Using Linguistic Hedges in $\mathbf{L}$-rough Concept Analysis

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Consider the following pairs of operators induced by an $\mathbf{L}$-context $\langle X, Y, I \rangle$. First, the pair $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ of operators ${}^{\uparrow} : \mathbf{L}^X \to \mathbf{L}^Y$ and ${}^{\downarrow} : \mathbf{L}^Y \to \mathbf{L}^X$ is defined by

$$A^{\uparrow}(y) = \bigwedge_{x \in X} A(x) \to I(x, y) \quad \text{and} \quad B^{\downarrow}(x) = \bigwedge_{y \in Y} B(y) \to I(x, y).$$

Second, the pair $\langle {}^{\cap}, {}^{\cup} \rangle$ of operators ${}^{\cap} : \mathbf{L}^X \to \mathbf{L}^Y$ and ${}^{\cup} : \mathbf{L}^Y \to \mathbf{L}^X$ is defined by

$$A^{\cap}(y) = \bigvee_{x \in X} A(x) \otimes I(x, y) \quad \text{and} \quad B^{\cup}(x) = \bigwedge_{y \in Y} I(x, y) \to B(y).$$

The $\mathbf{L}$-rough context induces two operators defined as follows. Let $\langle X, Y, \underline{I}, \overline{I} \rangle$ be an $\mathbf{L}$-rough context. Define $\mathbf{L}$-*rough concept-forming operators* as

$$\begin{aligned} A^{\vartriangle} &= \langle A^{\uparrow_{\underline{I}}}, A^{\cap_{\overline{I}}} \rangle, \\ \langle \underline{B}, \overline{B} \rangle^{\triangledown} &= \underline{B}^{\downarrow_{\underline{I}}} \cap \overline{B}^{\cup_{\overline{I}}} \end{aligned} \tag{1}$$

for $A \in \mathbf{L}^X, \underline{B}, \overline{B} \in \mathbf{L}^Y$. Fixed points of $\langle \vartriangle, \triangledown \rangle$, i.e. tuples $\langle A, \langle \underline{B}, \overline{B} \rangle \rangle \in \mathbf{L}^X \times (\mathbf{L} \times \mathbf{L}^{-1})^Y$ such that $A^{\vartriangle} = \langle \underline{B}, \overline{B} \rangle$ and $\langle \underline{B}, \overline{B} \rangle^{\triangledown} = A$, are called $\mathbf{L}$-*rough concepts*. The $\underline{B}$ and $\overline{B}$ are called *lower intent approximation* and *upper intent approximation*, respectively.

Linguistic Hedges

Truth-stressing hedges were studied from the point of fuzzy logic as logical connectives 'very true'.

A *truth-stressing hedge* is a mapping $^* : L \to L$ satisfying

$$1^* = 1, \quad a^* \leqslant a, \quad a \leqslant b \text{ implies } a^* \leqslant b^*, \quad a^{**} = a^* \qquad (2)$$

Truth-stressing hedges we for each $a, b \in L$. re used to parametrize antitone $\mathbf{L}$-Galois connections, and isotone $\mathbf{L}$-Galois connections.

On every complete residuated lattice $\mathbf{L}$, there are two important truth-stressing hedges:

(i) identity, i.e. $a^* = a \ (a \in L)$;

(ii) globalization, i.e.

$$a^* = \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Linguistic Hedges

A *truth-depressing hedge* is a mapping $^\square : L \to L$ such that following conditions are satisfied

$$0^\square = 0, \quad a \leqslant a^\square, \quad a \leqslant b \text{ implies } a^\square \leqslant b^\square, \quad a^{\square\square} = a^\square$$

for each $a, b \in L$. A truth-depressing hedge is a (truth function of) logical connective 'slightly true'.

On every complete residuated lattice $\mathbf{L}$, there are two important truth-depressing hedges:

(i) identity, i.e. $a^\square = a \ (a \in L)$;

(ii) antiglobalization, i.e.

$$a^\square = \begin{cases} 0, & \text{if } a = 0, \\ 1, & \text{otherwise}. \end{cases}$$

Let $\langle X, Y, I\rangle$ be an $\mathbf{L}$-context and let $\heartsuit, \diamondsuit$ be truth-stressing hedges on $\mathbf{L}$. The antitone concept-forming operators parametrized by $\heartsuit$ and $\diamondsuit$ induced by I are defined as

$$A^{\uparrow_\heartsuit}(y) = \bigwedge_{x \in X} A(x)^\heartsuit \rightarrow I(x, y),$$

$$B^{\downarrow_\diamondsuit}(x) = \bigwedge_{y \in Y} B(y)^\diamondsuit \rightarrow I(x, y)$$

for all $A \in \mathbf{L}^X, B \in \mathbf{L}^Y$.

