$$U_B = Q^\dagger U Q = Q^\dagger k_1 Q\, Q^\dagger A Q\, Q^\dagger k_2 Q = O_1 F O_2$$

3) elimination of the local part O_1

$$m = U_B^T U_B = O_2^T F O_1^T O_1 F O_2 = O_2^T F^2 O_2$$

$$O_k^T O_k = I$$

$$F^2 = diag\left\{e^{i(c_1 - c_2 + c_3)}, e^{i(c_1 + c_2 - c_3)}, e^{i(c_1 + c_2 + c_3)}, e^{i(-c_1 + c_2 + c_3)}\right\}$$

4) characteristic equation of m and elimination of O_2

$$\lambda^4 - \mathrm{tr}(m)\lambda^3 + \frac{1}{2}\left[\mathrm{tr}^2(m) - \mathrm{tr}\left(m^2\right)\right]\lambda^2 - \mathrm{tr}^*(m)\lambda + 1 = 0$$

F^2 determines the spectrum on the Makhlin matrix m: $\mathrm{tr}(m) = \mathrm{tr}(F^2)$

Local invariants

$$g_1 = \mathrm{Re}\left\{\frac{\mathrm{tr}^2(m)}{16}\right\},\ g_2 = \mathrm{Im}\left\{\frac{\mathrm{tr}^2(m)}{16}\right\},\ g_3 = \frac{\mathrm{tr}^2(m) - \mathrm{tr}\left(m^2\right)}{4}$$

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Local equivalence classes

Local invariants:

$$g_1 = \mathrm{Re}\left\{\frac{\mathrm{tr}^2(m)}{16}\right\},\ g_2 = \mathrm{Im}\left\{\frac{\mathrm{tr}^2(m)}{16}\right\},\ g_3 = \frac{\mathrm{tr}^2(m) - \mathrm{tr}\,(m^2)}{4}$$

Uniquelly characterize a class of gates that are equivalent up to local, single qubit, transformations; they define **local equivalence classes [*U*].**

Relation between the Cartan decomposition and local invariants:

$$\boldsymbol{\sigma(F^2) = \{} e^{i(c1-c2+c3)}, e^{i(c_1+c_2-c_3)}, e^{-i(c_1+c_2+c_3)}, e^{i(-c_1+c_2+c_3)} \}$$

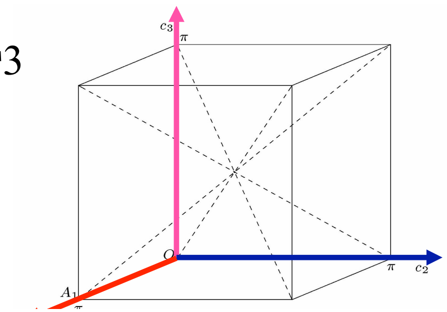
$$\begin{aligned} g_1 &= \frac{1}{4}\left[\cos(2c_1) + \cos(2c_2) + \cos(2c_3) + \cos(2c_1)\cos(2c_2)\cos(2c_3)\right] \\ g_2 &= \frac{1}{4}\sin(2c_1)\sin(2c_2)\sin(2c_3) \\ g_3 &= \cos(2c_1) + \cos(2c_2) + \cos(2c_3) \end{aligned}$$

Weyl chamber

Non-local factor A of the Cartan decomposition has the structure of three-torus

$$A = e^{\frac{i}{2}\left(c_1\, \sigma_x^1 \sigma_x^2 + c_2\, \sigma_y^1 \sigma_y^2 + c_3\, \sigma_z^1 \sigma_z^2 \right)}$$

T^3



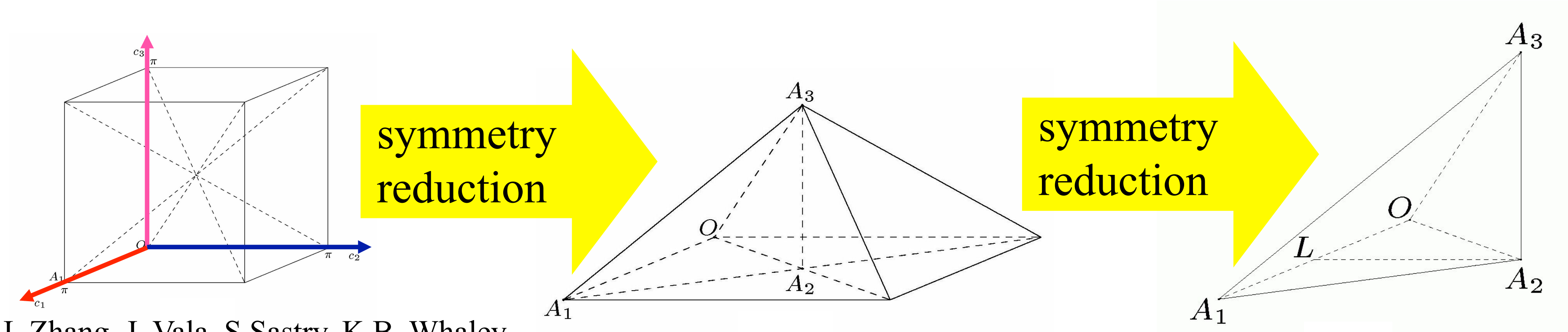
Local invariants
$$g_1 = \frac{1}{4}\left[\cos(2c_1) + \cos(2c_2) + \cos(2c_3) + \cos(2c_1)\cos(2c_2)\cos(2c_3)\right]$$
$$g_2 = \frac{1}{4}\sin(2c_1)\sin(2c_2)\sin(2c_3)$$
$$g_3 = \cos(2c_1) + \cos(2c_2) + \cos(2c_3)$$

