State-of-the-art on winning probability relations

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Contents

1. Intransitivity of indifference

The Sorites Paradox

Many versions of the Sorites Paradox:

- **The Bald Man Paradox**: there is no particular number of hairs whose loss marks the transition to boldness

- **The Heap Paradox**: no grain of wheat can be identified as making the difference between a heap and not being a heap

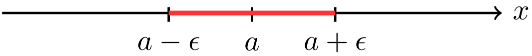
- **The Luce Paradox**: sugar in coffee example

The Poincaré Paradox

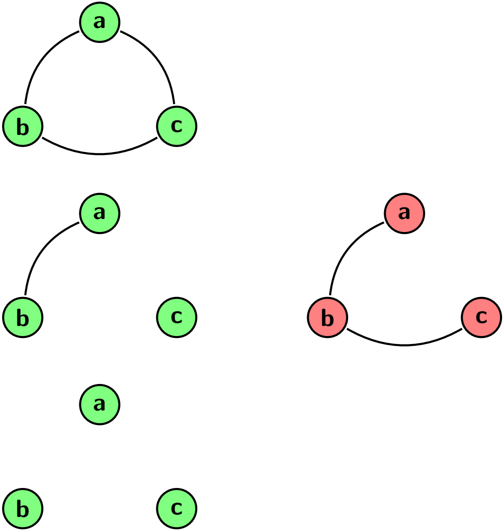
Approximate equality of real numbers is not transitive, i.e. stating that $a \in \mathbb{R}$ is similar to $b \in \mathbb{R}$ if

$$|a - b| \leq \epsilon$$

is **not transitive**



Possible symmetric configurations ($n = 3$)

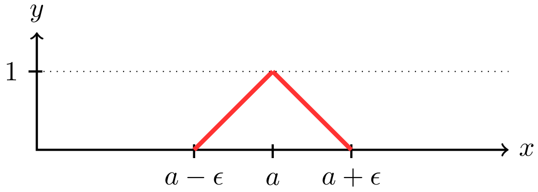


The Poincaré Paradox revisited

The fuzzy relation

$$E_\epsilon(a,b) = \max\left(1 - \frac{|a-b|}{\epsilon}, 0\right)$$

is **$T_{\mathbf{L}}$-transitive**, i.e. $E_\epsilon(a,b) + E_\epsilon(b,c) - 1 \leq E_\epsilon(a,c)$



The function $d_\epsilon = 1 - E_\epsilon$ is a metric: the **triangle inequality** holds

$$d_\epsilon(a,b) + d_\epsilon(b,c) \geq d_\epsilon(a,c)$$

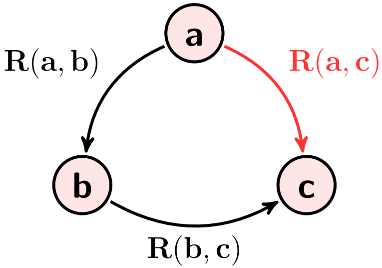
T-Transitivity of fuzzy relations

Fuzzy relation: $R : A^2 \to [0,1]$, with a **unipolar** semantics

- A fuzzy relation R on A is called **T-transitive**, with T a t-norm, if

$$T(R(a,b), R(b,c)) \leq R(a,c)$$

for any a, b, c in A



Triangular norms

Basic continuous t-norms:

minimum	$T_{\mathbf{M}}$	$\min(x, y)$
product	$T_{\mathbf{P}}$	xy
Łukasiewicz t-norm	$T_{\mathbf{L}}$	$\max(x + y - 1, 0)$

T-triplets

Consider three elements a_1, a_2 and a_3:

- A permutation (a_i, a_j, a_k) is called a **T-triplet** if

$$T(R(a_i, a_j), R(a_j, a_k)) \leq R(a_i, a_k)$$

- There can be at most 6 T-triplets
- T-transitivity expresses that there always are 6 T-triplets

2. Intransitivity of preference

Transitivity of preference

Transitivity of preference is a fundamental principle underlying most major rational, prescriptive and descriptive contemporary models of decision making

- **Rationality of individual and collective choice**: a transitive person, group or society that prefers choice option x to y and y to z must prefer x to z
- **Intransitive relations** are often perceived as something **paradoxical** and are associated with **irrational behaviour**
- Main argument: **money pump**

Intransitivity of preference

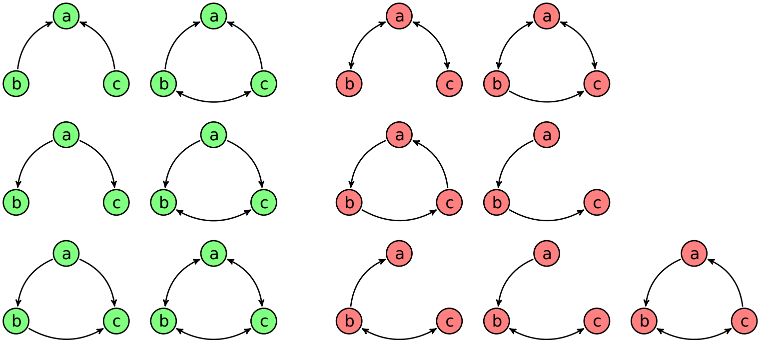
- **Transitivity** is expected to hold if preferences are based on a single scale (fitness maximization)
- **Intransitive choices** have been reported from both humans and other animals, such as **gray jays** (Waite, 2001) collecting food for storage

