Saturated and supremal directoids

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Introduction

It is superfluous to recall how important partially ordered sets, and in particular directed posets, are for the whole of mathematics. However, unlike other equally fundamental mathematical structures, such as groups or Boolean algebras, posets and directed posets are *relational structures*, not algebras, whence they do not lend themselves to be the objects of common algebraic constructions like quotients, products, subalgebras and the like. In fact, insofar as they exist at all for relational structures, these constructions admit of several competing variants, none of which enjoys a universal acclaim, and are generally recognised as more cumbersome and less efficient than in the algebraic case. In order to enable such algebraic constructions with ordered sets, J. Ježek and R. Quackenbush introduced the notion of *directoid*.

Basic notions

Recall that a partially ordered set (poset) $\mathbf{A} = \langle A, \leq \rangle$ is said to be *directed* in case any two $a, b \in A$ have a common upper bound, i.e. in case the upper corner $U(x, y) = \{z \in A : x, y \leq z\}$ is nonempty. Of course, if $\mathbf{A}$ has a greatest element 1, then it is directed. An *antitone involution* on a poset $\mathbf{A} = \langle A, \leq \rangle$ is a unary operation $'$ s.t., for any $a \in A$, $(a')' = a$, and if $a \leq b$ in $\mathbf{A}$, then $b' \leq a'$. $(a')'$ will be shortened to a'' hereafter. It is evident that, whenever a poset with antitone involution $\mathbf{D}$ has a greatest element 1, then it contains a smallest element too, namely, $1'$. In place of $1'$, we denote such an element by 0. Furthermore, observe that if $a \vee b$ exists in D, then the infimum $a' \wedge b' = (a \vee b)'$ also exists in D.

Directoids

A *directoid* (*commutative directoid*, in the usage of Ježek and Quackenbush) is a groupoid $\mathbf{D} = \langle D, \sqcup \rangle$ that satisfies the following axioms:

(D1) $x \sqcup x \approx x$;

(D2) $x \sqcup y \approx y \sqcup x$;

(D3) $x \sqcup ((x \sqcup y) \sqcup z) \approx (x \sqcup y) \sqcup z$.

If $\mathbf{D} = \langle D, \sqcup \rangle$ is a directoid, the partial order relation $\leq$ defined for all $a, b \in D$ by

$$a \leq b \quad \text{iff} \quad a \sqcup b = b$$

will be called *the order induced by* $\sqcup$ on $\mathbf{D}$, or its *induced order*, while the poset $\langle D, \leq \rangle$ will be called the *induced poset* of $\mathbf{D}$.

Involutive directoids

Any directed poset $\mathbf{A} = (A, \leq)$ can be turned into a directoid as follows:

- if $a \leq b$, then we set $a \sqcup b = b \sqcup a = b$;
- if a and b are incomparable, then $a \sqcup b = b \sqcup a$ is an arbitrary common upper bound of a, b.

The resulting directoid $\mathcal{D}(\mathbf{A}) = \langle A, \sqcup \rangle$ is such that its induced order coincides with the partial ordering of $\mathbf{A}$. In other words, the directoid fully retrieves the ordering of the original poset. However, it may happen that two incomparable elements $a, b \in A$ have a supremum $a \vee b$ that does not coincide with our choice of $a \sqcup b$. And this is a shortcoming under several respects. It is therefore our aim to prove that, for directed posets $\mathbf{A} = \langle A, \leq \rangle$ that admit an antitone involution, we can get around this difficulty.
An *involutive directoid* is an algebra $\mathbf{D} = \langle D, \sqcup, ' \rangle$ of type $(2, 1)$ s.t. $\langle D, \sqcup \rangle$ is a directoid and $'$ is an antitone involution on the induced poset of $\mathbf{D}$. Observe that:

Involutive directoids

Proposition

The class of involutive directoids is a variety.

Recall that two elements a, b of a directoid $\mathbf{D}$ are said to be *orthogonal* in case $a \leq b'$, or equivalently $b \leq a'$.

Theorem 1

Let $\mathbf{D} = \langle D, \sqcup, ' \rangle$ be an involutive directoid, and let $\leq$ be its induced order. The following conditions are equivalent:

1. for all $a, b \in D$, if a, b are orthogonal, then $a \sqcup b = a \vee b$;
2. $\mathbf{D}$ satisfies the identity

$$\text{(D4)} \quad (((x \sqcup z) \sqcup (y \sqcup z)')' \sqcup (y \sqcup z)') \sqcup z' \approx z'.$$

Bounded involutive directoids

The previous correspondence assumes a particularly interesting form when the poset in question is bounded, and the type includes two constants denoting the bounds. A case in point is given by *effect algebras*, which play a noteworthy role in quantum logic — in fact, they can be presented as bounded posets equipped with an antitone involution, such that the supremum $a \vee b$ exists for orthogonal elements a, b. We have that:

Corollary

Let $\mathbf{A} = \langle A, \leq, ', 0, 1\rangle$ be a bounded poset with antitone involution. The following conditions are equivalent:

1. *For $a, b \in A$, $a \vee b$ exists whenever a, b are orthogonal.*
2. *$\mathcal{D}(\mathbf{A}) = \langle A, \sqcup, ', 0, 1\rangle$ satisfies (D2)–(D4) and*

$$\text{(D5) } x \sqcup 0 \approx x.$$

Corollary entails that bounded involutive directoids, such that $a \vee b$ exists for orthogonal elements a, b, are completely characterised by the equations (D2)–(D5), and therefore form a variety of type $(2, 1, 0, 0)$.