$\{X \in \mathfrak{a} : c_1 - c_2 = 0\}$, $\{X \in \mathfrak{a} : c_1 + c_2 = \pi\}$

$\{X \in \mathfrak{a} : c_1 - c_3 = 0\}$, $\{X \in \mathfrak{a} : c_1 + c_3 = \pi\}$

$\{X \in \mathfrak{a} : c_2 - c_3 = 0\}$, $\{X \in \mathfrak{a} : c_2 + c_3 = \pi\}$

are invariant with interchanges of c_1, c_2, and c_3 with & without sign flips:



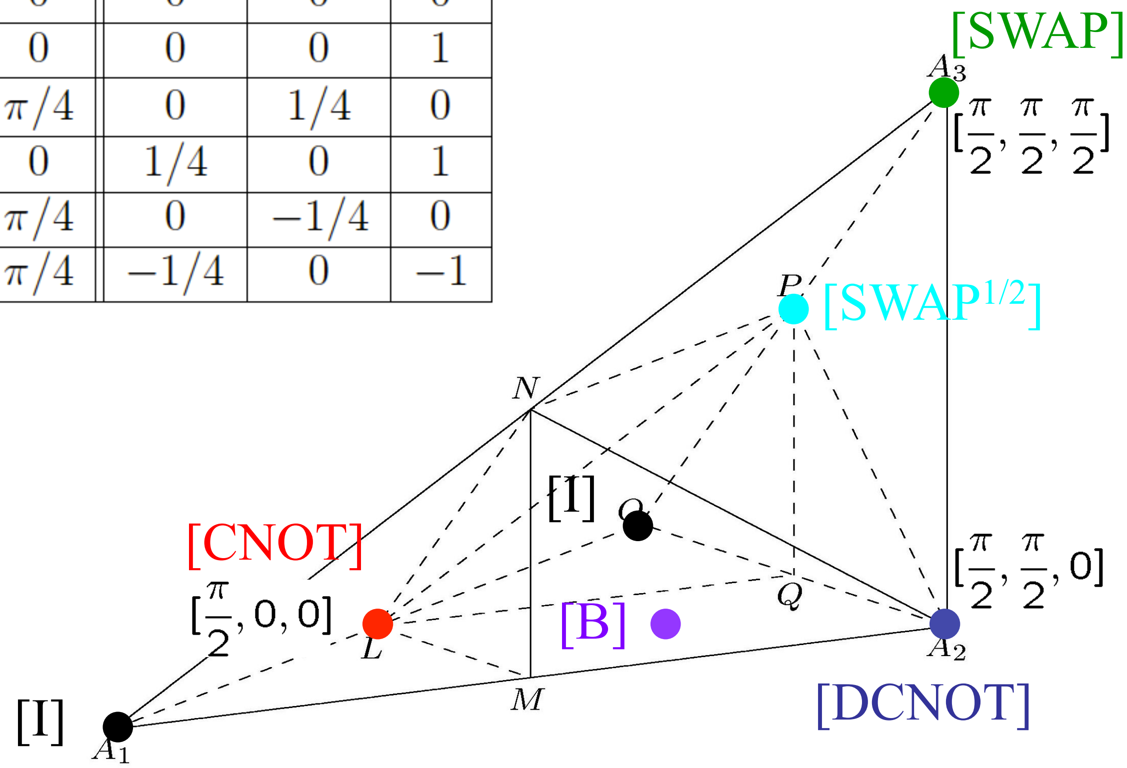
Weyl chamber

J. Zhang, J. Vala, S.Sastry, K.B. Whaley
Phys. Rev. A 67, 042313 (2003)

Examples

Each point inside of the Weyl chamber corresponds to one local equivalence class.
This is unique with except of the base of the Weyl chamber.

point (gate)	c_1	c_2	c_3	g_1	g_2	g_3
O, A_1 ([$\mathbb{1}$])	0, π	0	0	1	0	3
A_2 ([DCNOT])	$\pi/2$	$\pi/2$	0	0	0	-1
A_3 ([SWAP])	$\pi/2$	$\pi/2$	$\pi/2$	-1	0	-3
B ([B-Gate])	$\pi/2$	$\pi/4$	0	0	0	0
L ([CNOT])	$\pi/2$	0	0	0	0	1
P ([$\sqrt{\text{SWAP}}$])	$\pi/4$	$\pi/4$	$\pi/4$	0	$1/4$	0
Q, M	$\pi/4$, $3\pi/4$	$\pi/4$	0	$1/4$	0	1
N	$3\pi/4$	$\pi/4$	$\pi/4$	0	$-1/4$	0
R	$\pi/2$	$\pi/4$	$\pi/4$	$-1/4$	0	-1



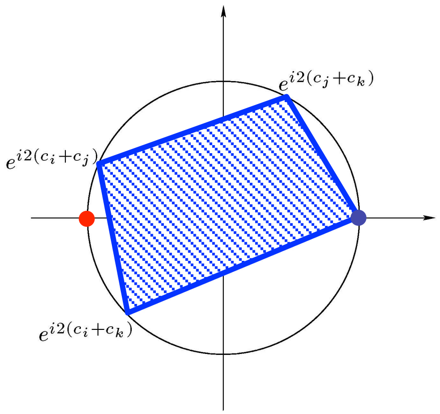
Perfect entanglers

Definition

A two qubit gate is called a **perfect entangler** if it can produce a maximally entangled state from a product state.

