Algebras assigned to ternary systems

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Introduction

In [2] and [3], there were shown that to certain relational systems $\mathcal{A} = (A; R)$, where R is a binary relation on $A \neq \emptyset$, there can be assigned a certain groupoid $\mathcal{G}(A) = (A; \circ)$ which captures the properties of R. Namely, $x \circ y = y$ if and only if $(x, y) \in R$.

Hence, there arises the natural question if a similar way can be used for ternary relational systems and algebras with one ternary relation.

In the following let A denote a fixed arbitrary non-empty set.

Basic notions

Definition

Let T be a ternary relation on A and $a, b \in A$. The set

$$Z_T(a,b) := \{x \in A \,|\, (a,x,b) \in T\}$$

is called the **centre of** (a,b) **with respect to** T. The ternary relation T on A is called **centred** if $Z_T(a,b) \neq \emptyset$ for all elements $a, b \in A$.

Definition

Let T be a ternary relation on A and $a, b, c \in A$. The set

$$M_T(a,b,c) := Z_T(a,b) \cap Z_T(b,c) \cap Z_T(c,a)$$

will be called the **median of** (a,b,c) **with respect to** T.

Basic notions

Now we show that to every centred ternary relation there can be assigned ternary operations.

Definition

Let T be a centred ternary relation on A and t a ternary operation on A satisfying

$$t(a,b,c) \begin{cases} = b & \text{if } (a,b,c) \in T \\ \in Z_T(a,c) & \text{otherwise.} \end{cases}$$

Such an operation t is called **assigned to** T.

Remark

By definition, if T is a centred ternary relation on A and t assigned to T then $(a, t(a,b,c), c) \in T$ for all $a, b, c \in A$.

Lemma

Let T be a centred ternary relation on A and t an assigned operation. Let $a, b, c \in A$. Then $(a, b, c) \in T$ if and only if $t(a, b, c) = b$.

Proof

By Definition 3, if $(a, b, c) \in T$ then $t(a, b, c) = b$. Conversely, assume $(a, b, c) \notin T$. Then $t(a, b, c) \in Z_T(a, c)$. Now $t(a, b, c) = b$ would imply $(a, b, c) = (a, t(a, b, c), c) \in T$ contradicting $(a, b, c) \notin T$. Hence $t(a, b, c) \neq b$.

Example . . .

Theorem

A ternary operation t on A is assigned to some centred ternary relation T on A if and only if it satisfies the identity

$$t(x, t(x, y, z), z) = t(x, y, z). \tag{I1}$$

Proof

Let $a, b, c \in A$.
Assume that T is a ternary relation on A and t an assigned operation. If $(a, b, c) \in T$ then $t(a, b, c) = b$ and hence $t(a, t(a, b, c), c) = t(a, b, c)$. If $(a, b, c) \notin T$ then $t(a, b, c) \in Z_T(a, c)$ and hence $(a, t(a, b, c), c) \in T$ which yields $t(a, t(a, b, c), c) = t(a, b, c)$. Thus t satisfies identity (I1).
Conversely, assume $t : A^3 \to A$ satisfies (I1) and define $T := \{(x, y, z) \in A^3 \mid t(x, y, z) = y\}$. If $(a, b, c) \in T$ then $t(a, b, c) = b$ and, if $(a, b, c) \notin T$ then $(a, t(a, b, c), c) \in T$ whence $t(a, b, c) \in Z_T(a, c)$, i. e. t is assigned to T.

Properties of ternary relations

Further, we get a characterization of some important properties of ternary relations by means of identities of their assigned operations.