Involutive directoids

Given an involutive directoid $\mathbf{D} = \langle D, \sqcup, ' \rangle$, we define

$$x \sqcap y := (x' \sqcup y')'.$$

It is not difficult to verify that $\langle D, \sqcap \rangle$ is again a directoid whose induced order is dual to the induced order of $\mathbf{D}$. Moreover, the absorption laws

$$x \sqcap (x \sqcup y) \approx x \quad \text{and} \quad x \sqcup (x \sqcap y) \approx x,$$

are satisfied. Therefore, since $x \leq x \sqcup y$, we have $x \sqcap (x \sqcup y) = x$. And since $x \sqcap y \leq x$, we also have $x \sqcup (x \sqcap y) = x$. Thus, we obtain the following theorem.

Theorem 2

Any variety of involutive directoids is congruence distributive, with majority term

$$M(x, y, z) := ((x \sqcap y) \sqcup (y \sqcap z)) \sqcup (x \sqcap z).$$

In the absence of the involution, Theorem 2 fails, because semilattices (a subvariety of directoids) satisfy no nontrivial lattice identity.

Complemented directoids

Let us call a bounded involutive directoid *complemented* in case it satisfies the equation $x \sqcup x' \approx 1$. Note that the class of complemented directoids satisfying Theorem 1 forms a variety that includes, for example, orthomodular lattices.

Recall that an algebra $\mathbf{A}$ with a distinguished element 1 is called 1-*regular* whenever any congruence on $\mathbf{A}$ is entirely determined by its 1-class: namely, for $\theta, \phi \in \mathrm{Con}(\mathbf{A})$, whenever $1/\theta = 1/\phi$, then $\theta = \phi$. Also, a variety $\mathcal{V}$ is said to be 1-*regular* if any algebra in $\mathcal{V}$ is 1-regular.

Theorem 3

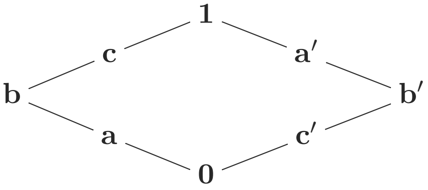
The variety of complemented directoids satisfying the equation

$$(PL)\ x \sqcup (x' \sqcap y) = x \sqcup y$$

is 1-regular.

Complemented directoids

(PL) is crucial in proving Theorem 3. The lattice



is a directoid where $x \sqcup y = x \vee y$, but it fails to be 1-regular. In fact, consider the congruences $\theta(b,c)$ and $\theta(a,b)$. We have that $1/\theta(b,c) = 1/\theta(a,b)$, yet $\theta(b,c) \neq \theta(a,b)$.

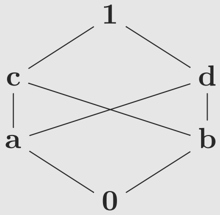
Saturated and supremal directoids

We have seen that there are directoids where $a \sqcup b = a \vee b$, at least for orthogonal or comparable elements a, b. Now we show that the classes of directoids where $x \sqcup y$ is minimal in the upper corner $U(x, y)$, or where $x \sqcup y = x \vee y$ in case $x \vee y$ exists, have a special significance. To this aim we introduce the following notions. A directoid $\mathbf{D} = \langle D, \sqcup \rangle$ is called *saturated* if $x \sqcup y$ is minimal in $U(x, y)$. $\mathbf{D}$ is *supremal* if $x \sqcup y = x \vee y$ in case $x \vee y$ exists.

Saturated and supremal directoids

Example

Consider the following ordered set:



If we set $a \sqcup b = c$ or $a \sqcup b = d$, and for $\{x, y\} \neq \{a, b\}$ we take $x \sqcup y = x \vee y$, then it is a saturated directoid. However, upon setting $a \sqcup b = 1$, on the same ordered set, the resulting directoid is no longer saturated, since 1 is not minimal in $U(a, b)$, even though it is still trivially supremal, because $a \vee b$ does not exist.

Saturated directoids

Note that every saturated directoid is supremal. In fact, if $x \vee y$ exists, then it is minimal in $U(x, y)$, whence $x \sqcup y = x \vee y$. The previous example shows that the converse is not true.