Theorem

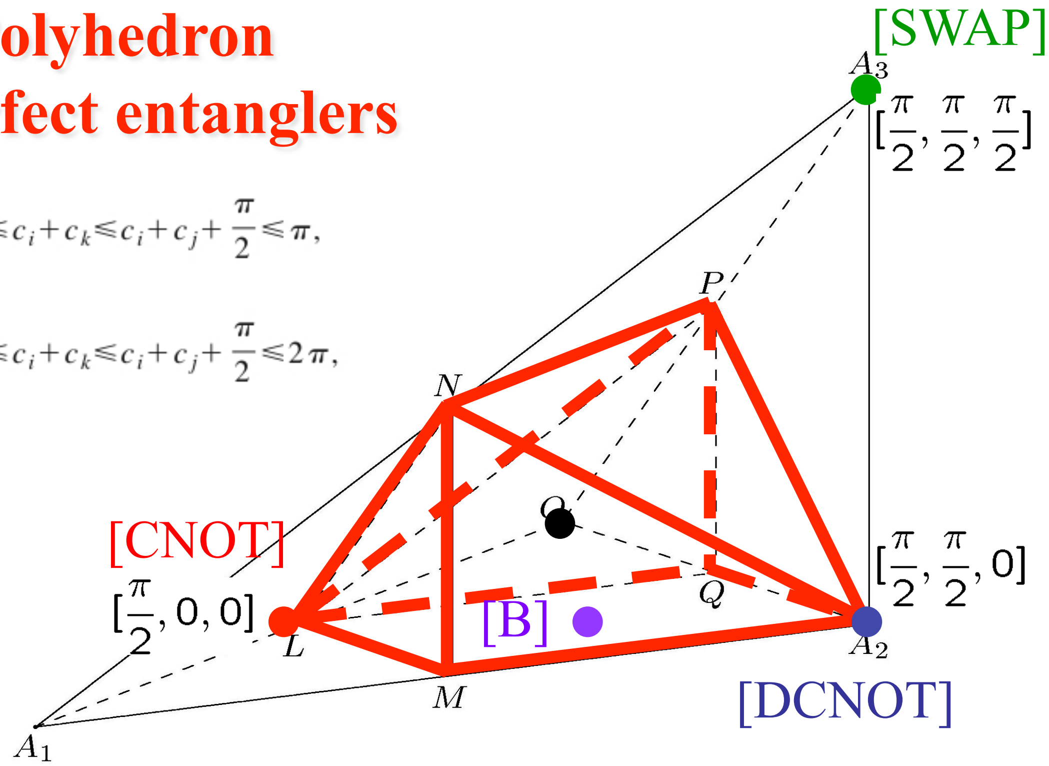
A two qubit gate *U* is a perfect entangler if and only if **the convex hull** of the eigenvalues of the Makhlin matrix *m(U)* containes zero.