Definition

Let T be a ternary relation on A. We call T

- **reflexive** if $|\{a,b,c\}| \leq 2$ implies $(a,b,c) \in T$;
- **symmetric** if $(a,b,c) \in T$ implies $(c,b,a) \in T$;
- **antisymmetric** if $(a,b,a) \in T$ implies $a = b$;
- **cyclic** if $(a,b,c) \in T$ implies $(b,c,a) \in T$;
- R-**transitive** if $(a,b,c), (b,d,e) \in T$ implies $(a,d,e) \in T$;
- t_1-**transitive** if $(a,b,c), (a,d,b) \in T$ implies $(d,b,c) \in T$;
- t_2-**transitive** if $(a,b,c), (a,d,b) \in T$ implies $(a,d,c) \in T$;
- R-**symmetric** if $(a,b,c) \in T$ implies $(b,a,c) \in T$;
- R-**antisymmetric** if $(a,b,c), (b,a,c) \in T$ implies $a = b$;
- **non-sharp** if $(a,a,b) \in T$ for all $a, b \in A$;
- **cyclically transitive** if $(a,b,c), (a,c,d) \in T$ implies $(a,b,d) \in T$.

Theorem 1/3

Let T be a centred ternary relation on A and t an assigned operation. Then (i) – (xi) hold:
(i) T is reflexive if and only if t satisfies the identities

$$t(x, x, y) = t(y, x, x) = t(y, x, y) = x.$$

(ii) T is symmetric if and only if t satisfies the identity

$$t(z, t(x, y, z), x) = t(x, y, z).$$

(iii) T is antisymmetric if and only if t satisfies the identity

$$t(x, y, x) = x.$$

(iv) T is cyclic if and only if t satisfies the identity

$$t(t(x, y, z), z, x) = z.$$

Theorem 2/3

(v) T is R-transitive if and only if t satisfies the identity

$$t(x, t(t(x, y, z), u, v), v) = t(t(x, y, z), u, v).$$

(vi) T is t_1-transitive if and only if t satisfies the identity

$$t(t(x, u, t(x, y, z)), t(x, y, z), z) = t(x, y, z).$$

(vii) T is t_2-transitive if and only if t satisfies the identity

$$t(x, t(x, u, t(x, y, z)), z) = t(x, u, t(x, y, z)).$$

(viii) T is R-symmetric if and only if t satisfies the identity

$$t(t(x, y, z), x, z) = x.$$

Theorem 3/3

(ix) If t satisfies the identity

$$t(t(x,y,z),x,z) = t(x,y,z)$$

then T is R-antisymmetric.
(x) T is non-sharp if and only if t satisfies the identity

$$t(x,x,y) = x.$$

(xi) T is cyclically transitive if and only if t satisfies the identity

$$t(x,t(x,y,t(x,z,u)),u) = t(x,y,t(x,z,u)).$$

Centred ternary relational system

By a **ternary relational system** is meant a couple $\mathcal{T} = (A; T)$ where T is a ternary relation on A. $\mathcal{T}$ is called **centred** if T is centred. As shown above, to every centred ternary relational system $\mathcal{T} = (A; T)$ there can be assigned an algebra $\mathcal{A}(T) = (A; t)$ with one ternary operation $t : A^3 \to A$ such that t is assigned to T. Now, we can introduce an inverse construction. It means that to every algebra $\mathcal{A} = (A; t)$ of type (3) there can be assigned a ternary relational system $\mathcal{T}(A) = (A; T_t)$ where T_t is defined by

$$T_t := \{(x, y, z) \in A^3 \,|\, t(x, y, z) = y\}. \tag{1}$$

Of course, an assigned ternary relational system $\mathcal{T}(A) = (A; T_t)$ need not be centred. However, if $\mathcal{T} = (A; T)$ is a centred ternary relational system and $\mathcal{A}(T) = (A; t)$ an assigned algebra then T_t is centred despite the fact that t is not determined uniquely. In fact, we have $(a, b, c) \in T_t$ if and only if $t(a, b, c) = b$ if and only if $(a, b, c) \in T$. Hence, we have proved the following

Centred ternary relational system

Lemma

Let $\mathcal{T} = (A; T)$ *be a centred ternary relational system,* $\mathcal{A}(T) = (A; t)$ *an assigned algebra and* $\mathcal{T}(\mathcal{A}(T)) = (A; T_t)$ *the ternary relational system assigned to* $\mathcal{A}(T)$*. Then* $\mathcal{T}(\mathcal{A}(T)) = \mathcal{T}$*.*

The best known correspondence between centred ternary relational systems and corresponding algebras of type (3) is the case of "betweenness"-relations and median algebras.

