Semilattices with section switching involutions

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The concept of basic algebra was introduced as a common generalization of an MV-algebra and an orthomodular lattice. Remember that MV-algebras serve as an algebraic axiomatization of the so-called Łukasiewicz many-valued logics and orthomodular lattices form an algebraic counterpart of the logic of quantum mechanics. Hence, basic algebras form a common algebraic axiomatization of both logics mentioned above.

Recall that a **basic algebra** is an algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following identities

(B1) $x \oplus 0 = x$

(B2) $\neg\neg x = x$

(B3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

(B4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$, where $1 = \neg 0$.

Every basic algebra has its second face, namely $\mathscr{A} = (A; \oplus, \neg, 0)$ can be organized into a bounded lattice $(A; \vee, \wedge)$, where

$$x \vee y = \neg(\neg x \oplus y) \oplus y \quad \text{and} \quad x \wedge y = \neg(\neg x \vee \neg y),$$

whose order is given by

$$x \leq y \quad \text{if and only if} \quad \neg x \oplus y = 1.$$

Of course, $0 \leq x \leq 1$ for each $x \in A$. Moreover, this lattice $(A; \vee, \wedge)$ is endowed by a set $({}^{a})_{a \in A}$ of so-called **section antitone involutions**, i.e. for each $a \in A$ there exists a mapping $x \mapsto x^{a}$ of the interval $[a, 1]$ (called **section**) into itself such that

$$x^{aa} = x \quad \text{and} \quad x \leq y \Rightarrow y^{a} \leq x^{a} \quad \text{for all } x, y \in [a, 1].$$

This system $\mathscr{L}(\mathscr{A}) = (L; \vee, \wedge, ({}^{a})_{a \in L}, 0, 1)$ is called a **lattice with section antitone involutions** assigned to $\mathscr{A} = (A; \oplus, \neg, 0)$. Also conversely, having a bounded lattice with section antitone involutions $\mathscr{L} = (L; \vee, \wedge, ({}^{a})_{a \in L}, 0, 1)$, one can convert it into a basic algebra $\mathscr{A}(\mathscr{L}) = (L; \oplus, \neg, 0)$, where

$$\neg x = x^{0} \quad \text{and} \quad x \oplus y = (\neg x \vee y)^{y}.$$

Moreover, the assignments $\mathscr{A} \to \mathscr{L}(\mathscr{A})$ and $\mathscr{L} \to \mathscr{A}(\mathscr{L})$ are one-to-one correspondences, i.e. $\mathscr{A}(\mathscr{L}(\mathscr{A})) = \mathscr{A}$ and $\mathscr{L}(\mathscr{A}(\mathscr{L})) = \mathscr{L}$.

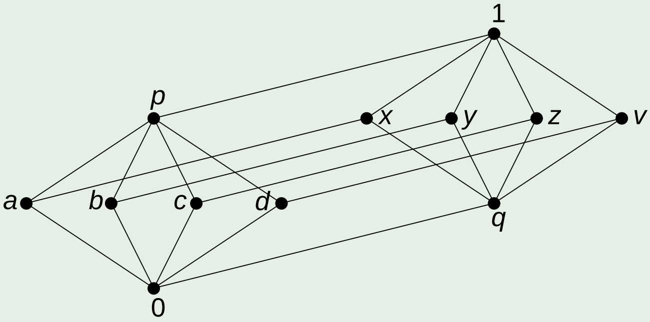
For MV-algebras, the underlying lattice can be induced in the same way as for basic algebras and for each element a, a mapping $x \mapsto \neg x \oplus a = x^a$ is a section antitone involution again.

For an orthomodular lattice $\mathscr{L} = (L; \vee, \wedge, ^{\perp}, 0, 1)$ and an arbitrary element $a \in L$, the mapping $x \mapsto x^a = x^{\perp} \vee a$ is an antitone involution in the section $[a, 1]$ which is, moreover, a section complementation.