Polyhedron of perfect entanglers

$$\frac{\pi}{2} \leq c_i + c_k \leq c_i + c_j + \frac{\pi}{2} \leq \pi,$$

$$\frac{3\pi}{2} \leq c_i + c_k \leq c_i + c_j + \frac{\pi}{2} \leq 2\pi,$$



Examples

CNOT

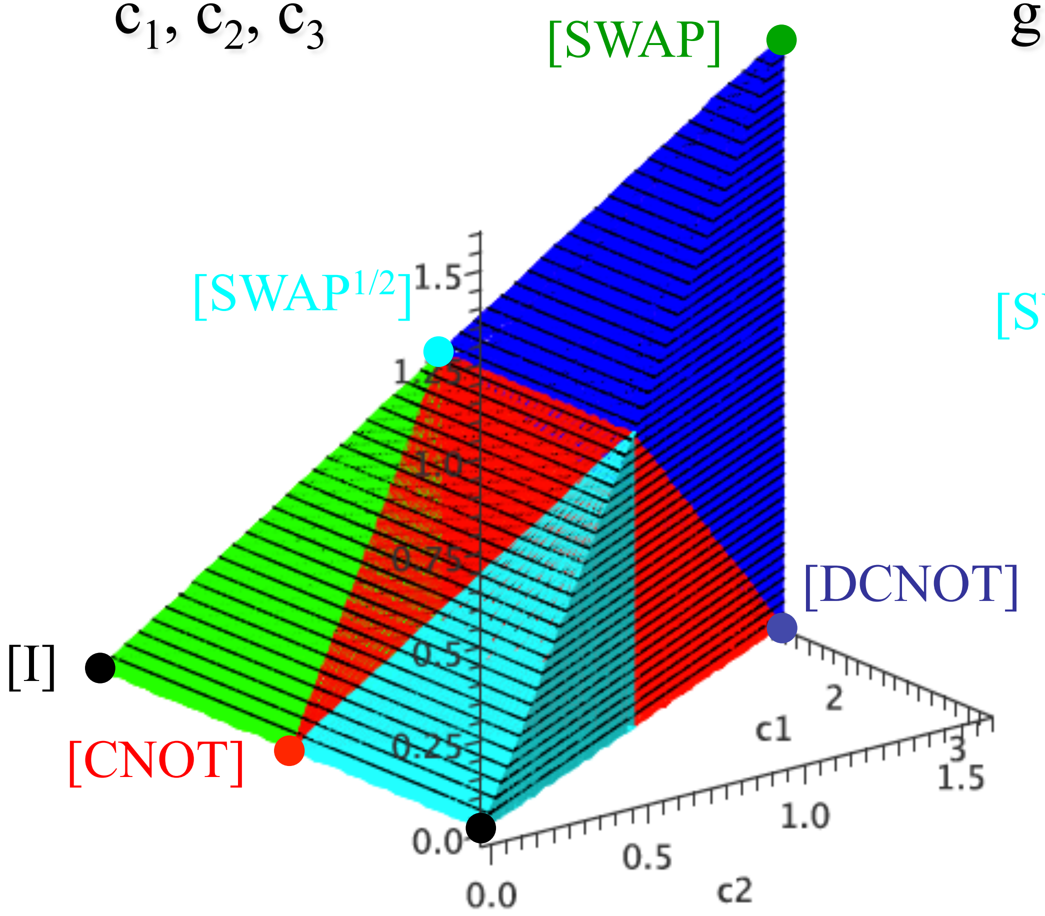
σ[m(CNOT)] = {1, 1, -1, -1}

J. Zhang, J. Vala, S.Sastry, K.B. Whaley
Phys. Rev. A 67, 042313 (2003)

Weyl chamber and local equivalence classes

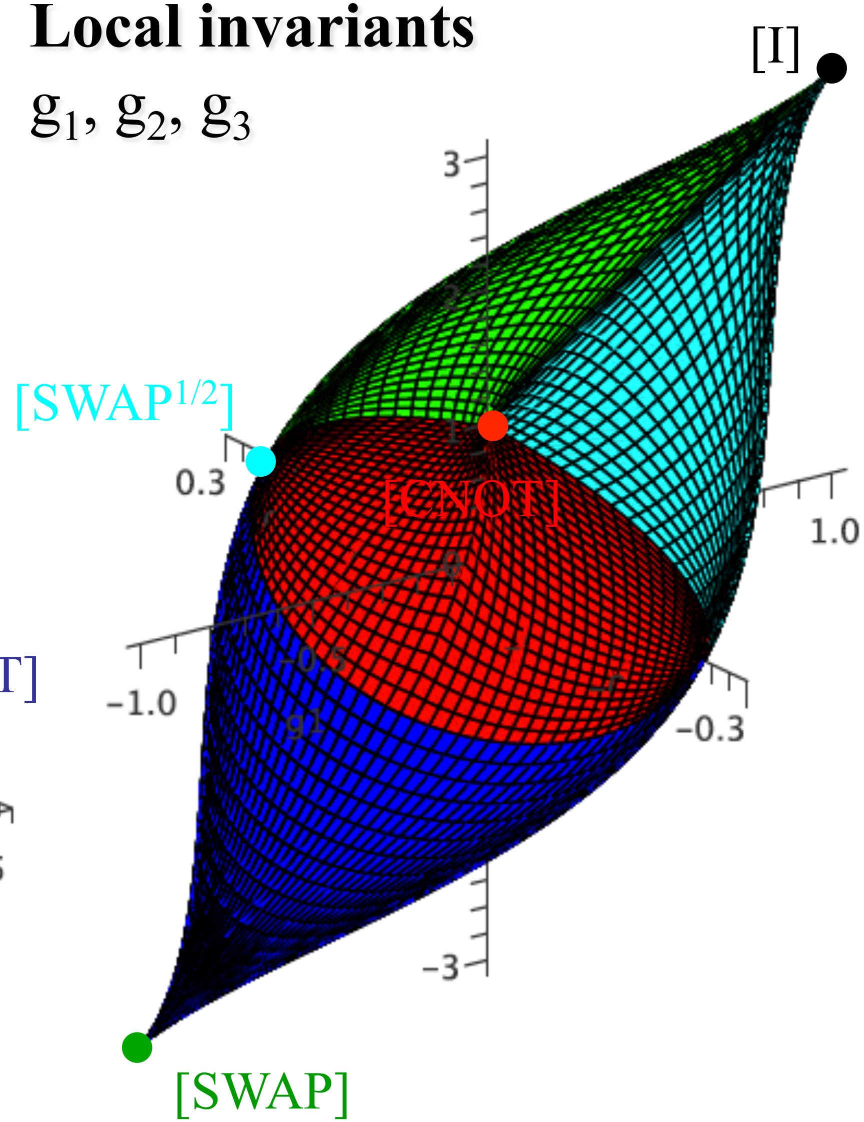
Weyl chamber coordinates

c_1, c_2, c_3



Local invariants

g_1, g_2, g_3



I. Applications

Jun Zhang
K. Birgitta Whaley
Shankar Sastry

Superconducting electronics
Phys. Rev. A **67**, 042313 (2003)

Vladimir Malinovsky

Trapped ions
Phys. Rev. A **89**, 032301 (2014).

Ignacio Sola

Application I: superconducting electronics

Josephson junction charge-coupled qubits

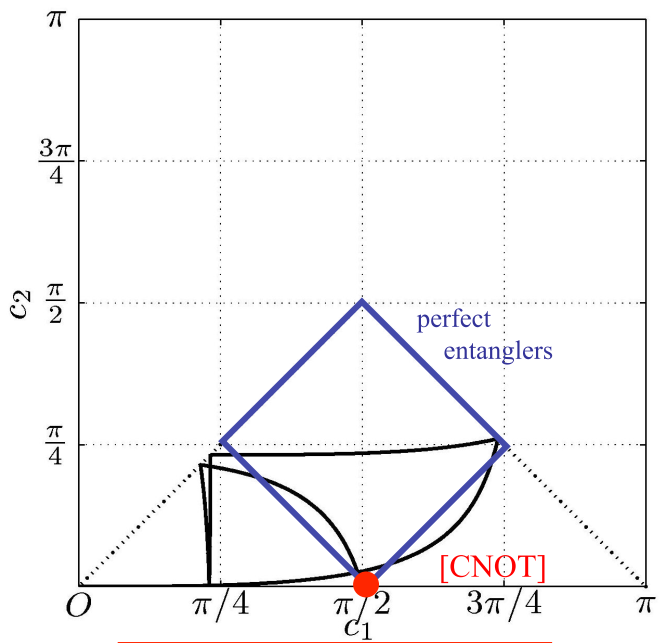
$$\hat{H} = -\frac{\alpha E_L}{2}\left(\sigma_x^1 + \sigma_x^2\right) + \alpha^2 E_L\,\sigma_y^1 \sigma_y^2$$

