Bounds on quantum process fidelity from quantum state fidelity measurements

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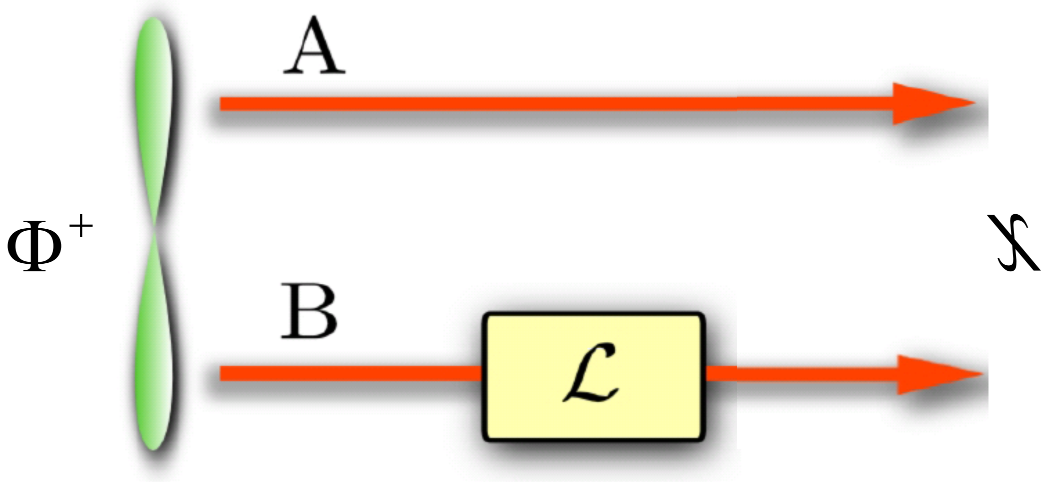
Quantum operations

Choi-Jamiolkowski isomorphism

$$\chi = I \otimes L(\Phi^+)$$



Maximally entangled probe state

$$|\Phi^+\rangle = \sum_{j=1}^{d} |jj\rangle$$

Transformation of input states

$$\rho_{\text{out}} = \text{Tr}_{\text{in}}[\rho_{\text{in}}^T \otimes \mathbb{I}_{\text{out}}\, \chi]$$

Complete positivity condition

$$\chi \geq 0$$

Trace preservation condition

$$\text{Tr}_{\text{out}}[\chi] = \mathbb{I}_{\text{in}}$$

Quantum gate fidelity

Choi matrix of a unitary operation U – density matrix of a pure maximally entangled state:

$$\chi_U = I \otimes U \left|\Phi^+\right\rangle\left\langle\Phi^+\right| I \otimes U^+$$

Quantum gate fidelity – defined as normalized overlap of Choi operators:

$$F_\chi = \frac{\mathrm{Tr}\,[\chi_U\,\chi\,]}{\mathrm{Tr}\,[\chi_\mathrm{U}]\mathrm{Tr}\,[\chi\,]}$$

Practical determination of gate fidelity:

- **quantum process tomography – full reconstruction of process matrix χ**
- **Monte Carlo sampling**
- **Hofmann lower and upper bounds on gate fidelity**

Hofmann bounds on gate fidelity

Determine average state fidelities for two conjugate bases $\left|\psi_j\right\rangle \quad \left|\varphi_k\right\rangle$

$$\left\langle\psi_k\middle|\psi_j\right\rangle=\delta_{jk} \qquad \left\langle\varphi_k\middle|\varphi_j\right\rangle=\delta_{jk} \qquad \left|\left\langle\varphi_k\middle|\psi_j\right\rangle\right|=\frac{1}{d}$$

Average state fidelities:

$$F_1=\frac{1}{d}\sum_{j=1}^{d}\left\langle\psi_j\middle|\rho_{j,out}\middle|\psi_j\right\rangle \qquad F_2=\frac{1}{d}\sum_{j=1}^{d}\left\langle\varphi_k\middle|\rho_{k,out}\middle|\varphi_k\right\rangle$$

$$\rho_{j,\mathrm{out}}=\mathrm{Tr}_{\mathrm{in}}\left[\left|\psi_j\right\rangle\left\langle\psi_j\right|\otimes I\chi\right] \qquad \rho_{k,\mathrm{out}}=\mathrm{Tr}_{\mathrm{in}}\left[\left|\varphi_k\right\rangle\left\langle\varphi_k\right|\otimes I\chi\right]$$

H.F. Hofmann, Phys. Rev. Lett. **94**, 160504 (2005).

