Minimal Solutions of Fuzzy Relational Equations

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Motivation

- medical motivation – we have (fuzzy) relation between pacients and their symptoms,
- we have known relation between symptoms and diseases established by an expert
- we want to infer which pacient have which disease (symptoms are obtained by medical personal)

$$(\text{Pacient} \times \text{Disease}) \circ (\text{Disease} \times \text{Symptoms}) = \text{Pacient} \times \text{Symptoms}$$

- $\circ$ is symbol for a (fuzzy) relation composition
- concept of fuzzy relational equation first appeared in Sanchez, E., 1976. Resolution of composite fuzzy relation equations. Information and Control 30, pp 38–48.
- this seminar is based on our article Bartl E. and Prochazka P. Do We Need Minimal Solutions of Fuzzy Relational Equations in Advance? Submitted to *IEEE Transactions on Fuzzy Systems*.

Fuzzy relational equation (FRE)

Fuzzy relation

(Binary) fuzzy relation R on a set X is a mapping $R : X \times X \to L$, where L is a residuated lattice.

Compositions of fuzzy relations $\circ$, $\triangleleft$

Let R and S be fuzzy relations on $X \times Y$ and $Y \times Z$, respectively. We define relations $(R \circ S)$, $(R \triangleleft S)$ on $X \times Z$ as

$$(R \circ S)(x, z) = \bigvee_{y \in Y} (R(x, y) \otimes S(y, z)),$$

$$(R \triangleleft S)(x, z) = \bigwedge_{y \in Y} (R(x, y) \to S(y, z)).$$

Fuzzy relational equation (FRE) cntd.

Fuzzy relational equation

Fuzzy relational equation is an expression

$$U \circ S = T,$$

where S and T are given fuzzy relations and U is unknown.

Solvability and the greatest solution

Solution to $U \circ S = T$ exists iff $(S \triangleleft T^{-1})^{-1}$ is its solution. If $(S \triangleleft T^{-1})^{-1}$ is a solution to $U \circ S = T$ then it is the greatest one.

Solution set structure

- $\mathrm{Sols} = \{R \in L^{X \times Y} | R \circ S = T\}$ is closed under suprema w.r.t. $\subseteq$
- **minimal solution** is a minimal item of Sols
- number of minimal solutions is exponentialy bounded from above by $2^{|X|}$
- there exists simple algorithm from P class to find a minimal solution

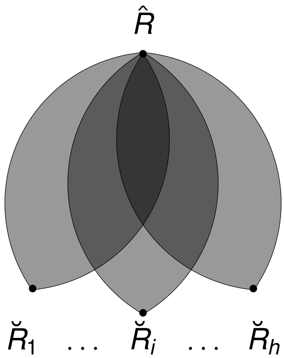
```
FRE_minsol <- function( S , T ){
  G <- FRE_greatest( S , T );
  prev <- null

  while( FRE_solution_p( G , S , T ) ){
    prev <- G
    decrease arbitrary non zero degree in G
  }

  return prev
}
```

Solution set structure cntd.

- $\langle \mathrm{Sols}, \subseteq \rangle$ is convex set: if $R_1, R_2 \in \mathrm{Sols}$ then $(\forall R) R_1 \subseteq R \subseteq R_2 \Rightarrow R \in \mathrm{Sols}$
- therefore $\langle \mathrm{Sols}, \subseteq \rangle$ can be represented as follows (union of intervals between minimal solutions and the greatest solution)



- disadvantage: we have to compute many duplicities (darker areas)

We use some well-known results

- we introduce an other representation of solution set avoiding previous disadvantage
- we use some well-known results from extremal set theory:
- Sperner's theorem
- Dilworth's theorem
- Hansel's theorem

Sperner's theorem

Sperner E. 1928. Ein Satz über Untermengen einer endlichen Menge. *Mathematische Zeitschrift* 27 (1): 544–548.