This motivated us to investigate a more general case. Consider e.g. a bounded modular lattice $\mathscr{L} = (L; \vee, \wedge, 0, 1)$ with a complementation $x \mapsto x'$. Let $a \in L$ and $x \in [a, 1]$. It is well-known that $x' \vee a$ is a complement of x in section $[a, 1]$. Hence, complemented modular lattices can be also considered as lattices with section involutions and we can consider a logic induced by them. However, there is an essential difference from the above mentioned cases, see the following.

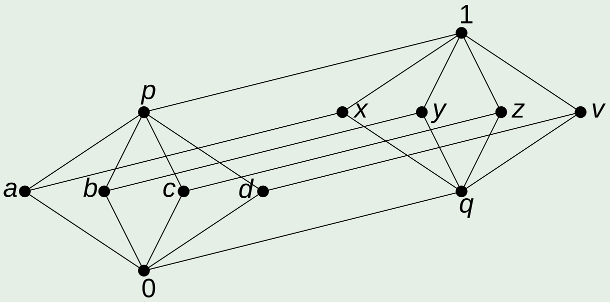
Example 1 (1/2)

Let $\mathscr{L} = (L; \vee, \wedge, 0, 1)$ be the modular lattice depicted in Fig. 1:



It is evident that $\mathscr{L}$ is isomorphic to the direct product of M_4 and a two-element chain C_2. Although every possible complementation in M_4 is antitone as well as the unique complementation in C_2, there exists a complementation in $\mathscr{L}$ which is not antitone (and hence it is not a direct product of any complementations in M_4 and C_2). We can get e.g. the following:

Example 1 (2/2)



t	0	a	b	c	d	x	y	z	v	p	q	1
t'	1	y	z	v	x	d	a	b	c	q	p	0

One can easily check that the complementation $t \mapsto t'$ is really an involution in $\mathscr{L}$ but it is not antitone:

$$a \leq x \quad \text{but} \quad a' = y \parallel d = x'.$$

Hence, although $\mathscr{L}$ is a complemented modular lattice, it is not orthocomplemented (with respect to the complementation by our choice).

It means that for complemented modular lattices we can study section involutions similarly as for basic algebras, MV-algebras and orthomodular lattices but we cannot assume that these involutions are antitone. Hence, for a conversion of complemented modular lattices into algebras similar to basic algebras, we have to consider lattices (or semilattices) whose section involutions need not be antitone but only switching the endpoints of the section.

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Consider a section $[a, 1]$ of an ordered set with greatest element 1. A mapping $x \mapsto x^a$ of $[a, 1]$ into itself is called a **section switching involution** if $x^{aa} = x$ for each $x \in [a, 1]$ and $a^a = 1, 1^a = a$. In general, we do not ask that this involution should be antitone; it only switches the endpoints of the section.

Now, we can consider a bounded lattice with section switching involutions $\mathscr{L} = (L; \vee, \wedge, (^a)_{a \in L}, 0, 1)$ and study what an algebra can be obtained by using a similar construction as that for basic algebras. For our purposes, we will consider only a semilattice since the operation meet is not applied in the construction of the operations of the new algebra.

Theorem 1

Let $\mathscr{S} = (S; \vee, (^a)_{a \in S}, 0, 1)$ be a bounded semilattice with section switching involutions. Define $\neg x = x^0$ and $x \oplus y = (x^0 \vee y)^y$. Then the algebra $\mathscr{A}(\mathscr{S}) = (S; \oplus, \neg, 0)$ assigned to $\mathscr{S}$ satisfies the following identities:

(P1) $\neg x \oplus x = 1$

(P2) $x \oplus 0 = x$

(P3) $\neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y$

(P4) $\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z = \neg(\neg(\neg(\neg y \oplus z) \oplus z) \oplus x) \oplus x.$

If, moreover, the involution $x \mapsto x^0$ is antitone, then $\mathscr{A}(\mathscr{S})$ satisfies the identity

(PA) $(\neg(x \oplus y) \oplus y) \oplus x = 1.$

Remark 1 (1/2)

(a) Every bounded semilattice $\mathscr{S} = (S; \vee, 0, 1)$ can be considered as a semilattice with section switching involutions. Namely, for each $a \in S$ one can define a switching involution on $[a, 1]$ as follows: $a^a = 1, 1^a = a$ and $x^a = x$ for each $x \in [a, 1], a \neq x \neq 1$.