Strong homomorphism

By a **subsystem** of $\mathcal{T} = (A; T)$ is meant a couple of the form $(B, T|B)$ with a non-empty subset B of A and $T|B := T \cap B^3$. One can easily see that this need not be a subalgebra of $\mathcal{A}(T) = (A; t)$.

By a **homomorphism** of a ternary relational system $\mathcal{T} = (A; T)$ into a ternary relational system $\mathcal{S} = (B; S)$ is meant a mapping $h : A \to B$ satisfying

$$(a, b, c) \in T \quad \Longrightarrow \quad (h(a), h(b), h(c)) \in S.$$

A homomorphism h is called **strong** if for each triple $(p, q, r) \in S$ there exists $(a, b, c) \in T$ such that $(h(a), h(b), h(c)) = (p, q, r)$.

t-homomorphism

Definition

A t-**homomorphism** from a centred ternary relational system $\mathcal{T} = (A; T)$ to a ternary relational system $\mathcal{S} = (B; S)$ is a homomorphism from $\mathcal{T}$ to $\mathcal{S}$ such that there exists an algebra $(A; t)$ assigned to $\mathcal{T}$ such that $a, b, c, a', b', c' \in A$ and $(h(a), h(b), h(c)) = (h(a'), h(b'), h(c'))$ together imply $h(t(a, b, c)) = h(t(a', b', c'))$.

Theorem

Let $\mathcal{T} = (A; T)$ and $\mathcal{S} = (B; S)$ be centred ternary relational systems and $\mathcal{A}(T) = (A; t)$ and $\mathcal{B}(S) = (B; s)$ assigned algebras. Then every homomorphism from $\mathcal{A}(T)$ to $\mathcal{B}(S)$ is a t-homomorphism from $\mathcal{T}$ to $\mathcal{S}$.

The theorem says that every homomorphism of assigned algebras is a t-homomorphism of the original relational systems. Now we can show under which conditions the converse assertion becomes true.

Theorem

Let $\mathcal{T} = (A; T)$ and $\mathcal{S} = (B; S)$ be centred ternary relational systems. Then for every strong t-homomorphism h from $\mathcal{T}$ to $\mathcal{S}$ with assigned algebra $\mathcal{A}(T) = (A; t)$ there exists an algebra $\mathcal{B}(S) = (B; s)$ assigned to $\mathcal{S}$ such that h is a homomorphism from $\mathcal{A}(T)$ to $\mathcal{B}(S)$.

Proof

Let h be a strong t-homomorphism from $\mathcal{T}$ to $\mathcal{S}$. By definition there exists an algebra $\mathcal{A}(T) = (A; t)$ assigned to $\mathcal{T}$ such that for all $a, b, c, a', b', c' \in A$ with $(h(a), h(b), h(c)) = (h(a'), h(b'), h(c'))$ it holds $h(t(a, b, c)) = h(t(a', b', c'))$. Define a ternary operation s on B as follows: $s(h(x), h(y), h(z)) := h(t(x, y, z))$ for all $x, y, z \in A$. Since h is strong and a t-homomorphism, s is correctly defined. For $a, b, c \in A$, if $(h(a), h(b), h(c)) \in S$ then there exists $(d, e, f) \in T$ such that $(h(d), h(e), h(f)) = (h(a), h(b), h(c))$. Now

$$s(h(a), h(b), h(c)) = h(t(a, b, c)) = h(t(d, e, f)) = h(e) = h(b).$$

If $(h(a), h(b), h(c)) \notin S$ then $(a, b, c) \notin T$ since h is a homomorphism from $\mathcal{T}$ to $\mathcal{S}$ and hence $t(a, b, c) \in Z_T(a, c)$, i. e. $(a, t(a, b, c), c) \in T$. Thus $(h(a), h(t(a, b, c)), h(c)) \in S$, i. e. $(h(a), s(h(a), h(b), h(c)), h(c)) \in S$ whence $s(h(a), h(b), h(c)) \in Z_S(h(a), h(c))$. This shows that $\mathcal{B}(S)$ is an algebra assigned to $\mathcal{B}$. It is easy to see that h is a homomorphism from $\mathcal{A}(T)$ to $\mathcal{B}(S)$.