curvature translation

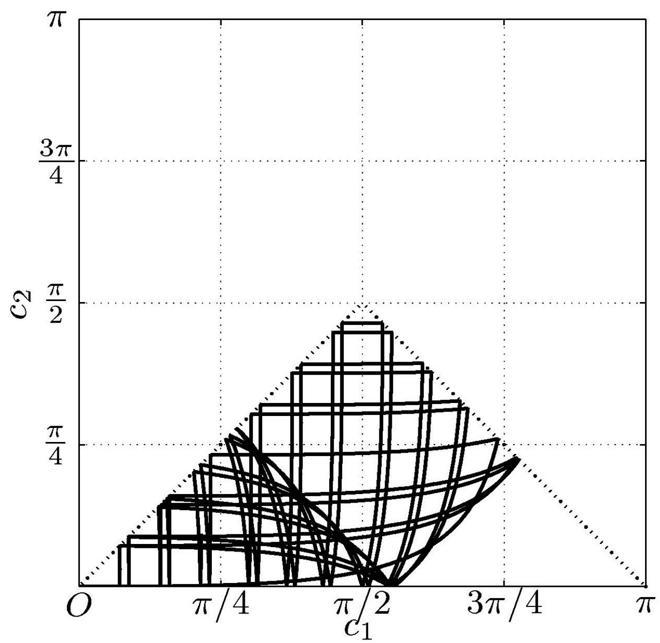
Weyl chamber trajectory:

$$c_1(t) = \alpha^2 E_L t - \omega(\alpha, t),$$
$$c_2(t) = \alpha^2 E_L t + \omega(\alpha, t),$$
$$c_3(t) = 0.$$

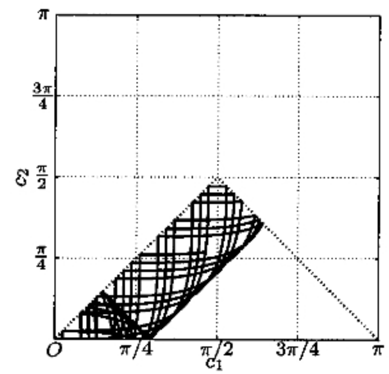
$\omega(\alpha, t) = tan^{-1}(\frac{\alpha^2+1}{2})$



$\alpha = 1.1991,\ t = 2.7309$



$\alpha = 1.1991,\ t = 20$



$\alpha = 0.5,\ t = 80$

Application II: quantum computation with trapped ions

Two distingushable qubits, ions, in one-dimensional harmonic trap

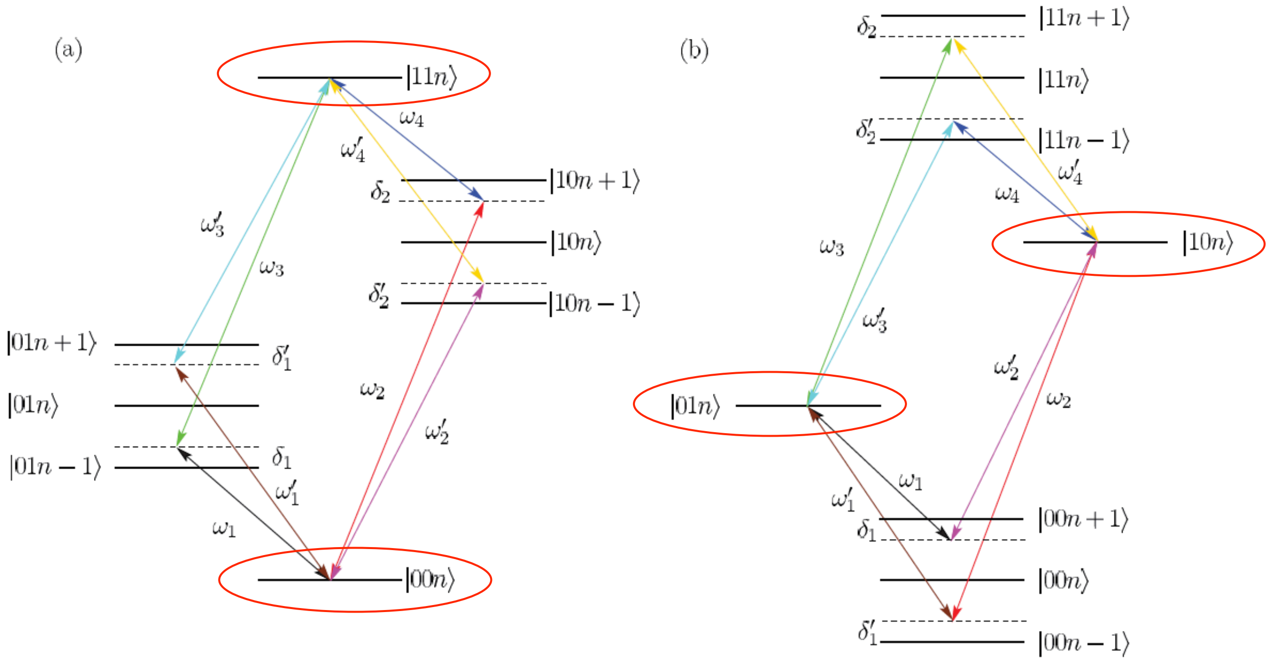
$$H_0 = \hbar\nu\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \sum_i \frac{E_2^{(i)}}{2}(\boldsymbol{I} - \boldsymbol{\sigma}_{zi})$$