Hofmann bounds on gate fidelity

Average state fidelities:

$$F_1 = \frac{1}{d}\sum_{j=1}^{d} \langle \psi_j | \rho_{j,\mathrm{out}} | \psi_j \rangle$$

$$F_2 = \frac{1}{d}\sum_{j=1}^{d} \langle \varphi_k | \rho_{k,\mathrm{out}} | \varphi_k \rangle$$

$$\rho_{j,\mathrm{out}} = \mathrm{Tr}_{\mathrm{in}}\left[|\psi_j\rangle\langle\psi_j| \otimes I\chi \right]$$

$$\rho_{k,\mathrm{out}} = \mathrm{Tr}_{\mathrm{in}}\left[|\varphi_k\rangle\langle\varphi_k| \otimes I\chi \right]$$

Lower bound on quantum gate fidelity

$$F_\chi \geq F_1 + F_2 - 1$$

Upper bound on quantum gate fidelity

$$F_\chi \leq \min\left(F_1, F_2\right)$$

Minimum number of probe states

To obtain a nontrivial bound on quantum gate fidelity, it suffices to probe the quantum gate with d+1 pure probe states:

Computational basis states:

$$|j\rangle,\ j = 0,\ldots,d-1$$

Superposition state:

$$|+\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}|j\rangle$$

Average state fidelity F:

$$F = \frac{1}{d}\mathrm{Tr}\left[|j\rangle\langle j|\otimes|j\rangle\langle j|\chi\right]$$

State fidelity G:

$$G = \mathrm{Tr}\left[|+\rangle\langle +|\otimes|+\rangle\langle +|\chi\right]$$

Determine a lower bound on quantum gate fidelity given F and G.

D.M. Reich, G. Gualdi, and C.P. Koch, Phys. Rev. A **88**, 042309 (2013).

Two-qubit gates

Construction of a specific quantum operation that will be proven to minimize the gate fidelity for given fixed state fidelitites F and G:

$$\tilde{\chi} = (\mathbb{I} \otimes U)\, \tilde{\chi}_S\, (\mathbb{I} \otimes U^{\dagger})$$

$$\tilde{\chi}_S = \sum_{m=0}^{3} |\chi_m\rangle\langle\chi_m|$$

$$|\chi_0\rangle = a\, Z_{00}|\Phi_2^{+}\rangle + b\,|{+}{+}\rangle|{+}{+}\rangle$$

$$|\chi_1\rangle = c\, Z_{01}|\Phi_2^{+}\rangle + d\,|{+}{+}\rangle|{+}{-}\rangle$$

$$|\chi_2\rangle = c\, Z_{10}|\Phi_2^{+}\rangle + d\,|{+}{+}\rangle|{-}{+}\rangle$$

$$|\chi_3\rangle = c\, Z_{11}|\Phi_2^{+}\rangle + d\,|{+}{+}\rangle|{-}{-}\rangle$$