Antichain in an n-element set X – family of pairwise incomparable subsets of X

Sperner's theorem

Number of all elements of every antichain does not exceed

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Remark:

- longest antichain consits of $\lfloor \frac{n}{2} \rfloor$-element subsets of X

Dilworth's theorem

R. P. Dilworth 1948. A decomposition theorem for partially ordered sets. *Annals of Mathematics* Vol. 51, No. 1, January, 1950.

Independency of a set

Let have set $\langle P, \leq \rangle$ and some $S \subseteq 2^P$, S is **independent** if every two distinct elements of S are non-comparable, otherwise S is **comparable**.

Dilworth's theorem

Let every set of $k + 1$ elements of a partially ordered set P be dependent while at least one set of k element is independent. Then P is a set sum of k disjoint chains.

In other words:

- for every partially ordered set P there exists an antichain $A \subseteq P$ such that P can be decomposed into a family of $|A|$ disjoint chains

Christmas tree pattern

Christmas tree of order 1

$$0 \quad 1$$

Christmas tree of order 2

$$\begin{array}{ccc} & 10 & \\ 00 & 01 & 11 \end{array}$$

Christmas tree of order $n+1$, replace every row of christmas tree of order 2 which is in form:

$$\sigma_1\sigma_2\ldots\sigma_s$$

by

$$\begin{array}{ccccc} & \sigma_2 0 & \ldots & \sigma_s 0 & \\ \sigma_1 0 & \sigma_1 1 & \ldots & \sigma_{s-1} 1 & \sigma_s 1 \end{array}$$

The first of these rows is ommited when $s = 1$.

Christmas tree pattern

Example

```
                    101010
             101000 101001 101011
                    101100
             100100 100101 101101
             100010 100110 101110
      100000 100001 100011 100111 101111
                    110010
             110000 110001 110011
                    110100
             010100 010101 110101
             010010 010110 110110
      010000 010001 010011 010111 110111
                    111000
             011000 011001 111001
             001010 011010 111010
      001000 001001 001011 011011 111011
             001100 011100 111100
      000100 000101 001101 011101 111101
      000010 000110 001110 011110 111110
000000 000001 000011 000111 001111 011111 111111
```

Christmas tree pattern properties

- rows are formed by chains, neigboring items in a row differ in exactly one 1
- columns are antichains
- width of the *n*-order christmas tree is equal to $n + 1$
- height is equal to
$$\binom{n}{\lfloor \frac{n}{2} \rfloor}$$
- suppose we have a row like this $\ldots \sigma_{i-1}\sigma_i\sigma_{i+1} \ldots$, then item $\sigma_{i-1} \oplus \sigma_i \oplus \sigma_{i+1}$ lies in an previous row (above) in the same column, where $\oplus$ is symbol for exclusive or

Hansel's lemma

Hansel Georges 1966. Sur le nombre des fonctions boolénenes monotones de *n* variables. *Logique mathématique* C. R. Acad. Sc. Paris, t. 262

- we suppose Boolean non-decreasing function $f : \{0,1\}^n \to \{0,1\}$

Hansel's lemma

The minimal number $\Psi(n)$ of evaluations of f to establish all thresholds of f is

$$\Psi(n) = \binom{n}{\lfloor \frac{n}{2} \rfloor} + \binom{n}{\lfloor \frac{n}{2} \rfloor + 1}.$$

Problem to establish all solutions to Boolean relational equation may be translated to evaluation of corresponding Boolean non-decreasing function.

(n, k)-Christmas tree pattern

Let $n \geq 2$, $k \geq 1$ be natural numbers, and L be a k-element chain

$$\{0 = a_0, a_1, \dots, a_{k-2}, a_{k-1} = 1\},$$

where $a_i = \frac{i}{k-1}$ for $i = 0, \dots, k-1$. We define (n, k)-Christmas tree pattern as a collection of all fuzzy sets from L^n arranged in the following way: for $n = 1$ we define $(1, k)$-Christmas tree pattern:

$$0 \quad a_2 \quad \dots \quad a_{k-2} \quad 1$$

For $n > 1$, we take every row $\sigma_1 \dots \sigma_s$ of the $(n-1, k)$-Christmas tree pattern and replace it by $\min\{s, k\}$ rows. If $s < k$ then:

$$\begin{array}{cccccccccc}
 & & & \sigma_s\,0 & \dots & \sigma_s\,a_{k-s} & & & & \\
 & & & & & & \ddots & & & \\
 & & \sigma_3\,0 & \dots & \sigma_3\,a_{k-4} & \sigma_3\,a_{k-3} & \dots & \sigma_s\,a_{k-3} & & \\
 & \sigma_2\,0 & \sigma_2\,a_1 & \dots & \sigma_2\,a_{k-3} & \sigma_2\,a_{k-2} & \sigma_3\,a_{k-2} & \dots & \sigma_s\,a_{k-2} & \\
\sigma_1\,0 & \sigma_1\,a_1 & \sigma_1\,a_2 & \dots & \sigma_1\,a_{k-2} & \sigma_1\,1 & \sigma_2\,1 & \sigma_3\,1 & \dots & \sigma_s\,1
\end{array}$$

(n,k)-Christmas tree cntd.

Otherwise, we proceed as in the previous case until we obtain triangular pattern with the first row being singleton $\sigma_k 0$, then we continue as follows:

$$
\begin{array}{ccccccccccc}
 & & & & \sigma_k 0 & \ldots & \sigma_s 0 & & & & \\
 & & & \sigma_{k-1} 0 & \sigma_{k-1} a_1 & \sigma_k a_1 & \ldots & \sigma_s a_1 & & & \\
 & & \iddots & \vdots & \vdots & \vdots & \ddots & & \ddots & & \\
 & \sigma_2 0 & \ldots & \sigma_2 a_{k-1} & \sigma_2 a_{k-2} & \sigma_3 a_1 & \ldots & \sigma_k a_{k-2} & \ldots & \sigma_s a_{k-2} & \\
\sigma_1 0 & \sigma_1 a_1 & \ldots & \sigma_1 a_{k-2} & \sigma_1 1 & \sigma_2 1 & \ldots & \sigma_{k-1} 1 & \sigma_k 1 & \ldots & \sigma_s 1
\end{array}
$$

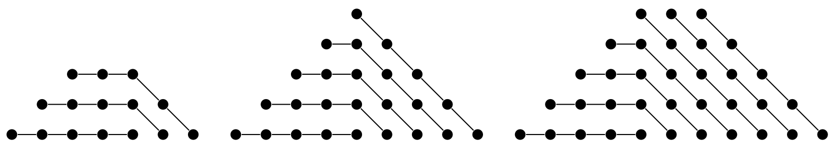


Figure : Construction of (n,k)-Christmas tree pattern for $s<k$ (left), $s=k$ (center) and $s>k$ (right).

(n, k)-Christmas tree example

Example

For tree-element chain $L = \{0, 0.5, 1\}$ and $n = 2$ we get:

		1.0 0.0		
	0.5 0.0	0.5 0.5	1.0 0.5	
0.0 0.0	0.0 0.5	0.0 1.0	0.5 1.0	1.0 1.0

For if $n = 3$, we obtain $(3, 3)$-Christmas tree pattern:

		1.0 0.0 0.0	1.0 0.0 0.5	1.0 0.0 1.0		
			1.0 0.5 0.0			
		0.5 0.5 0.0	0.5 0.5 0.5	1.0 0.5 0.5		
	0.5 0.0 0.0	0.5 0.0 0.5	0.5 0.0 1.0	0.5 0.5 1.0	1.0 0.5 1.0	
		0.0 1.0 0.0	0.5 1.0 0.0	1.0 1.0 0.0		
	0.0 0.5 0.0	0.0 0.5 0.5	0.0 1.0 0.5	0.5 1.0 0.5	1.0 1.0 0.5	
0.0 0.0 0.0	0.0 0.0 0.5	0.0 0.0 1.0	0.0 0.5 1.0	0.0 1.0 1.0	0.5 1.0 1.0	1.0 1.0 1.0

