Fuzzy logic and knowledge structures

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Purpose of the talk

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- introduce fuzzy logic [2,3]

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- introduce fuzzy logic [2,3]
- show a possibility of using it in knowledge structures [1]

Uncertainty

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- . . . others

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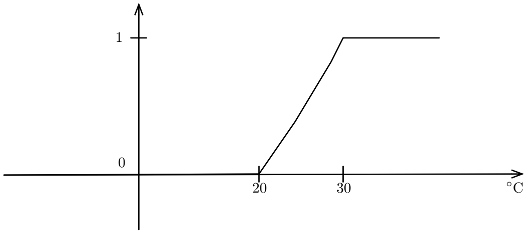

A fuzzy set of high temperatures

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- Differences between fuzziness and probability!

Residuated lattices

Definition

A complete residuated lattice: algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$

1. $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice
2. $\langle L, \otimes, 1\rangle$ is a commutative monoid
3. $\otimes$ and $\rightarrow$ satisfy adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$.

Examples on $[0, 1]$ and its subsets

$\mathbf{L} = \langle [0, 1], \min, \max, \otimes, \rightarrow, 0, 1\rangle$,

- Łukasiewicz: $a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$.
- Gödel (minimum): $a \otimes b = \min(a, b)$, $a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$
- Goguen (product): $a \otimes b = a \cdot b$, $a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ \frac{b}{a} & \text{otherwise.} \end{cases}$

Fuzzy sets ($\mathbf{L}$-sets)

An $\mathbf{L}$*-set* A *in a universe* X: mapping $A\colon X \to L$

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Operations with $\mathbf{L}$-sets

- *Intersection:* an $\mathbf{L}$-set $A \cap B$ such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$
- *Union:* an $\mathbf{L}$-set $A \cup B$ such that $(A \cup B)(x) = A(x) \vee B(x)$ for each $x \in X$

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Subsethood relation

– $A \subseteq B$ if $A(x) \leq B(x)$ for each x
– More generally: *subsethood degree*

$$S(A, B) = \bigwedge_{x \in X} A(x) \to B(x)$$

Example

- $A(x)$: x understands German.
- $B(x)$: x understands Dresden dialect.

	$A(x)$	$B(x)$	$A(x) \wedge B(x)$	$A(x) \vee B(x)$	$A(x) \otimes B(x)$	$A(x) \to B(x)$
x_1	0.1	0.2	0.1	0.2	0.0	1.0
x_2	1.0	0.8	0.8	1.0	0.8	0.8
x_2	0.6	0.5	0.5	0.6	0.1	0.9

In particular, we have

$$S(A, B) = \bigwedge_{x \in X} A(x) \to B(x) = 0.8$$

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They allow fuzzy reasoning about items, knowledge states, skills. . .

- If a student cannot use past tense then he cannot use irregular verbs as well.

Graded knowledge structures

Definition

Graded knowledge state on Y is graded (fuzzy) set K in Y. *Graded knowledge structure* on Y is any family $\mathcal{K} \subseteq \mathbf{L}^Y$ of graded knowledge states which contains $\emptyset$ and Y.

- Y ... set of problems/questions/items
- $K(y) \in \mathbf{L}$ ... the degree to which an individual in knowledge state K has mastered problem y

Definition

Graded knowledge space on Y is graded (fuzzy) knowledge structure $\mathcal{K}$ satisfying:

(i) if $K_i \in \mathcal{K}$, $i \in I$, then $\bigcup_{i \in I} K_i \in \mathcal{K}$ (closed under union),

(ii) if $K \in \mathcal{K}$ and $a \in L$, then $a^* \otimes K \in \mathcal{K}$ (closed under $\otimes$-multiplication).

First results on graded knowledge structures and graded knowledge spaces can be found in [1].

References

[1] Bartl E., Belohlavek R.: Knowledge spaces with graded knowledge states. Information Sciences 181(8)(2011), 1426-1439.

[2] Belohlavek, R.: Fuzzy Relational Systems: Foundations and Principles. Kluwer Academic Publishers, Norwell, USA, 2002.

[3] Klir G. J., Yuan B.: Fuzzy Sets and Fuzzy Logic. Theory and Applications. PrenticeHall, Upper Saddle River, NJ, 1995.