- **Bounded rationality**: intransitive choices are a suboptimal byproduct of heuristics that usually perform well in real-world situations (Kahneman and Tversky, 1969)
- Intransitive choices can result from decision strategies that maximize fitness (Houston, McNamara and Steer, 2007), as a kind of insurance against a run of bad luck

Intransitivity in life

Life provides many examples of intransitive relations, they often seem to be necessary and play a positive role

- sports: team A which defeated team B, which in turn won from C, can be overcome by C
- 13 love triangles:

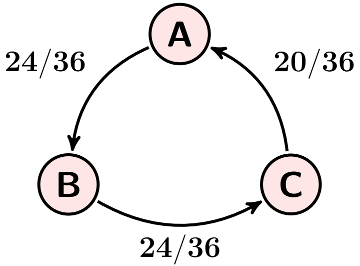


The God-Einstein-Oppenheimer dice puzzle
(New York Times, 30-03-09)

Integers 1–18 distributed over **3 dice**:

A	1	2	13	14	15	16
B	7	8	9	10	11	12
C	3	4	5	6	17	18

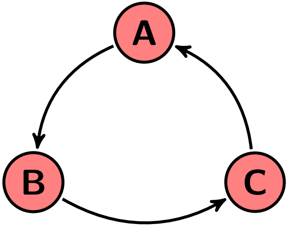
Winning probabilities:



Statistical preference

Statistical preference: X is preferred to Y if $\text{Prob}\{X > Y\} > \frac{1}{2}$

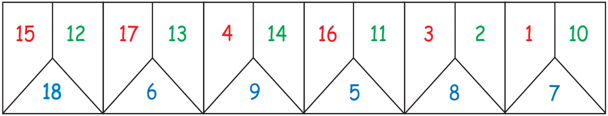
- May lead to cycles (Steinhaus and Trybuła, 1959):



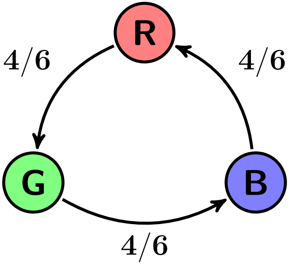
- There exist 10.705 cyclic distributions of the numbers 1–18 and 15 of them constitute a cycle of the highest equal probability $21/36 = 7/12$

A single die variant

Integers 1–18 distributed over **1 die**: 3 numbers on each face



Winning probabilities:

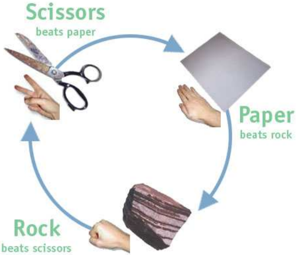


The single die can be seen as 3 coupled dice

Rock-Paper-Scissors

Cyclic dice are a type of **Rock-Paper-Scissors** (RPS):
(ancient children's game, *jan-ken-pon*, *rochambeau*)

- rock defeats scissors
- scissors defeat paper
- rock loses to paper



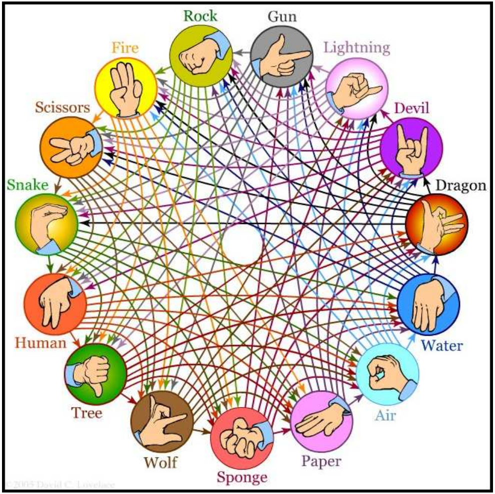
Rock-Paper-Scissors

The Rock-Paper-Scissors game:

- is often used as a **selection method** in a way similar to coin flipping, drawing straws, or throwing dice
- unlike truly random selection methods, RPS can be played with a **degree of skill**: recognize and exploit the non-random behaviour of an opponent
- **World RPS Society**:

  "Serving the needs of decision makers since 1918"

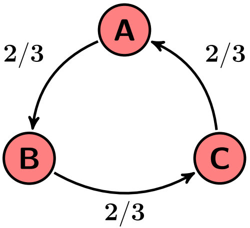
Rock-Paper-Scissors



RPS in voting

The voting paradox of Condorcet (Marquis de Condorcet, 1785)

voter 1:	A	$>$	B	$>$	C
voter 2:	B	$>$	C	$>$	A
voter 3:	C	$>$	A	$>$	B



Inspiration to **Arrow's impossibility theorem**: there is no choice procedure meeting the democratic assumptions

RPS in evolutionary biology: lizards

Common side-blotched **lizard** mating strategies (Sinervo and Lively, Nature, 1996) depending on the colour of throats of males

RPS in evolutionary biology: Survival of the Weakest

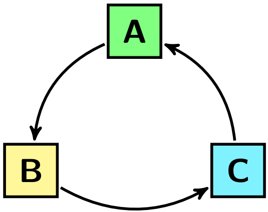
Cyclic competitions in spatial ecosystems (Reichenbach et al., 2007; Frey, 2009) (alternative to Lotka-Volterra equations, computer simulations using cellular automata)

- in large populations, the weakest species would - with very high probability - come out as the victor
- biodiversity in RPS games is negatively correlated with the rate of migration: critical rate of migration ϵ_{crit} above which biodiversity gets lost

Simulating microbial competition

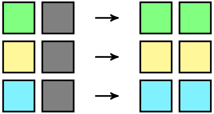
Simulation setting:

- three subpopulations: A, B, C
- initial population density: 25 % A, 25 % B, 25 % C, 25 % ■
- cellular automaton on a square grid
- environmental conditions discarded

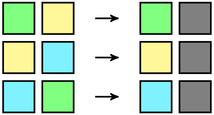


Simulating microbial competition: mechanisms

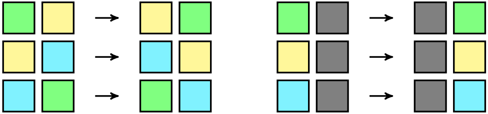
- Reproduction (μ):



- Selection (σ):



- Migration (ϵ):



Simulation experiment 1

$\epsilon < \epsilon_c$

Simulation experiment 2

$\epsilon > \epsilon_c$

3. Reciprocal relations

Reciprocal relations

Reciprocal relation: $Q : A^2 \to [0,1]$, with a **bipolar** semantics, satisfying

$$Q(a,b) + Q(b,a) = 1$$

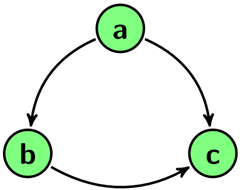
- Example 1: **3-valued representation** of a **complete** relation R

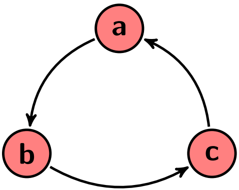
$$Q(a,b) = \begin{cases} 1 & \text{, if } R(a,b) = 1 \text{ and } R(b,a) = 0 \\ 1/2 & \text{, if } R(a,b) = R(b,a) = 1 \\ 0 & \text{, if } R(a,b) = 0 \text{ and } R(b,a) = 1 \end{cases}$$

- Example 2: **winning probabilities** associated with a random vector $(X_1, X_2, \ldots, X_n)$

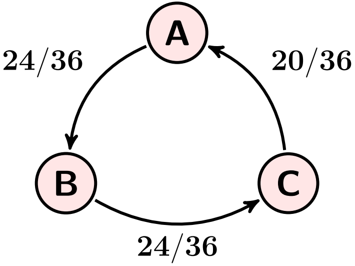
$$Q(X_i, X_j) = \mathrm{Prob}\{X_i > X_j\} + \frac{1}{2}\,\mathrm{Prob}\{X_i = X_j\}$$

Possible complete asymmetric configurations ($n = 3$)





Oppenheimer's set of dice



Reciprocal relation:

$$Q = \begin{pmatrix} 1/2 & 24/36 & 16/36 \\ 12/36 & 1/2 & 24/36 \\ 20/36 & 12/36 & 1/2 \end{pmatrix}$$

T-transitivity of reciprocal relations

Although not compatible with the bipolar semantics, T-transitivity can be imposed formally

Theorem

Consider a reciprocal relation on three elements:

- There are either **3**, **5** or **6** **$T_{\mathbf{M}}$-triplets**
- There are either **3**, **4**, **5** or **6** **$T_{\mathbf{P}}$-triplets**
- There are either **3** or **6** **$T_{\mathbf{L}}$-triplets**

$T_{\mathbf{L}}$-transitivity of reciprocal relations

$T_{\mathbf{L}}$-transitivity of a reciprocal relation = "triangle inequality":

$$Q(a,b) + Q(b,c) \geq Q(a,c)$$

Theorem

The **winning probability relation** associated with a random vector satisfies the **triangle inequality**

Stochastic transitivity of reciprocal relations

A reciprocal relation Q is called **g-stochastic transitive** if

$$(Q(a,b) \geq 1/2 \;\wedge\; Q(b,c) \geq 1/2) \Rightarrow g(Q(a,b), Q(b,c)) \leq Q(a,c)$$

- **weak** stochastic transitivity ($g = 1/2$): iff 1/2-cut of Q is transitive
- **moderate** stochastic transitivity ($g = \min$): iff all α-cuts (with $\alpha \geq 1/2$) are transitive
- **strong** stochastic transitivity ($g = \max$)

A reciprocal relation Q is called **partially stochastic transitive** if

$$(Q(a,b) > 1/2 \;\wedge\; Q(b,c) > 1/2) \Rightarrow \min(Q(a,b), Q(b,c)) \leq Q(a,c) \;;$$

iff all α-cuts (with $\alpha > 1/2$) are transitive

4. Dice games: independent RV

A probabilistic viewpoint

Three random variables X_1, X_2 and X_3:

$$\mathrm{Prob}\{X_1 > X_2 \wedge X_2 > X_3\} \leq \mathrm{Prob}\{X_1 > X_3\}$$

Even if they are independent, then not necessarily

$$\mathrm{Prob}\{X_1 > X_2\}\,\mathrm{Prob}\{X_2 > X_3\} \leq \mathrm{Prob}\{X_1 > X_3\}$$

How close are winning probabilities to being $T_{\mathbf{P}}$-transitive

$$Q(a,b)Q(b,c) \leq Q(a,c)\ ?$$

Oppenheimer's set of dice

Reciprocal relation:

$$Q = \begin{pmatrix} 1/2 & 24/36 & 16/36 \\ 12/36 & 1/2 & 24/36 \\ 20/36 & 12/36 & 1/2 \end{pmatrix}$$

Four product-triplets, the only conditions **not** fulfilled are

$$Q(b,c)Q(c,a) \leq Q(b,a) \quad \text{and} \quad Q(c,a)Q(a,b) \leq Q(c,b)$$

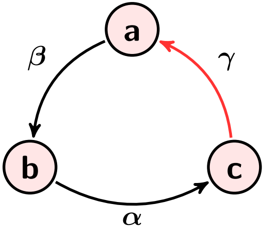
since

$$\frac{20}{36} \times \frac{24}{36} = \frac{12}{36} + \frac{1}{27} > \frac{12}{36}$$