J. Fiurášek and M. Sedlák, Phys. Rev. A **89**, 012323 (2014).

Parameters of the quantum operation

$$\tilde{\chi}_S = \sum_{m=0}^{3} |\chi_m\rangle\langle\chi_m|$$

$$|\chi_0\rangle = a\, Z_{00}|\Phi_2^+\rangle + b\,|++\rangle|++\rangle$$

$$|\chi_1\rangle = c\, Z_{01}|\Phi_2^+\rangle + d\,|++\rangle|+-\rangle$$

$$|\chi_2\rangle = c\, Z_{10}|\Phi_2^+\rangle + d\,|++\rangle|-+\rangle$$

$$|\chi_3\rangle = c\, Z_{11}|\Phi_2^+\rangle + d\,|++\rangle|--\rangle$$

Determined from the trace-preservation condition and from the fixed state fidelitites:

$$a = \frac{2}{3}\left[(8F-5)\sqrt{G} - 4\sqrt{(1-F)(4F-1)(1-G)}\right]$$

$$b = \sqrt{G} - \frac{a}{2},$$

$$c = \sqrt{\frac{4-a^2}{3}},$$

$$d = \sqrt{\frac{1-G}{3}} - \frac{1}{2}\sqrt{\frac{4-a^2}{3}}.$$

Analytical formula for quantum gate fidelity of this operation:

$$\tilde{F}_\chi = \left[(2F-1)\sqrt{G} - \sqrt{(4F-1)(1-F)}\sqrt{1-G}\right]^2$$

Generalized Hofmann lower bound

Define a threshold fidelity

$$F_{\text{th}} = \frac{1}{8}\left(5 - G + \sqrt{9 - 10G + G^2}\right)$$

If $F>F_{th}$ then the lower bound on gate fidelity reads

$$\tilde{F}_\chi = \left[(2F-1)\sqrt{G} - \sqrt{(4F-1)(1-F)}\sqrt{1-G}\right]^2$$

If $F<F_{th}$ then the lower bound is zero:

$$F_\chi \geq 0$$

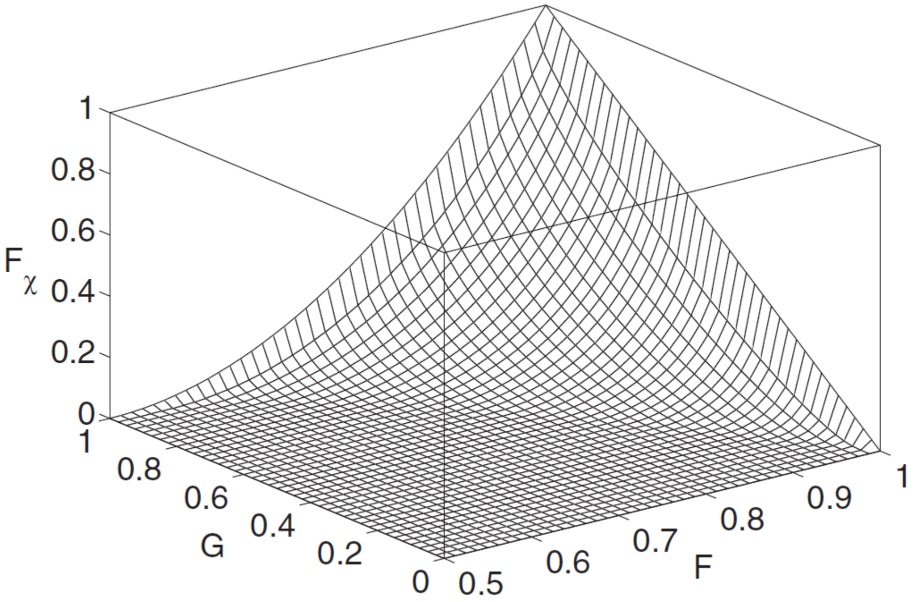


FIG. 1. Lower bound on quantum process fidelity F_χ of a two-qubit quantum operation plotted as a function of state fidelities F and G.