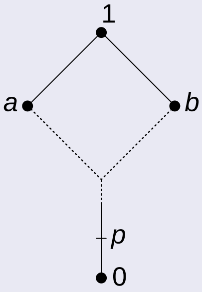
(b) If the involution $x \mapsto x^0$ is antitone, then $\mathscr{S} = (S; \vee, 0, 1)$ is in fact a lattice due to the DeMorgan laws because

$$x \wedge y = (x^0 \vee y^0)^0.$$

(c) There exist bounded semilattices with section switching involutions which are not lattices.

Remark 1 (2/2)

Such a semilattice $\mathscr{K}$, call a “kite” is visualized in Fig. 2:



It is an ordinal sum of an infinite chain C with the least element 0 and without a greatest element and a three element semilattice $\{a, b, 1\}$, i.e. $p \leq a, b$ for each $p \in C$. Then $\mathscr{K}$ is a $\vee$-semilattice which is not a lattice since $\inf\{a, b\}$ does not exists. Moreover, for each $c \in \mathscr{K}$ we define $c^c = 1, 1^c = c$ and $x^c = x$ for $x \in [c, 1]$, $c \neq x \neq 1$.

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We have shown that to every bounded semilattice $\mathscr{S}$ with section switching involutions we can assign an algebra $\mathscr{A}(\mathscr{S}) = (S; \oplus, \neg, 0)$ satisfying (P1)–(P4). We are going to show that algebras satisfying (P1)–(P4) are interesting for their own sake.

Definition 1

An algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities (P1)–(P4) will be called a **pseudo basic algebra**. If, moreover, $\mathscr{A}$ satisfies also the identity (PA), it will be called a **strict pseudo basic algebra**.

Since both pseudo basic algebras and strict pseudo basic algebras are determined by identities, their classes are in fact varieties.

Lemma 1

Every pseudo basic algebra satisfies the following identities:

(i) $0 \oplus x = x$

(ii) $\neg\neg x = x$

(iii) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

(iv) $\neg x \oplus (y \oplus x) = 1$, where $1 = \neg 0$

(v) $x \oplus \neg x = 1$

(vi) $1 \oplus x = 1 = x \oplus 1$.

Theorem 2

The axioms (P1)–(P4) are independent.

Our next task is to show that also conversely, every pseudo basic algebra can be organized into a bounded semilattice with section switching involutions.

Theorem 3

Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a pseudo basic algebra. Define $1 = \neg 0$, $x \vee y = \neg(\neg x \oplus y) \oplus y$ and for any $a \in A$, let $x^a = \neg x \oplus a$. Then

(a) $(A; \vee)$ is a join-semilattice with least element 0 and greatest element 1

(b) $x \leq y$ if and only if $\neg x \oplus y = 1$ is the induced order of the semilattice $(A; \vee)$

(c) for each $a \in A$ and $x \in [a, 1]$, the mapping $x \mapsto x^a = \neg x \oplus a$ is a section switching involution on the section $[a, 1]$.

If, moreover, $\mathscr{A}$ is a strict pseudo basic algebra, then $(A; \vee)$ is a lattice where $x \wedge y = \neg(\neg x \vee \neg y)$.

We can show that the assignment between pseudo basic algebras and bounded semilattices with section switching involutions is a one-to-one correspondence.