interacting via electromagnetic field

$$V_{\text{int}}(t) = -\hbar\sum_{j,i}\Omega_j(t)\cos[\omega_j t + \phi_j - \eta_j(\hat{a}^\dagger + \hat{a})]\,\boldsymbol{\sigma}_{xi} + \text{H.c.}$$

$$\delta_1 = \delta_2'$$

$$\delta_2 = \delta_1'$$



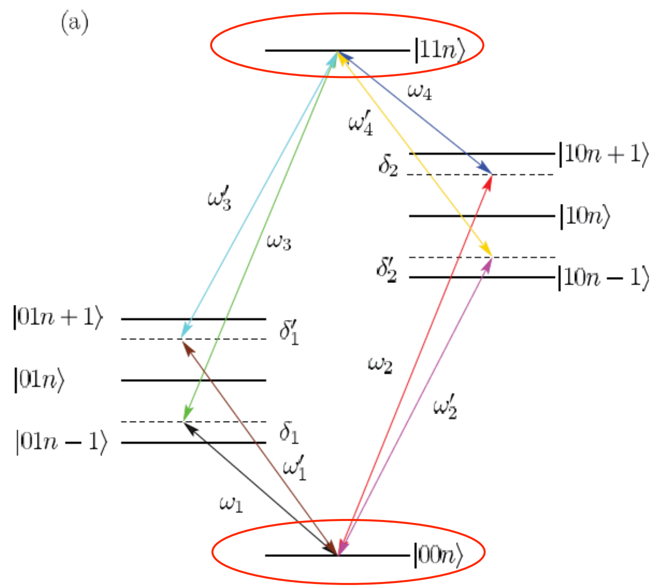
Effective Hamiltonian

$$H_{\text{eff}} = -\hbar \begin{pmatrix} \Omega_{\text{ac}}(t) & \Omega_e(t) \\ \Omega_e^*(t) & \Omega_{\text{ac}}(t) \end{pmatrix}$$

a) |00n>, |11n>:

$$\Omega_{\text{ac}}(t) = \eta^2 \Omega_0^2(t)[n/\delta_1 + (n+1)/\delta_2]/2$$

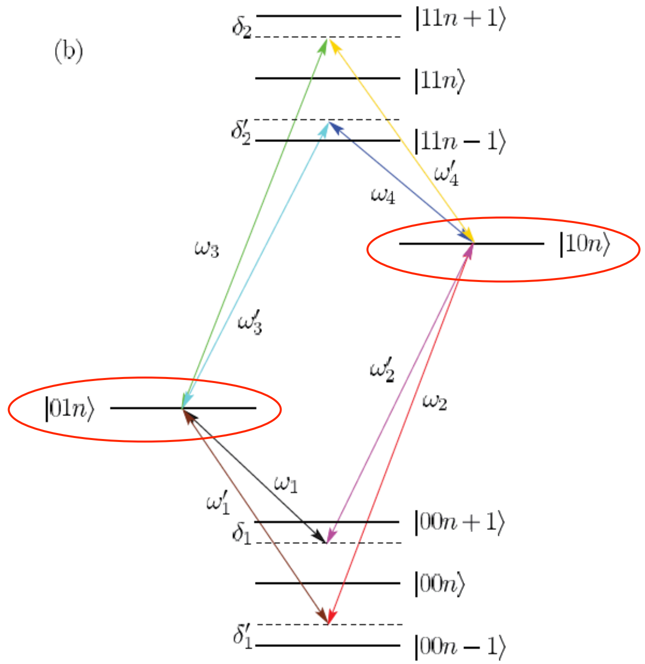
$$\Omega_e(t) = \Omega_{14}(t) = \eta^2 \Omega_0^2(t)[n(e^{i\phi_{13}} + e^{i\phi'_{24}})/\delta_1 + (n+1)(e^{i\phi_{24}} + e^{i\phi'_{13}})/\delta_2]/4$$



b) |01n>, |10n>:

$$\Omega_{\text{ac}}(t) = \eta^2 \Omega_0^2(t)[1/\delta_2 - 1/\delta_1]/4$$

$$\Omega_e(t) = \Omega_{23}(t) = \eta^2 \Omega_0^2(t)\left[\frac{n+1}{\delta_2} e^{i(\phi_3 - \phi'_4)} - \frac{n+1}{\delta_1} e^{i(\phi'_2 - \phi_1)} - \frac{n}{\delta_2} e^{i(\phi_2 - \phi'_1)} + \frac{n}{\delta_1} e^{i(\phi'_3 - \phi_4)}\right] \Big/ 4$$



Effective Hamiltonian with two laser fields

Two ions excited by two laser fields:

$$\delta_2 = -\delta_1 = \delta_0 \qquad \phi_2 \equiv \phi_1' \equiv \phi_3 \equiv \phi_4'$$
$$\phi_1 \equiv \phi_2' \equiv \phi_4 \equiv \phi_3'$$