t-subsystem

Definition

Let $\mathcal{T} = (A; T)$ be a centred ternary relational system. A subset B of A is called a t-**subsystem** of $\mathcal{T}$ if there exists an algebra $\mathcal{A}(T) = (A; t)$ assigned to $\mathcal{T}$ such that $(B; t)$ is a subalgebra of $\mathcal{A}(T)$.

Example

Consider $A = \{a, b, c, d\}$ and the ternary relation T on A defined as follows: $T := A \times \{d\} \times A$. Then $d \in Z_T(x, y)$ for each $x, y \in A$ and hence T is centred and its median is non-empty, in fact $M_T(x, y, z) = \{d\}$ for all $x, y, z \in A$. For $B = \{a, b, c\}$, $\mathcal{B} = (B; T|B)$ is a subsystem of $\mathcal{A} = (A; T)$ but it is not a t-subsystem. Namely, for every $x, y, z \in A$ t can be defined in the unique way as follows: $t(x, y, z) := d$. Hence, $(\{a, b, c\}; t)$ is not a subalgebra of $(A; t)$. On the contrary, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ are t-subsystems of $\mathcal{A}$.

t-homomorphism

Remark

Let $\mathcal{A} = (A; t)$, $\mathcal{B} = (B; s)$ be algebras of type (3) and $h : A \to B$ a homomorphism from $\mathcal{A}$ to $\mathcal{B}$. Put $\mathcal{T}(A) := (A; T_t)$ and $\mathcal{S}(B) := (B; S_s)$ where T_t, S_s are defined by (1). Then h need not be a t-homomorphism of $\mathcal{T}(A)$ to $\mathcal{S}(B)$, see the following example.

t-homomorphism

Example

Let $A = \{-1, 0, 1\}$, $B = \{1, 0\}$ and $t(x, y, z) = x \cdot y$, $s(x, y, z) = x \cdot y$, where "$\cdot$"is the multiplication of integers. Let $h : A \to B$ be defined by $h(x) = |x|$. Then h is clearly a homomorphism from $\mathcal{A} = (A; t)$ to $\mathcal{B} = (B; s)$ and

$$T_t = (A \times \{0\} \times A) \cup (\{1\} \times A^2).$$

There exists exactly one algebra $(A; t^*)$ assigned to $\mathcal{T}(A)$, namely where

$$t^*(x, y, z) := \begin{cases} y & \text{if } y = 0 \text{ or } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now $h(-1) = h(1)$ but $h(t^*(-1, -1, 1)) = h(0) = 0 \neq 1 = h(1) = h(t^*(1, 1, 1))$. Thus h is not a t-homomorphism.

t-homomorphism

We can prove the following:

Theorem

If $\mathcal{A} = (A; t)$ and $\mathcal{B} = (B; s)$ are algebras of type (3), $\mathcal{A}$ satisfies the identity

$$t(x, t(x, y, z), z) = t(x, y, z)$$

and $\mathcal{T}(A) = (A; T_t)$ and $\mathcal{S}(B) = (B; S_s)$ denote the relational systems corresponding to $\mathcal{A}$ and $\mathcal{B}$, respectively, as defined by (1) then every homomorphism h from $\mathcal{A}$ to $\mathcal{B}$ is a t-homomorphism from $\mathcal{T}(A)$ to $\mathcal{S}(B)$.

Median algebra

The concept of a median algebra was introduced in [1] as follows: An algebra $\mathcal{A} = (A; t)$ of type (3) is called a **median algebra** if it satisfies the following identities:

(M1) $t(x, x, y) = x$;

(M2) $t(x, y, z) = t(y, x, z) = t(y, z, x)$;

(M3) $t(t(x, y, z), v, w) = t(x, t(y, v, w), t(z, v, w))$.