Comparison with standard Hofmann bound

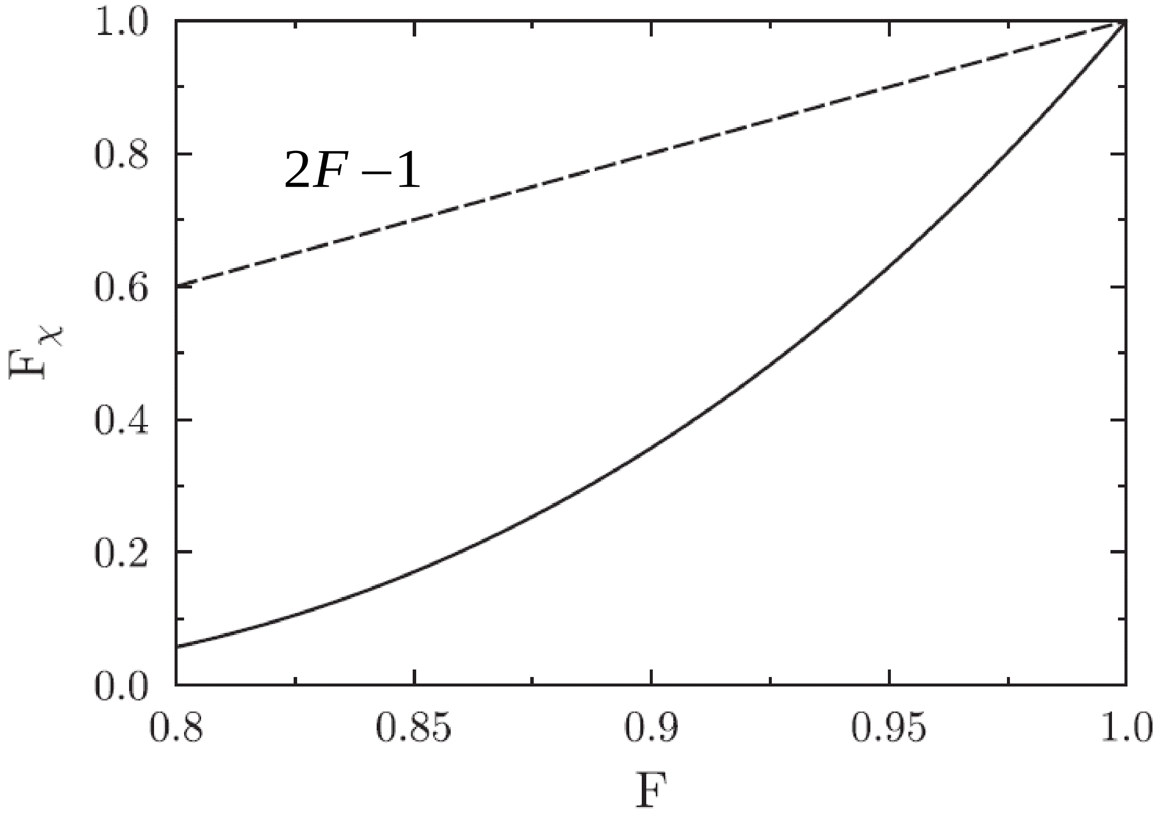


FIG. 2. Lower bound on quantum process fidelity F_χ determined from the knowledge of state fidelities F and G (solid line) and the Hofmann lower bound on quantum process fidelity (dashed line) plotted for two-qubit quantum operations assuming $G = F' = F$.

Optimality proof

The proof exploits semidefinite programming techniques.

Define operator

$$M = \frac{1}{4}|\Phi_2^+\rangle\langle\Phi_2^+| + x R_F + w R_G + y\mathbb{I} + z|{+}{+}\rangle\langle{+}{+}| \otimes \mathbb{I}_{\mathrm{out}}$$

where

$$R_F = \frac{1}{4}\sum_{j,k=0}^{1} |jk\rangle\langle jk| \otimes |jk\rangle\langle jk| \qquad R_G = |{+}{+}\rangle\langle{+}{+}| \otimes |{+}{+}\rangle\langle{+}{+}|$$

and find Lagrange multipliers *x,w,y,z* such that

$$M \geqslant 0 \qquad M\,\tilde{\chi}_S = 0$$

Optimality proof II

$$M = \frac{1}{4}|\Phi_2^+\rangle\langle\Phi_2^+| + x R_F + w R_G + y\mathbb{I} + z|++\rangle\langle++| \otimes \mathbb{I}_{\text{out}}$$