Properties of (n, k)-Christmas tree

- rows are again chains and columns are antichains
- width of (n, k)-Christmas tree is $n(k-1)+1$
- neighboring items of a row differs exactly by $|\sigma_i| - |\sigma_{i-1}| = \frac{1}{k-1}$

Decomposition of solution set

- we use denotation:

$$\binom{a}{b} = \begin{cases} \frac{a!}{b!\cdot(a-b)!}, & \text{for } a \geq b, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem

The set of all solutions to fuzzy relational equation $X \circ S = T$ defined on k-element chain is decomposable to a family of at most $h(n,k)$ disjoint chains, where

$$h(n,k) = \sum_{j=0}^{n} (-1)^j \cdot \binom{n}{j} \cdot \binom{[\frac{n}{2}](k-1) - jk + n - 1}{n-1},$$

and

$$[\tfrac{n}{2}] = \max\{|R|; R \in L^X, |X| = n, |R| \leq \tfrac{n}{2}\}.$$

Decomposition of solution set

- for L being a 2-element chain, the upper bound $h(n,k)$ from the previous theorem equals to the cardinality of the largest Sperner family over n-element set, i.e., the largest antichain defined on n-element set

Corollary

We have

$$h(n,2) = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Identifying All Solutions in (n, k)-Christmas Tree Pattern

Theorem says that there are at most $h(n, k)$ disjoint chains of solutions to a given fuzzy relational equation but yet we do not know how to identify them in (n, k)-Christmas tree pattern. In this section we introduce a method of finding all solutions in the pattern.

Reduction to Inclusion Set Cover Problem

Inclusion Set Cover Problem (ISCP) as optimization problem

Let have a set M, some $C \subseteq 2^M$ such that $\cup C = M$. Our goal is to find minimal (w.r.t. $\subseteq$) $C' \subseteq C$ such that $\cup C' = M$. Cost function is defined as

$$\mathrm{cost}(C') = \begin{cases} 1, & \text{if } C' \text{ is } \subseteq\text{-minimal solution,} \\ 2, & \text{otherwise.} \end{cases}$$

Minimal solution to FRE (MINSOL)

Is problem to find an arbitrary $\subseteq$-minimal solution of

$$U \circ S = T$$

For solution R *cost* function is defined as follows

$$\mathrm{cost}(R) = \begin{cases} 1, & \text{if } R \text{ is } \subseteq\text{-minimal solution,} \\ 2 & \text{otherwise.} \end{cases}$$

An approximation factor preserving reduction

Recall:

An approximation factor preserving reduction

Approximation factor preserving reduction of optimization problems Π_1 to Π_2 consists of two polynomial time computable functions, f and g, satisfying: (i) for each instance I_1 of Π_1, $f(I_1) \in N_2$ (i.e., $f(I_1)$ is an instance of Π_2) and $\mathrm{opt}_{\Pi_2}(f(I_1)) \leq \mathrm{opt}_{\Pi_1}(I_1)$; (ii) for each $\alpha \in \mathrm{sol}_{\Pi_2}(f(I_1))$, $g(I_1, \alpha) \in \mathrm{sol}_{\Pi_1}(I_1)$ and $\mathrm{cost}_{\Pi_1}(I_1, g(I_1, \alpha)) \leq \mathrm{cost}_{\Pi_2}(f(I_1), \alpha)$.

Reduction of MINSOL to ISCP

Theorem

There is approximation factor preserving reduction of MINSOL to ISCP.