It is well-known (see e.g. [1], [5]) that the ternary relation T_t on A assigned to t via (1) is centred and, moreover, $|M_{T_t}(a, b, c)| = 1$ for all $a, b, c \in A$. In fact, $t(a, b, c) \in M_{T_t}(a, b, c)$. In particular, having a distributive lattice $\mathcal{L} = (L; \vee, \wedge)$ then $m(x, y, z) = M(x, y, z)$ and putting $t(x, y, z) := m(x, y, z)$, one obtains a median algebra. Conversely, every median algebra can be embedded into a distributive lattice. Moreover, the assigned ternary relation T_t is the so-called "betweenness", see [7] and [8].
In what follows, we focus on the case when $M_T(a, b, c) \neq \emptyset$ for all $a, b, c \in A$ and $t(a, b, c) \in M_T(a, b, c)$ also in case $|M_T(a, b, c)| \geq 1$.

Median-like algebra

Definition

A **median-like algebra** is an algebra $(A; t)$ of type (3) where t satisfies (M1) and (M2) and where there exists a centred ternary relation T on A such that $t(x, y, z) \in M_T(x, y, z)$ for all $x, y, z \in A$.

Theorem

An algebra $\mathcal{A} = (A; t)$ of type (3) is median-like if t satisfies (M1), (M2) and

$$t(x, t(x, y, z), y) = t(y, t(x, y, z), z) = t(z, t(x, y, z), x) = t(x, y, z).$$

Lemma

Every median algebra is a median-like algebra.

Median-like algebra

Example

Put $A := \{1,2,3,4,5\}$, let t denote the ternary operation on A defined by $t(x,x,y) = t(x,y,x) = t(y,x,x) := x$ for all $x,y \in A$ and $t(x,y,z) := \min(x,y,z)$ for all $x,y,z \in A$ with $x \neq y \neq z \neq x$ and put $T := \{(x,x,y) \mid x,y \in A\} \cup \{(y,x,x) \mid x,y \in A\} \cup \{(x,y,z) \in A^3 \mid y < x < z\} \cup \{(x,y,z) \in A^3 \mid y < z < x\}$. Then t satisfies (M1) and (M2) and $t(x,y,z) \in M_T(x,y,z)$ for all $x,y,z \in A$. This shows that $(A;t)$ is median-like. However, this algebra is not a median algebra since

$$t(t(1,3,4),2,5) = t(1,2,5) = 1 \neq 2 = t(1,2,2) = t(1,t(3,2,5),t(4,2,5))$$

and hence (M3) is not satisfied.

Example . . .

Median-like algebra

Theorem

Let $\mathcal{L} = (L; \vee, \wedge)$ *be a lattice. Define* $t_1(x, y, z) := m(x, y, z)$, $t_2(x, y, z) := M(x, y, z)$. *Then* $\mathcal{A}_1 := (L; t_1)$ *and* $\mathcal{A}_2 := (L; t_2)$ *are median-like algebras. Moreover, the following conditions are equivalent*

(a) $\mathcal{A}_1 = \mathcal{A}_2$;

(b) $\mathcal{A}_1$ *is a median algebra;*

(c) $\mathcal{L}$ *is distributive.*

Proof

Since both $m(x,y,z)$ and $M(x,y,z)$ satisfy (M1) and (M2) and $m(x,y,z), M(x,y,z) \in [m(x,y,z), M(x,y,z)] = M_T(x,y,z)$ for $(x,y,z) \in L^3$ and $T := \{(x,y,z) \in L^3 \mid x \wedge z \leq y \leq x \vee z\}$, $\mathcal{A}_1, \mathcal{A}_2$ are median-like algebras. It is well-known that $m(x,y,z) = M(x,y,z)$ if and only if $\mathcal{L}$ is distributive which proves $(a) \Leftrightarrow (c)$. The implication $(c) \Rightarrow (b)$ is well-known (see e.g. [1], [5]). Finally, we prove $(b) \Rightarrow (c)$. Assume that (b) holds but (c) does not. Then $\mathcal{L}$ contains either $\mathcal{M}_3 = (\{0,a,b,c,1\}; \vee, \wedge)$ or $\mathcal{N}_5 = (\{0,a,b,c,1\}; \vee, \wedge)$ (with $a < c$) as a sublattice. In the first case we have