Cycle-transitivity

Reciprocal relation Q:

α_{abc}	$\min\{Q(a,b), Q(b,c), Q(c,a)\}$
β_{abc}	$\text{median}\{Q(a,b), Q(b,c), Q(c,a)\}$
γ_{abc}	$\max\{Q(a,b), Q(b,c), Q(c,a)\}$



$T_{\mathbf{P}}$-transitivity

A reciprocal relation Q is $T_{\mathbf{P}}$-transitive if and only if $\alpha\beta \leq 1 - \gamma$ (both clockwise and counter-clockwise)

Pairwise independent random variables

Theorem (characterization for $n = 3$ and rational numbers)

The **winning probability relation** $Q^{\mathbf{P}}$ associated with **pairwise independent** random variables is **weakly $T_{\mathbf{P}}$-transitive** (dice-transitive), i.e.

$$\beta\gamma \leq 1 - \alpha$$

(both clockwise and counter-clockwise)

Interpretation

The winning probability relation $Q^{\mathbf{P}}$ is **at least** $\frac{4}{6} \times 100\%$ **$T_{\mathbf{P}}$-transitive**

Some interesting numbers for 3 dice

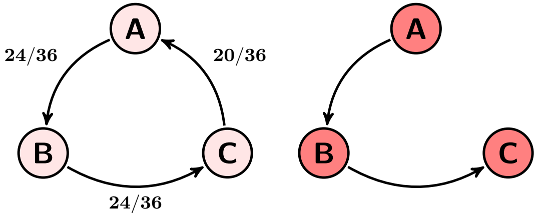
	4 faces	5 faces	6 faces	7 faces
4 $T_{\mathbf{P}}$-triplets	8.66%	1.67%	0.325%	0.060%
5 $T_{\mathbf{P}}$-triplets	14.01%	7.98%	4.2 %	2.31 %
6 $T_{\mathbf{P}}$-triplets	85.90%	92.00%	95.8%	97.68%
total number	5.78E+03	1.26E+05	2.86E+06	6.65+07

Avoiding cycles

- The **strict ϕ-cut** of $Q^{\mathbf{P}}$, with ϕ the **golden section**:

$$\frac{22}{36} < \phi = \frac{\sqrt{5}-1}{2} < \frac{23}{36}$$

  contains no cycles of length 3
- The 3/4-**cut** of $Q^{\mathbf{P}}$ is acyclic



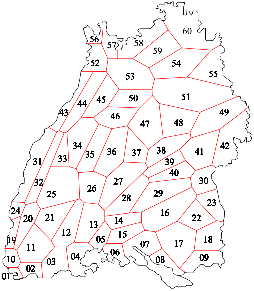
5. Poset ranking: coupled RV

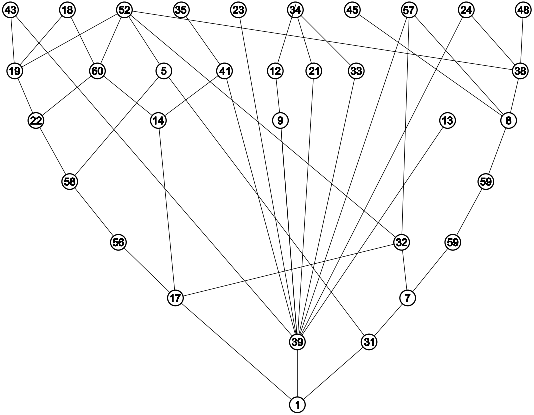
Partially ordered sets

Partially ordered sets (**posets**) are witnessing an increased interest:

- multi-criteria analysis without a common scale
- allow for incomparability
- usually based on product ordering in a multi-dimensional setting
- the **Hasse diagram technique** in **environmetrics** and **chemometrics**

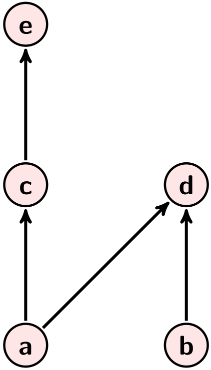
Real-world example: pollution in Baden-Württemberg

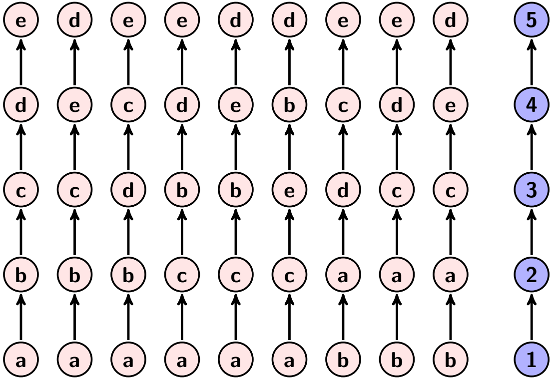




Toy example: a poset and its linear extensions

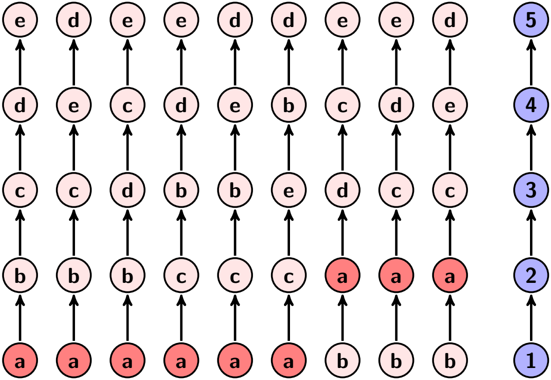
Linear extension: an **order-preserving permutation** of the elements