Theorem 4

Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a pseudo basic algebra, $\mathscr{S}(\mathscr{A})$ its assigned semilattice with section switching involutions. Then $\mathscr{A}(\mathscr{S}(\mathscr{A})) = \mathscr{A}$.
Let $\mathscr{S} = (S; \vee, (^a)_{a \in S}, 0, 1)$ be a bounded semilattice with section switching involutions, $\mathscr{A}(\mathscr{S})$ its assigned pseudo basic algebra. Then $\mathscr{S}(\mathscr{A}(\mathscr{S})) = \mathscr{S}$.

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In this section, we reveal several interesting properties of strict and/or commutative pseudo basic algebras.

Lemma 2

Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a pseudo basic algebra and $\mathscr{S}(\mathscr{A})$ its assigned semilattice. Then $\neg a \vee a = 1$ in $\mathscr{S}(\mathscr{A})$ if and only if $a \oplus a = a$ in $\mathscr{A}$.

As mentioned in (b) of Remark 1, if a pseudo basic algebra $\mathscr{A}$ is strict, then the assigned semilattice $\mathscr{S}(\mathscr{A})$ is a lattice $(A; \vee, \wedge)$ where $x \wedge y = \neg(\neg x \vee \neg y)$. In what follows, we will use this fact and $\mathscr{S}(\mathscr{A})$ will be called an **assigned lattice**.

Theorem 5

Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a strict pseudo basic algebra. Then $\neg$ is a complementation in the induced lattice $(A; \vee, \wedge, 0, 1)$ if and only if $\mathscr{A}$ satisfies the identity $x \oplus x = x$.

A pseudo basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ is called **commutative** if it satisfies the identity $x \oplus y = y \oplus x$ and $\mathscr{A}$ is called **associative** if it satisfies the identity $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.
An interesting connection is given by the following.

Theorem 6

(a) Every commutative pseudo basic algebra is strict.

(b) A pseudo basic algebra is an MV-algebra if and only if it is associative.

It was proved by M. Botur and R. Halaš that every finite commutative basic algebra is in fact an MV-algebra. Hence, it is a natural question if there really exist commutative pseudo basic algebras which are not basic algebras. The answer is positive also for a finite pseudo basic algebra.

Example 2 (1/2)

Consider the commutative pseudo basic algebra $\mathscr{A} = (\{0, a, b, c, d, e, 1\}; \oplus, \neg, 0)$, where the operations $\neg$ and $\oplus$ are given by the tables

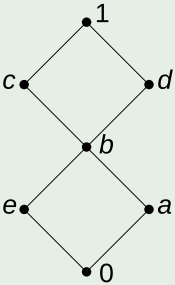
x	0	a	b	c	d	e	1
$\neg x$	1	c	b	a	e	d	0

$\oplus$	0	a	b	c	d	e	1
0	0	a	b	c	d	e	1
a	a	c	d	1	d	b	1
b	b	d	1	1	1	c	1
c	c	1	1	1	1	c	1
d	d	d	1	1	1	1	1
e	e	b	c	c	1	d	1
1	1	1	1	1	1	1	1

By Theorem 6 (a), $\mathscr{A}$ is strict and its induced lattice is visualized in Fig. 3.

Example 2 (2/2)

Fig. 3:



The section switching involutions are as follows:
In $[0,1]$, $x^0 = \neg x$.
In $[e,1]$ it is: $e^e = 1$, $b^e = c$, $c^e = b$, $d^e = d$, $1^e = e$.
In $[a,1]$ it is: $a^a = 1$, $b^a = d$, $d^a = b$, $c^a = c$, $1^a = a$.
In $[b,1]$ it is: $b^b = 1$, $c^b = d$, $d^b = c$ and $1^b = b$.
In $[c,1],[d,1]$ and $[1,1]$ it is determined uniquely. One can easily check that $x \mapsto x^e$ and $x \mapsto x^a$ are not antitone since e.g.

$$b \leq d \quad \text{but} \quad b^e = c \parallel d = d^e.$$

Hence, $\mathscr{A}$ cannot be a basic algebra.