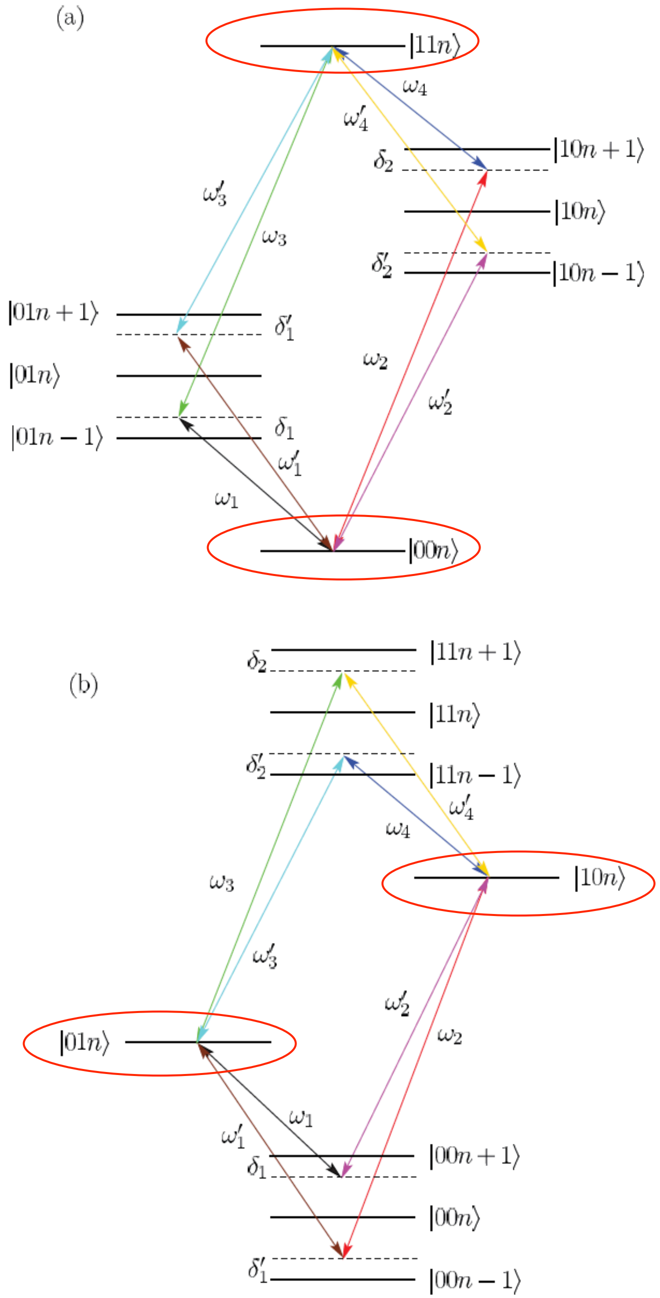
a) |00n>, |11n>: $\Omega_{\mathrm{ac}}(t) = \eta^2\Omega_0^2(t)[n/\delta_1 + (n+1)/\delta_2]/2$

$$\Omega_e(t) = \Omega_{14}(t) = \eta^2\Omega_0^2(t)[n(e^{i\phi_{13}} + e^{i\phi_{24}'})/\delta_1 + (n+1)(e^{i\phi_{24}} + e^{i\phi_{13}'})/\delta_2]/4$$

b) |01n>, |10n>: $\Omega_{\mathrm{ac}}(t) = \eta^2\Omega_0^2(t)[1/\delta_2 - 1/\delta_1]/4$

$$\Omega_e(t) = \Omega_{23}(t) = \eta^2\Omega_0^2(t)\left[\frac{n+1}{\delta_2}e^{i(\phi_3-\phi_4')} - \frac{n+1}{\delta_1}e^{i(\phi_2'-\phi_1)} - \frac{n}{\delta_2}e^{i(\phi_2-\phi_1')} + \frac{n}{\delta_1}e^{i(\phi_3'-\phi_4)}\right]\Big/4$$

$$\Omega_{\mathrm{ac}}(t) = \eta^2\Omega_0^2(t)/(2\delta_0)$$
$$\Omega_{14}(t) = \eta^2\Omega_0^2(t)e^{i(\phi_1+\phi_2)}/(2\delta_0)$$
$$\Omega_{23}(t) = \eta^2\Omega_0^2(t)/(2\delta_0)$$



Solution and decomposition

$$H_T(t) = -\hbar\eta^2\Omega_0^2(t)/(2\delta_0)[\cos(\phi_+/2)\boldsymbol{\sigma}_x - \sin(\phi_+/2)\boldsymbol{\sigma}_y] \otimes [\cos(\phi_+/2)\boldsymbol{\sigma}_x - \sin(\phi_+/2)\boldsymbol{\sigma}_y]$$

$$\phi_+ = \phi_1 + \phi_2$$

$$U_t(\xi) = \exp\left[-\frac{i}{\hbar}\int_0^t H_T(t')dt'\right] = \exp\{i\xi(t)[\cos(\phi_+/2)\boldsymbol{\sigma}_x - \sin(\phi_+/2)\boldsymbol{\sigma}_y] \otimes [\cos(\phi_+/2)\boldsymbol{\sigma}_x - \sin(\phi_+/2)\boldsymbol{\sigma}_y]\}$$

$$= \begin{pmatrix} \cos\xi(t) & 0 & 0 & ie^{i\phi_+}\sin\xi(t) \\ 0 & \cos\xi(t) & i\sin\xi(t) & 0 \\ 0 & i\sin\xi(t) & \cos\xi(t) & 0 \\ ie^{-i\phi_+}\sin\xi(t) & 0 & 0 & \cos\xi(t) \end{pmatrix}$$