Positive semidefinitenes of M implies

$$\mathrm{Tr}[M\chi] \geq 0 \quad \Rightarrow \quad F_\chi + xF + wG + 4y + z \geqslant 0$$

$$M\tilde{\chi}_S = 0 \quad \Rightarrow \quad \tilde{F}_\chi = -xF - wG - 4y - z$$

Altogether we have $F_\chi \geqslant \tilde{F}_\chi$

Optimality proof III

$$M = \tfrac{1}{4}|\Phi_2^+\rangle\langle\Phi_2^+| + xR_F + wR_G + y\mathbb{I} + z|{+}{+}\rangle\langle{+}{+}| \otimes \mathbb{I}_{\text{out}}$$

Lagrange multipliers:

$$x = \frac{\sqrt{4-a^2}\,(3a+2\sqrt{G})}{2a\sqrt{1-G} - 2\sqrt{G(4-a^2)}},$$

$$w = -\left(3\sqrt{4-a^2} + 2\sqrt{1-G}\right)\frac{3a+2\sqrt{G}}{64\sqrt{G(1-G)}},$$

$$y = \frac{1}{32}\left(3a+2\sqrt{G}\right)\frac{3\sqrt{4-a^2}+2\sqrt{1-G}}{\sqrt{G(4-a^2)} - a\sqrt{1-G}},$$

$$z = \frac{\sqrt{4-a^2} - 2\sqrt{1-G}}{2\sqrt{1-G}}\,y.$$

$$a = \frac{2}{3}\left[(8F-5)\sqrt{G} - 4\sqrt{(1-F)(4F-1)(1-G)}\right]$$

Eigenvalues of M:

$$\lambda_1 = y,$$
$$\lambda_2 = \tfrac{1}{8}(A - \sqrt{B}), \quad \lambda_3 = \tfrac{1}{8}(A + \sqrt{B}),$$
$$\lambda_4 = \tfrac{1}{8}(C - \sqrt{D}), \quad \lambda_5 = \tfrac{1}{8}(C + \sqrt{D}).$$

$$A = x + 8y + 4z, \quad B = x^2 - 4xz + 16z^2,$$
$$C = 1 + 4w + x + 8y + 4z,$$

$$D = 1 + 16w^2 + 2x + x^2 - 4w(1 + x - 8z) - 4z - 4xz + 16z^2.$$

It can be proven analytically that all the eigenvalues are non-negative.

N-qubit gates

Construction of the specific quantum operation can be extended to N-qubit gates but no proof of optimality is available.

$$\tilde{\chi}_S = \sum_{j=0}^{2^N-1} |\chi_j\rangle\langle\chi_j|$$

$$|\chi_0\rangle = a|\Phi_N^+\rangle + b|s\rangle|s\rangle$$

$$|\chi_j\rangle = \mathbb{I} \otimes V_j(c\,|\Phi_N^+\rangle + d\,|s\rangle|s\rangle)$$

$$|\Phi_N^+\rangle = \frac{1}{\sqrt{2^N}} \sum_{j=0}^{2^N-1} |j\rangle|j\rangle \qquad V_j = \bigotimes_{k=1}^{N} \sigma_Z^{j_k}$$

Anyway, this construction yields an upper bound on the generalized Hofmann lower bound for this case:

$$\tilde{F}_\chi = \Big\{[1-(1-F)2^{N-1}]\sqrt{G} - \sqrt{(1-F)(1-G)}\sqrt{2^N - 1 - (1-F)2^{2N-2}}\Big\}^2$$

$$F_{\rm th} = 1 - \frac{1}{2^{N-1}} + \frac{1-G}{2^{2N-1}} + \frac{2}{2^{2N}}\sqrt{(1-G)[(2^N-1)^2 - G]}$$

The bound is 0 when $F<F_{th}$.

The fidelity F must be exponentially close to 1 to obtain a nontrivial bound.