$$t(t(a,b,c),a,1) = t(0,a,1) = a \neq 1 = t(a,1,1) = t(a,t(b,a,1),t(c,a,1))$$

whereas in the second case

$$t(t(c,b,a),a,1) = t(a,a,1) = a \neq c = t(c,1,a) = t(c,t(b,a,1),t(a,a,1))$$

which shows that (M3) does not hold. This is a contradiction to (b). Hence (c) holds.

Median-like algebra

Let us mention that median-like algebras form a variety because they are defined by identities. Moreover, this variety is congruence distributive, i. e. $\mathrm{Con}\mathcal{A}$ is distributive for every median-like algebra $\mathcal{A}$, because the operation t is a majority term, i. e. it satisfies by (M1) and (M2)

$$t(x,x,y) = t(x,y,x) = t(y,x,x) = x.$$

Theorem

Let $\mathcal{L} = (L;\vee,\wedge)$ be a lattice and t a ternary operation on L satisfying (M1) and (M2) and $t(x,y,z) \in [m(x,y,z), M(x,y,z)]$ for all $x,y,z \in A$. Then $\mathcal{A} := (L;t)$ is a median-like algebra.

Cyclic order

Apart from the "betweenness" relation, another ternary relation plays an important role in mathematics. It is the so-called **cyclic order**, see e.g. [4], [6].

Definition

A ternary relation T on A is called **asymmetric** if

$$(a,b,c) \in T \text{ for } a \neq b \neq c \quad \text{implies} \quad (c,b,a) \notin T. \tag{2}$$

A ternary relation C on A is called a **cyclic order** if it is cyclic, asymmetric, cyclically transitive and satisfies $(a,a,a) \in C$ for each $a \in A$.

Remark

Let C be a cyclic order on a set A. Then $(a,b,a) \notin C$ for all $a,b \in A$ with $a \neq b$. Namely, if $(a,b,a) \in C$ then, by (2), $(a,b,a) \notin C$, a contradiction. Since C is cyclic, we have also $(a,a,b),(b,a,a) \notin C$.

Cyclic algebra

Applying (2), we derive immediately

Lemma

A centred ternary relation T on A is asymmetric if and only if any assigned ternary operation t satisfies the implication:

$$(t(x,y,z) = y \text{ and } x \neq y \neq z) \implies t(z,y,x) \neq y. \tag{3}$$

Similarly as for "betweenness" relations, we can derive an algebra of type (3) for a centred cyclic order by means of its assigned operation.

Definition

A **cyclic algebra** is an algebra assigned to a cyclic relation.

Cyclic algebras can be characterized by certain identities and the implication (3) as follows.

Cyclic algebra

Theorem

An algebra $\mathcal{A} = (A; t)$ *of type (3) is a cyclic algebra if and only if it satisfies (3) and*

$$\begin{aligned} &t(x, t(x, y, z), z) = t(x, y, z), \\ &t(t(x, y, z), z, x) = z, \\ &t(x, t(x, y, t(x, z, u)), u) = t(x, y, t(x, z, u)), \\ &t(x, x, x) = x. \end{aligned}$$

Cyclic algebra

Example

Let K be a circle in a plane with a given direction.
Define a ternary relation C on K as follows:

$(a, a, a) \in C$ for each $a \in K$ and

$(a, b, c) \in C$ if $a \to b$ and $b \to c$ for $a \neq b \neq c$.

It is an easy exercise to check that C is a cyclic order on K. If $a, b \in K$ then either $a = b$ and hence $Z_C(a, a) = \{a\}$ or $a \neq b$ thus $Z_C(a, b)$ equals the arc of K between a and b, i. e. it contains a continuum of points. Hence C is centred. For any assigned operation t, the algebra $\mathcal{A}(C) = (K; t)$ is a cyclic algebra.

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Thanks

Thank you for your attention!