$$\xi(t) = \eta^2 \int_0^t dt'\Omega_0^2(t')/(2\delta_0)$$

$$U_t(\xi) = C_1 \otimes C_2 e^{i\xi(t)\sigma_x \otimes \sigma_x} C_1^{-1} \otimes C_2^{-1} = C_1 \otimes C_2 \begin{pmatrix} \cos\xi(t) & 0 & 0 & i\sin\xi(t) \\ 0 & \cos\xi(t) & i\sin\xi(t) & 0 \\ 0 & i\sin\xi(t) & \cos\xi(t) & 0 \\ i\sin\xi(t) & 0 & 0 & \cos\xi(t) \end{pmatrix} C_1^{-1} \otimes C_2^{-1}$$

$$C_{1,2} = e^{i\phi_+\sigma_z/4}$$

$$U_t(\xi) = k_1(\phi_+)U_{xx}(\xi)k_2(\phi_+)$$

The entangling capabilities of the gate $U_t(\xi)$ are invariant to systematic variations of the phase ϕ_+, including phase errors in applied laser fields or imperfections of the physical implementations.

Phase error invariant subspace

$$H_T(t) = -\hbar\eta^2\Omega_0^2(t)/(2\delta_0)[\cos(\phi_+/2)\boldsymbol{\sigma}_x - \sin(\phi_+/2)\boldsymbol{\sigma}_y] \otimes [\cos(\phi_+/2)\boldsymbol{\sigma}_x - \sin(\phi_+/2)\boldsymbol{\sigma}_y]$$

$$= -\hbar\eta^2\Omega_0^2(t)/(2\delta_0)\begin{pmatrix} 0 & e^{i\phi_+/2} \\ e^{-i\phi_+/2} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & e^{i\phi_+/2} \\ e^{-i\phi_+/2} & 0 \end{pmatrix}, \qquad \phi_+ = \phi_1 + \phi_2$$

$$U_t(\xi) = C_1 \otimes C_2 e^{i\xi(t)\sigma_x \otimes \sigma_x} C_1^{-1} \otimes C_2^{-1} = C_1 \otimes C_2 \begin{pmatrix} \cos\xi(t) & 0 & 0 & i\sin\xi(t) \\ 0 & \cos\xi(t) & i\sin\xi(t) & 0 \\ 0 & i\sin\xi(t) & \cos\xi(t) & 0 \\ i\sin\xi(t) & 0 & 0 & \cos\xi(t) \end{pmatrix} C_1^{-1} \otimes C_2^{-1}$$

$$C_{1,2} = e^{i\phi_+\sigma_z/4}$$

$$C_1 \otimes C_2 = \begin{pmatrix} e^{i\phi_+/4} & 0 \\ 0 & e^{-i\phi_+/4} \end{pmatrix} \otimes \begin{pmatrix} e^{i\phi_+/4} & 0 \\ 0 & e^{-i\phi_+/4} \end{pmatrix} = \begin{pmatrix} e^{i\phi_+/2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-i\phi_+/2} \end{pmatrix}$$

The subspace spanned by the basis {|01>, |10>} is invariant to **random** fluctuations of the phase of the two laser fields.

CNOT

$$U_t(\xi) = C_1 \otimes C_2 e^{i\xi(t)\sigma_x \otimes \sigma_x} C_1^{-1} \otimes C_2^{-1} = C_1 \otimes C_2 \begin{pmatrix} \cos\xi(t) & 0 & 0 & i\sin\xi(t) \\ 0 & \cos\xi(t) & i\sin\xi(t) & 0 \\ 0 & i\sin\xi(t) & \cos\xi(t) & 0 \\ i\sin\xi(t) & 0 & 0 & \cos\xi(t) \end{pmatrix} C_1^{-1} \otimes C_2^{-1}$$

$$C_{1,2} = e^{i\phi_+ \sigma_z/4}$$

CNOT gate is obtained using additional single qubit operations when $\xi(t) = \pi/4$:

$$U_{\mathrm{CNOT}} = e^{-i\pi/4} B_1 \otimes \boldsymbol{I}\, C_1^{-1} \otimes C_2^{-1} U_t\left(\frac{\pi}{4}\right) C_1 \otimes C_2 A_1 \otimes A_2$$

$$B_1 = \frac{i}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad A_1 = \frac{1}{2} \begin{pmatrix} 1-i & 1+i \\ -1+i & 1+i \end{pmatrix} \qquad A_2 = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$= e^{-i\pi/4} \frac{1}{\sqrt{2}} B_1 \otimes \boldsymbol{I} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix} A_1 \otimes A_2$$

Conclusions

Geometric theory of two-qubit gates

- provides powerful representation of two-qubit local equivalence classes;
- allows insights into structure and properties of perfect entanglers;
- gives intuitive picture of two-qubit quantum evolution;
- enables analytical construction of two-qubit quantum circuits;
- leads to new gates and implementations.

Optimal control applications

- relaxing constraints on the optimization target relaxes constraints on physical interactions, optimization process and implementation;
- optimization to a given local equivalence class converges faster and more reliably;
- optimization to the set of perfect entanglers promises to maximize entanglement generation, preliminary results are quite encouraging

Metric properties

- derived expressions for the invariant length element and volume in the representation particularly suitable for quantum information processing;
- true size of optimization targets; the largest in the center of the Weyl chamber;
- perfect entanglers are (almost) everywhere!

Thank you!