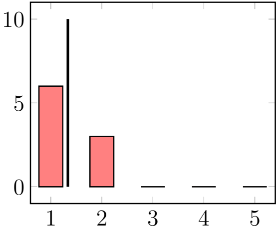




Toy example: average rank

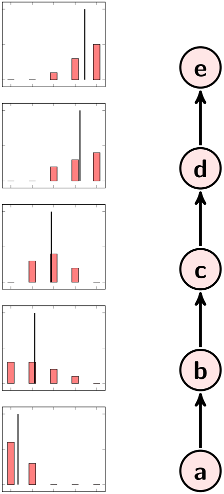
Discrete random variable X_a describing the position of a in a random linear extension





Toy example: poset ranking

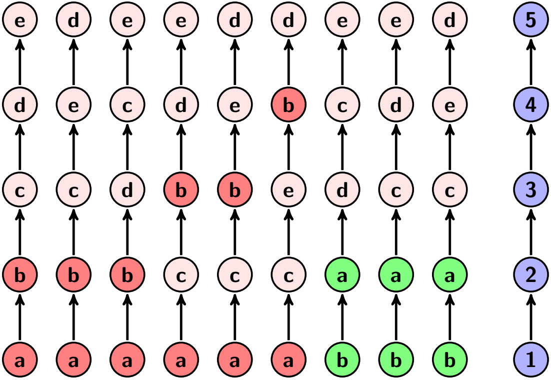
Ranking the elements according to their average rank (weak order)



Toy example: mutual rank probabilities

Fraction of linear extensions in which *a* is ranked above *b*:

$$\text{Prob}\{X_a > X_b\} = \frac{3}{9}$$



Mutual rank probability relation

Mutual rank probability relation: reciprocal relation expressing the probability that x_i is ranked above x_j

$$Q(x_i, x_j) = \mathrm{Prob}\{X_i > X_j\}$$

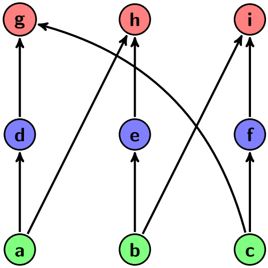
Toy example:

$$Q = \begin{pmatrix} 1/2 & 3/9 & 0 & 0 & 0 \\ 6/9 & 1/2 & 3/9 & 0 & 1/9 \\ 1 & 6/9 & 1/2 & 2/9 & 0 \\ 1 & 1 & 7/9 & 1/2 & 4/9 \\ 1 & 8/9 & 1 & 5/9 & 1/2 \end{pmatrix}$$

Linear extension majority cycles

Linear Extension Majority: x_i is ranked above x_j if $\mathrm{Prob}\{X_i > X_j\} > \frac{1}{2}$

- May lead to cycles ($n \geq 9$): only 5 out of 183 231 posets of size 9 contain LEM 3-cycles, none of them contains longer LEM cycles



$$Q(g,h) = Q(h,i) = Q(i,g) = \frac{720}{1431}$$
$$Q(d,e) = Q(e,f) = Q(f,d) = \frac{720}{1431}$$
$$Q(a,b) = Q(b,c) = Q(c,a) = \frac{720}{1431}$$

- Yu (1998): α-cuts of Q_P are transitive for

$$\alpha > \frac{1}{2}\left(1 + (\sqrt{2}-1)\sqrt{2\sqrt{2}-1}\right) \approx 0.78$$

Transitivity

Theorem

The mutual rank probability relation is **moderately $T_{\mathbf{P}}$-transitive**, i.e.

$$\alpha\gamma \leq 1 - \beta$$

(both clockwise and counter-clockwise)

Interpretation

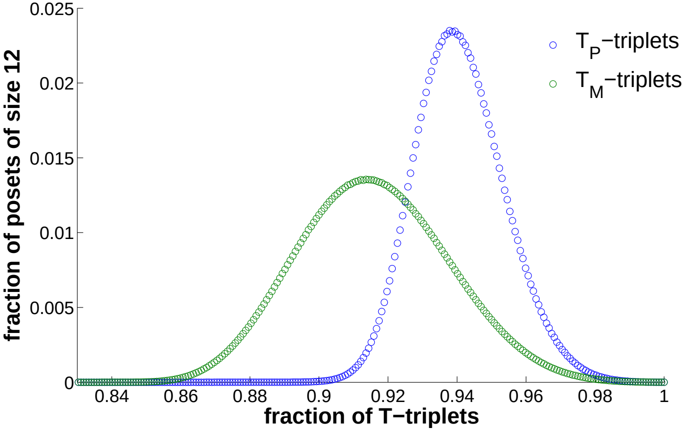
The mutual rank probability relation is **at least $\frac{5}{6} \times 100\%$ $T_{\mathbf{P}}$-transitive**

Avoiding 3-cycles

The strict ϕ-cut of Q_P, with ϕ the **golden section**, contains no cycles of length 3

Product-triplets and min-triplets

There are 1 104 891 746 non-isomorphic posets of 12 elements



6. Graded stochastic dominance: artificially coupled RV

Stochastic dominance

Aim:

- to define a **partial order relation** on a set of real-valued RV
- **semantics**: RV taking higher values are preferred

Application areas:

- economics and finance
- social statistics
- decision making under uncertainty
- machine learning and multi-criteria decision making

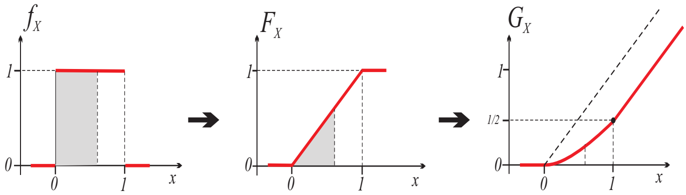
Stochastic dominance

General principle:

- **pairwise** comparison of RV
- pointwise comparison of **performance functions**

The **cumulative distribution function** (CDF) F_X:

$$F_X(x) = \mathrm{Prob}\{X \leq x\}$$



First order stochastic dominance (FSD)

- First order stochastic dominance relation (FSD):

$$X \succeq_{\mathrm{FSD}} Y \stackrel{\text{def}}{\Leftrightarrow} F_X \leq F_Y$$

  or, equivalently,

$$\mathbf{E}[u(X)] \geq \mathbf{E}[u(Y)]$$

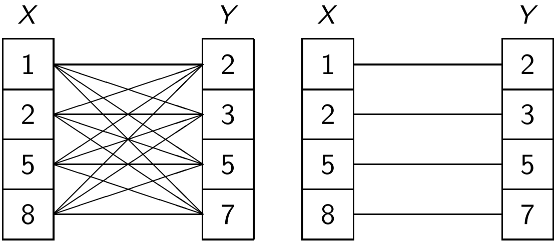
  for any **increasing** function u

- FSD implies weak **statistical preference**: $Q^{\mathbf{P}}(X, Y) \geq 1/2$

Shortcomings

- **no tolerance** for small deviations, **no grading**
- usually **sparse** graphs

Dice games versus co-monotone comparison



$Q^{\mathsf{P}}(X, Y) = 7/16$ $\qquad$ $Q^{\mathsf{M}}(X, Y) = 3/8$

Proportional expected difference

- Reciprocal relation: $Q^{\mathbf{M}}(X,Y) = \dfrac{1}{n}\sum_{k=1}^{n} \delta_k^{\mathbf{M}}$

  with

$$\delta_k^{\mathbf{M}} = \begin{cases} 1 & , \text{ if } x_k > y_k \\ 1/2 & , \text{ if } x_k = y_k \\ 0 & , \text{ if } x_k < y_k \end{cases}$$

- **Proportional expected difference relation**:

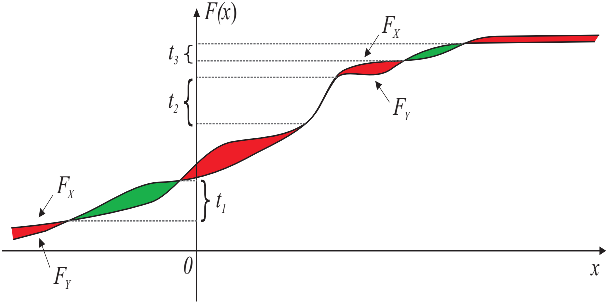
$$Q^{\mathrm{PED}}(X,Y) = \frac{\dfrac{1}{n}\displaystyle\sum_{k=1}^{n}(x_k - y_k)_+}{\dfrac{1}{n}\displaystyle\sum_{k=1}^{n}|x_k - y_k|} = \frac{\mathbf{E}[(X-Y)_+]}{\mathbf{E}[|X-Y|]}$$

  with $Q^{\mathrm{PED}}(X,Y) = 1$ if and only if $X \succ_{\mathrm{FSD}} Y$

Proportional expected difference

The case of continuous RV:

$$Q^{\mathrm{PED}}(X, Y) = \frac{\int (F_Y(x) - F_X(x))_+ \, \mathrm{d}x}{\int |F_Y(x) - F_X(x)| \, \mathrm{d}x}$$



Transitivity

Theorem

The proportional expected difference relation Q^{PED} is **partially stochastic transitive**

Use

- The strict 1/2-cut of Q^{PED} yields the strict order relation characterized by

$$Q^{\mathrm{PED}}(X,Y) > \frac{1}{2} \quad \Leftrightarrow \quad \mathbf{E}[X] > \mathbf{E}[Y]$$

- Any α-cut (with $\alpha > 1/2$) yields a **strict order relation**: with increasing α the graph (Hasse diagram) becomes more and more sparse (Hasse tree)

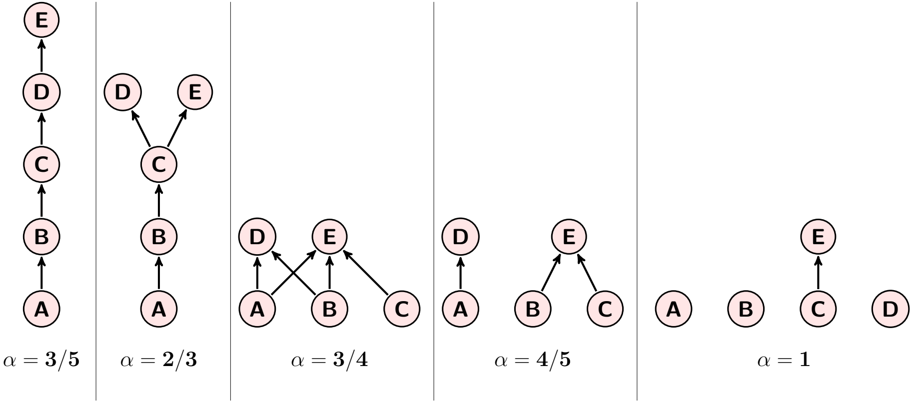
Example

Integers 1–9 distributed over **5 dice**:

A	1	4	9
B	3	4	8
C	3	6	7
D	2	7	8
E	5	6	7

$$Q^{\mathrm{PED}} = \begin{pmatrix} 1/2 & 1/3 & 1/3 & 1/5 & 1/4 \\ 2/3 & 1/2 & 1/3 & 1/4 & 1/5 \\ 2/3 & 2/3 & 1/2 & 1/3 & 0 \\ 4/5 & 3/4 & 2/3 & 1/2 & 2/5 \\ 3/4 & 4/5 & 1 & 3/5 & 1/2 \end{pmatrix}$$

Example


E
D
C
B
A
$\alpha = 3/5$
D
E
C
B
A
$\alpha = 2/3$
D
E
A
B
C
$\alpha = 3/4$
D
E
A
B
C
$\alpha = 4/5$
E
A
B
C
D
$\alpha = 1$

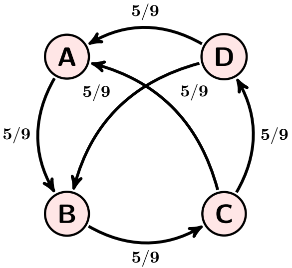
7. More dice games: beyond transitivity

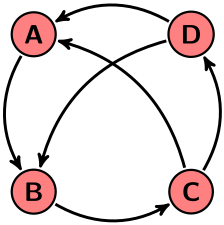
Rock-Paper-Scissors-Lizard

Integers 1–12 distributed over **4 dice**:

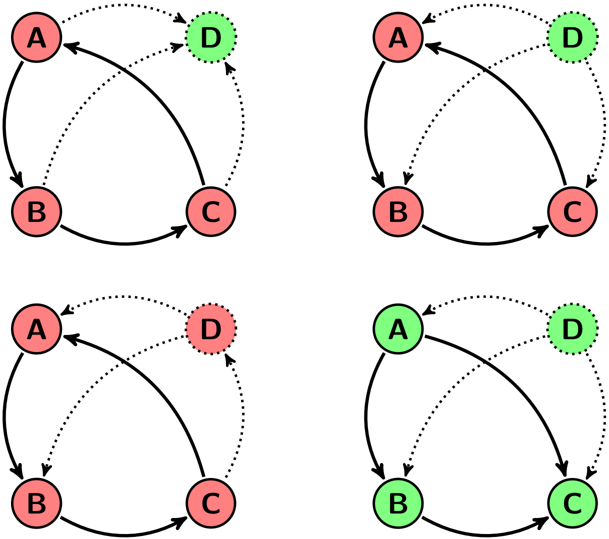
A	1	6	12
B	4	5	10
C	3	8	9
D	2	7	11

Statistical preference: 4-cycle $ABCD$ and two 3-cycles ABC and BCD





Possible complete asymmetric configurations ($n = 4$)



Product-triplets ($n = 4$)

Interpretation

The winning probability relation $Q^{\mathbf{P}}$ is **at least** $\frac{4}{6} \times 100\%$ $T_{\mathbf{P}}$**-transitive**

Some figures: number of product-triplets for 4 dice

	4 faces	5 faces	6 faces
16 triplets	-	-	-
17 triplets	-	-	0.000001 %
18 triplets	0.001%	0.00004%	0.000003 %
19 triplets	0.010%	0.0013%	0.0001%
20 triplets	0.26%	0.080%	0.018 %
21 triplets	3.37%	1.51%	0.54 %
22 triplets	17.45%	9.48%	4.91 %
23 triplets	10.63%	8.23%	5.35 %
24 triplets	68.28%	80.69%	89.18%
total number	2.63E+06	4.89E+08	9.30E+10

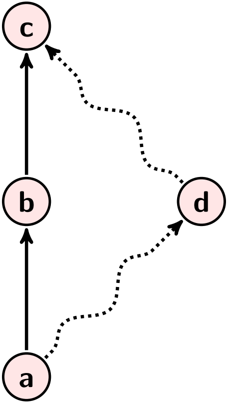
At least 16 product-triplets it is!

Integers 1–36 distributed over **4 dice**:

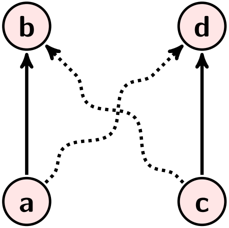
A	4	5	6	7	8	9	10	34	35
B	11	12	13	14	15	16	17	18	36
C	1	19	20	21	22	23	24	25	26
D	2	3	27	28	29	30	31	32	33

Semi-transitivity and the Ferrers property

Semi-transitivity:
if aRb and bRc, then aRd or dRc



The Ferrers property:
if aRb and cRd, then aRd or cRb



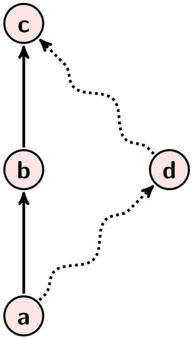
Key property of methods for **ranking fuzzy intervals (numbers)**, rather than transitivity!