Reduction:
We define mapping f which assigns to the given equation $U \circ S = T$ the pair $\langle M, C \rangle$ consisting of the set $M = Y$ and the collection $C = \{C_x \subseteq M \mid x \in X\}$ such that

$$y \in C_x \text{ iff } \hat{R}(x) \otimes S(x, y) = T(y).$$

For a $C' \in \mathrm{sol}_{\mathrm{ISCP}}(\langle M, C \rangle)$ we define a mapping g such that

$$(g(U \circ S = T, C))(x) = \begin{cases} \hat{R}(x), & \text{if } C_x \in C, \\ 0, & \text{otherwise.} \end{cases}$$

Properties of reduction

- function f computes for a given equation $U \circ S = T$ corresponding instance $f(U \circ S = T)$ of ISCP, while the mapping g for a feasible solution of $f(U \circ S = T)$ computes a solution to $U \circ S = T$
- we define for $\langle M, \{C_1, \ldots, C_n\}\rangle = f(U \circ S = T)$ the Boolean function with n variables:

$$b(v_1, \ldots, v_n) = \begin{cases} 1, & \text{if } \{C_i \mid v_i = 1\} \text{ covers } M, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

- obviously, function b is non-decreasing
- we compute all feasible solutions of $f(U \circ S = T)$ by evaluating the least possible number of inputs of corresponding (ordinary) Christmas tree pattern
- these solutions can be then transformed by function g to solutions of $U \circ S = T$

Example

Example

We suppose 3-element chain L, fuzzy relations $U \in L^X$, $S \in L^{X \times Y}$, $T \in L^Y$, where $X \in \{x_1, \dots, x_5\}$, $Y \in \{y_1, \dots, y_4\}$, $U \circ S = T$ with

$$S = \begin{pmatrix} 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \end{pmatrix}, \tag{2}$$

and

$$T = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 \end{pmatrix}. \tag{3}$$

The greatest solution to $U \circ S = T$ is

$$\hat{R} = (S \triangleleft T^{-1})^{-1} = \begin{pmatrix} 0.5 & 0.5 & 1 & 1 & 1 \end{pmatrix}. \tag{4}$$

$\langle M, \mathcal{S} \rangle = f(U \circ X = T)$ such that $M = Y = \{y_1, y_2, y_3, y_4\}$ and $\mathcal{S} = \{C_{x_1}, C_{x_2}, C_{x_3}, C_{x_4}, C_{x_5}\}$ where $C_{x_1} = \{y_1, y_2, y_4\}$, $C_{x_2} = \{y_1, y_3, y_4\}$, $C_{x_3} = C_{x_4} = C_{x_5} = \{y_1, y_2, y_3, y_4\}$.

Example - solution

Example

C_{x_1}	1	1	0	1
C_{x_2}	1	0	1	1
C_{x_3}	1	1	1	1
C_{x_4}	1	1	1	1
C_{x_5}	1	1	1	1
M	1	1	1	1

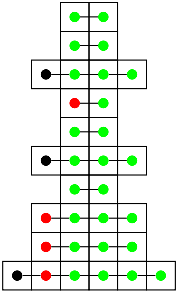
Table : Characteristic vectors of $C_{x_1}, \ldots, C_{x_5}$, and M.

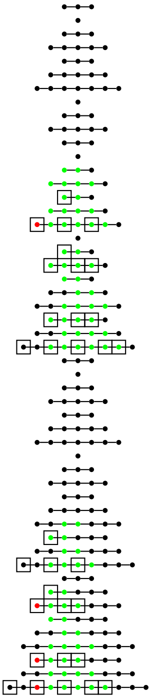
From 4 we can easily see minimal soulutons of ISCP – $\{C_{x_1}, C_{x_2}\}$, $\{C_{x_3}\}$, $\{C_{x_4}\}$, and $\{C_{x_5}\}$. They might be obtained by a Hansel-based algorithm or by Bartl's algorithm on a greatly reduced data (nk-Christmas tree versus Christmas tree) and transformed back to MINSOL point of view by g mapping.

Example - solution cntd.

Example

$$g(U \circ S = T, \{C_{x_1}, C_{x_2}\}) = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 \end{pmatrix},$$
$$g(U \circ S = T, \{C_{x_3}\}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$
$$g(U \circ S = T, \{C_{x_4}\}) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$
$$g(U \circ S = T, \{C_{x_5}\}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$





Thank you for your attention!