Theorem 4

A directoid $\mathbf{D} = \langle D, \sqcup \rangle$ is saturated if and only if it satisfies the quasi-identity:

$$\text{(Q)}\ (x \sqcup z \approx z \approx y \sqcup z) \quad \& \quad (z \sqcup (x \sqcup y) \approx x \sqcup y) \quad \Rightarrow \quad z \approx x \sqcup y.$$

Observe that the quasi-identity (Q) is in fact equivalent to the condition

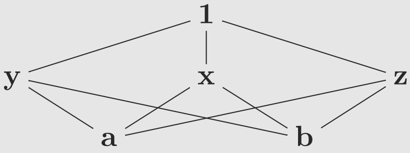
$$x, y \leq z \leq x \sqcup y \quad \Rightarrow \quad z \approx x \sqcup y.$$

By Theorem 4, the class of saturated directoids is a quasivariety. The next example shows that it is not a variety, because it is not closed under quotients.

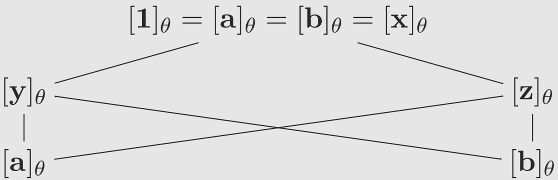
Saturated directoids

Example

Let $\mathbf{D}$ be the directoid given by the following diagram:



where $a \sqcup b = x$, and $p \sqcup q = p \vee q$ for the remaining elements. $\mathbf{D}$ is a saturated directoid. Consider the congruence $\theta(x, 1)$. Then, we obtain the quotient



where $[a]_\theta \sqcup [b]_\theta$ is not minimal in $U\left([a]_\theta, [b]_\theta\right)$.

Saturated and involutive directoids

Note that the variety of join semilattices is a nontrivial class strictly included in the quasivariety of saturated directoids. For involutive directoids, we can provide a sufficient condition for saturation formulated in the form of an identity.

Theorem 5

Let $\mathbf{D} = \langle D, \sqcup, ' \rangle$ be an involutive directoid. If $\mathbf{D}$ satisfies

$$\text{(D6)} \quad ((x \sqcap ((x \sqcup y) \sqcap z)) \sqcup (y \sqcap ((x \sqcup y) \sqcap z))) \sqcup z \approx z$$

then $\mathbf{D}$ is saturated.

The converse of Theorem 5, however, does not hold. Observe that the variety of involutive directoids satisfying (D6) contains all the involutive lattices. We can also characterise the quasivariety of supremal directoids.

Supremal directoids

Theorem 6

A directoid is supremal if and only if it satisfies the quasiequation

$$x, y \leq w \quad \& \quad w \leq x \sqcup y \quad \& \quad x, y, w \leq z \quad \Rightarrow \quad w \approx x \sqcup y.$$

Let us note that the quasi-identity of Theorem 6 can be easily expressed as a quasi-identity in the language of directoids.

Strong amalgamation property

A *V-formation* is a tuple $(\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2, i, j)$ such that $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2$ are similar algebras, and $i : \mathbf{A} \to \mathbf{B}_1, j : \mathbf{A} \to \mathbf{B}_2$ are embeddings. A class $\mathcal{K}$ of similar algebras is said to have the *amalgamation property* if for every V-formation with $\mathbf{A}, \mathbf{B_1}, \mathbf{B}_2 \in \mathcal{K}$ and $A \neq \emptyset$ there exists an algebra $\mathbf{D} \in \mathcal{K}$ and embeddings $h : \mathbf{B}_1 \to \mathbf{D}, k : \mathbf{B}_2 \to \mathbf{D}$ such that $k \circ j = h \circ i$. In such an event, we also say that k and h *amalgamate* the V-formation $(\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2, i, j)$. $\mathcal{K}$ is said to have the *strong amalgamation property* if, in addition, such embeddings can be taken s.t. $k \circ j\,(\mathbf{A}) = h\,(\mathbf{B}_1) \cap k\,(\mathbf{B}_2)$.

$$\begin{tikzcd}
 & \mathbf{B}_2 \arrow[dr, hook', dashed, "k"] & \\
\mathbf{A} \arrow[ur, hook, "j"] \arrow[dr, hook', "i"'] & & \mathbf{D} \\
 & \mathbf{B}_1 \arrow[ur, hook, dashed, "h"'] &
\end{tikzcd}$$

Strong amalgamation property

Amalgamations were first considered for groups by Schreier in the form of amalgamated free products. The general form of the AP was first formulated by Fraïsse, and the significance of this property to the study of algebraic systems was further demonstrated in Jónsson's pioneering work on the topic. The added interest in the AP for algebras of logic is due to its relationship with various syntactic interpolation properties.

Theorem 7

The varieties of directoids, bounded directoids, involutive directoids, bounded involutive directoids, and complemented directoids have the strong amalgamation property.

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Thanks

Thank you for your attention!