T-semi-transitivity

A fuzzy relation R on A is called **T-semi-transitive**, with T a t-norm and T^* its dual t-conorm, if

$$T(R(a,b),R(b,c)) \leq T^*(R(a,d),R(d,c))$$

for any a, b, c, d in A

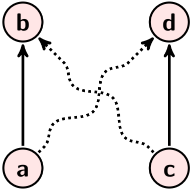


T-Ferrers property

A fuzzy relation R on A is called **T-Ferrers**, with T a t-norm and T^* its dual t-conorm, if

$$T(R(a,b),R(c,d)) \leq T^*(R(a,d),R(c,b))$$

for any a, b, c, d in A



Reciprocal relations

- **Complete relations**: transitivity implies semi-transitivity and the Ferrers property
- **Reciprocal relations**: if T is 1-Lipschitz continuous, then
  - T-transitivity implies T-semi-transitivity
  - T-transitivity implies the T-Ferrers property

$T_{\mathbf{L}}$-Ferrers

The **winning probability relation** associated with a random vector is $T_{\mathbf{L}}$-Ferrers

The Ferrers property

Four **independent** random variables X_1, X_2, X_3 and X_4:

$$\mathrm{Prob}\{X_1 > X_2\}\mathrm{Prob}\{X_3 > X_4\}$$

$$\leq \mathrm{Prob}\{X_1 > X_4\} + \mathrm{Prob}\{X_3 > X_2\} - \mathrm{Prob}\{X_1 > X_4\}\mathrm{Prob}\{X_3 > X_2\}$$

Theorem

The **winning probability relation** $Q^{\mathbf{P}}$ associated with pairwise independent random variables is $T_{\mathbf{P}}$**-Ferrers**

A stronger version of the $T_{\mathbf{P}}$-Ferrers property

Weak $T_{\mathbf{P}}$-transitivity and the $T_{\mathbf{P}}$-Ferrers property revisited

- A reciprocal relation Q is weakly $T_{\mathbf{P}}$-transitive (dice-transitive) if and only if for any 3 consecutive weights (t_1, t_2, t_3) it holds that

$$t_1 + t_2 + t_3 - 1 \geq \min(t_1 t_2, t_2 t_3, t_3 t_1)$$

- A reciprocal relation Q is $T_{\mathbf{P}}$-Ferrers if and only if for any 4 consecutive weights (t_1, t_2, t_3, t_4) it holds that

$$t_1 + t_2 + t_3 + t_4 - 1 \geq t_1 t_3 + t_2 t_4$$

4-cycle condition

The winning probability relation $Q^{\mathbf{P}}$ associated with pairwise independent random variables satisfies for any for any 4 consecutive weights (t_1, t_2, t_3, t_4)

$$t_1 + t_2 + t_3 + t_4 - 1 \geq t_1 t_3 + t_2 t_4 + \min(t_1, t_3) \min(t_2, t_4)$$

What if God does throw dice?

Integers 1–20 distributed over **5 dice**:

A	1	5	12	20
B	2	6	15	18
C	3	9	14	17
D	4	8	11	19
E	7	10	13	16

Whatever X, Y selected by Oppenheimer and Einstein, God can select Z such that

$$\mathrm{Prob}\{Z > \max(X, Y)\} > \mathrm{Prob}\{X > \max(Y, Z)\}$$

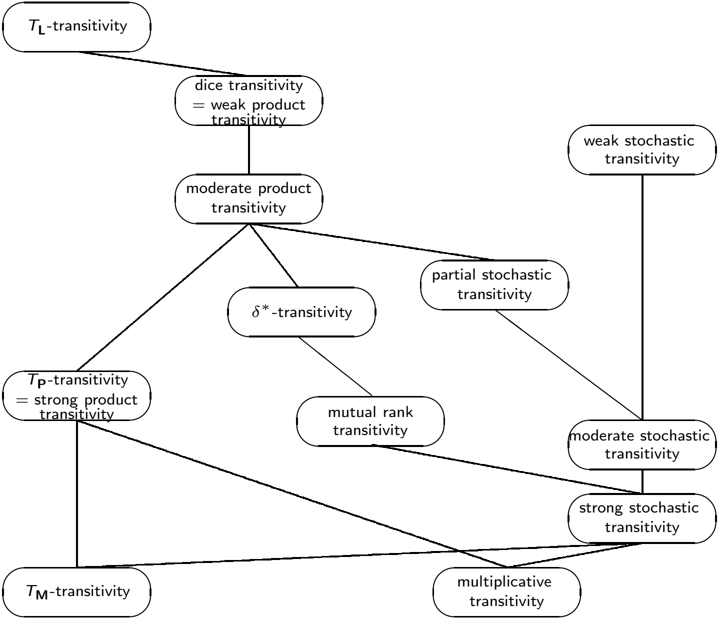
$$\mathrm{Prob}\{Z > \max(X, Y)\} > \mathrm{Prob}\{Y > \max(X, Z)\}$$

This cannot be realized with 3 or 4 dice

Conclusion

Conclusion

- **Cyclic phenomena** are not necessarily incompatible with transitivity, but arise due to the **granularity** considered
- **Cycle-transitivity** yields a general framework for studying the transitivity of **reciprocal relations**
- **Frequentist interpretation** of the transitivity of **winning probabilities** in terms of product-transitivity
- Alternative theories of **stochastic dominance**
- **In silico species competition** and coexistence
- In **machine learning**, the **AUC** (area under the ROC curve) in a 1-versus-1 multi-class classification scheme form a reciprocal relation



$T_{\mathbf{L}}$-transitivity
dice transitivity = weak product transitivity
weak stochastic transitivity
moderate product transitivity
partial stochastic transitivity
δ^*-transitivity
$T_{\mathbf{P}}$-transitivity = strong product transitivity
mutual rank transitivity
moderate stochastic transitivity
strong stochastic transitivity
$T_{\mathbf{M}}$-transitivity
multiplicative transitivity

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