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Now we turn our attention to congruence properties of pseudo basic algebras. Recall that an algebra $\mathscr{A}$ with a constant 1 is **weakly regular** if for any two congruences $\Theta, \Phi \in \mathrm{Con}\mathscr{A}$ we have

$$[1]_\Theta = [1]_\Phi \quad \text{implies} \quad \Theta = \Phi,$$

i.e. each congruence on $\mathscr{A}$ is uniquely determined by its 1-class. An algebra $\mathscr{A}$ is called **congruence regular** if for any two congruence $\Theta, \Phi \in \mathrm{Con}\mathscr{A}$ and any element $a \in A$ we have

$$[a]_\Theta = [a]_\Phi \quad \text{implies} \quad \Theta = \Phi,$$

i.e. every congruence $\mathscr{A}$ is uniquely determined by each class.
A variety $\mathscr{V}$ is called **weakly regular** or **congruence regular** if every $\mathscr{A} \in \mathscr{V}$ has this property.

Theorem 7

The variety of pseudo basic algebras is weakly regular. The variety of strict pseudo basic algebras is congruence regular.

An algebra $\mathscr{A}$ is **arithmetical** if its congruence lattice $\mathrm{Con}\mathscr{A}$ is distributive and $\Theta \circ \Phi = \Phi \circ \Theta$ for each two congruences $\Theta, \Phi \in \mathrm{Con}\mathscr{A}$ (i.e. congruences are permutable). A variety $\mathscr{V}$ is arithmetical if each $\mathscr{A} \in \mathscr{V}$ has this property.

Theorem 8

The variety of strict pseudo basic algebras is arithmetical.

As shown above, pseudo basic algebras are equivalent to bounded join-semilattices with section switching involutions. However, every section is a semilattice, i.e. it is again a bounded semilattice with section switching involutions. Hence, it can be converted into a pseudo basic algebra. How to organize its operations is shown in the following.

Theorem 9

Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a pseudo basic algebra, let $\leq$ be its induced order and $p \in A$. The section $[p, 1]$ can be organized into a pseudo basic algebra $([p, 1]; \oplus_p, \neg_p, p)$ as follows:

$$\neg_p x = \neg x \oplus p \quad \text{and} \quad x \oplus_p y = \neg(\neg x \oplus p) \oplus y$$

for $x, y \in [p, 1]$.

Appendix

Now, we show the number of non-isomorhic models of a given algebras of a given number of elements. Specifically, we focus on the MV-algebras, basic algebras (BA), pseudo basic algebras (PBA), commutative PBA and strict PBA. The numerical values in the following table were calculated using the program Prover9 and Mace4, see `http://www.cs.unm.edu/~mccune/mace4/`. The values in the fields marked with "–" values are not known, due to excessive computational complexity (time and/or memory).

	2	3	4	5	6	7	8	9	10
MV-algebras	1	1	2	1	2	1	3	2	2
BA	1	1	3	4	11	15	53	81	305
PBA	1	1	4	23	330	11516	–	–	–
com. PBA	1	1	2	1	2	2	5	3	5
strict PBA	1	1	3	5	25	164	4698	–	–

The table columns correspond to sizes of a given algebras (number of their elements). The first row contains the numbers of non-isomorphic MV-algebras. The second row contains the numbers of non-isomorphic basic algebras, ...

We can see from the table, for instance, that:

- there exists a 4-element basic algebra which is not an MV-algebra and that there exists a 4-element pseudo basic algebra which is not a basic algebra. It is 4-element chain, where $0 < a < b < 1$, $\neg 0 = 1$, $\neg a = a$, $\neg b = b$ and $\neg 1 = 0$, whence the corresponding involution is switching, but not antitone.
- there are 11516 non-isomorphic 7-element pseudo basic algebras, but only 164 of them are strict, and only two of them are commutative. Moreover, one of this two commutative pseudo basic algebras is not an MV-algebra (see Fig. 3).

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Thank you for your attention.