Let $\heartsuit$ and $\spadesuit$ be truth-stressing hedge and truth-depressing hedge on $\mathbf{L}$, respectively. The isotone concept-forming operators parametrized by $\heartsuit$ and $\spadesuit$ induced by I are defined as

$$A^{\cap_\heartsuit}(y) = \bigvee_{x \in X} A(x)^\heartsuit \otimes I(x, y),$$

$$B^{\cup_\spadesuit}(x) = \bigwedge_{y \in Y} I(x, y) \rightarrow B(y)^\spadesuit$$

for all $A \in \mathbf{L}^X, B \in \mathbf{L}^Y$.

Let $\heartsuit, \diamondsuit$ be truth-stressing hedges on $\mathbf{L}$ and let $\spadesuit$ be a truth-depressing hedge on $\mathbf{L}$. We parametrize the $\mathbf{L}$-rough concept-forming operators as

$$A^{\blacktriangle} = \langle A^{\uparrow_{\heartsuit}}, A^{\cap_{\heartsuit}} \rangle \quad \text{and} \quad \langle \underline{B}, \overline{B} \rangle^{\blacktriangledown} = \underline{B}^{\downarrow_{\diamondsuit}} \cap \overline{B}^{\cup_{\spadesuit}} \tag{3}$$

for $A \in \mathbf{L}^X, \underline{B}, \overline{B} \in \mathbf{L}^Y$.

Theorem

The pair $\langle {}^{\blacktriangle}, {}^{\blacktriangledown} \rangle$ of $\mathbf{L}$-rough concept-forming operators parametrized by hedges has the following properties.

(a) $A^{\blacktriangle} = A^{\heartsuit\vartriangle} = A^{\heartsuit\blacktriangle}$ *and* $\langle \underline{B}, \overline{B} \rangle^{\blacktriangledown} = \langle \underline{B}^{\diamondsuit}, \overline{B}^{\spadesuit} \rangle^{\triangledown} = \langle \underline{B}^{\diamondsuit}, \overline{B}^{\spadesuit} \rangle^{\blacktriangledown}$

(b) $A^{\vartriangle} \subseteq A^{\blacktriangle}$ *and* $\langle \underline{B}, \overline{B} \rangle^{\triangledown} \subseteq \langle \underline{B}, \overline{B} \rangle^{\blacktriangledown}$

(c) $S(A_1^{\heartsuit}, A_2^{\heartsuit}) \leqslant S(A_2^{\blacktriangle}, A_1^{\blacktriangle})$ *and*
$S(\langle \underline{B_1}, \overline{B_1} \rangle, \langle \underline{B_2}, \overline{B_2} \rangle) \leqslant S(\langle \underline{B_2}, \overline{B_2} \rangle^{\blacktriangledown}, \langle \underline{B_1}, \overline{B_1} \rangle^{\blacktriangledown})$

(d) $A^{\heartsuit} \subseteq A^{\blacktriangle\blacktriangledown}$ *and* $\langle \underline{B}^{\diamondsuit}, \overline{B}^{\spadesuit} \rangle \subseteq \langle \underline{B}, \overline{B} \rangle^{\blacktriangledown\blacktriangle}$*;*

(e) $A_1 \subseteq A_2$ *implies* $A_2^{\blacktriangle} \subseteq A_1^{\blacktriangle}$ *and* $\langle \underline{B_1}, \overline{B_1} \rangle \subseteq \langle \underline{B_2}, \overline{B_2} \rangle$ *implies* $\langle \underline{B_2}, \overline{B_2} \rangle^{\blacktriangledown} \subseteq \langle \underline{B_1}, \overline{B_1} \rangle^{\blacktriangledown}$

(f) $S(A^{\heartsuit}, \langle \underline{B}, \overline{B} \rangle^{\blacktriangledown}) = S(\langle \underline{B}^{\diamondsuit}, \overline{B}^{\spadesuit} \rangle, A^{\blacktriangle})$

(g) $(\bigcup_{i \in I} A_i^{\heartsuit})^{\blacktriangle} = \bigcap_{i \in I} A_i^{\blacktriangle}$ *and* $(\langle \bigcup_{i \in I} \underline{B_i}^{\diamondsuit}, \bigcap_{i \in I} \overline{B_i}^{\spadesuit} \rangle)^{\blacktriangledown} = \bigcap_{i \in I} \langle \underline{B_i}, \overline{B_i} \rangle^{\blacktriangledown}$

(h) $A^{\blacktriangle\blacktriangledown} = A^{\blacktriangle\blacktriangledown\blacktriangle\blacktriangledown}$ *and* $\langle \underline{B}, \overline{B} \rangle^{\blacktriangledown\blacktriangle} = \langle \underline{B}, \overline{B} \rangle^{\blacktriangledown\blacktriangle\blacktriangledown\blacktriangle}$.

Theorem

Let $\heartsuit, \heartsuit, \diamondsuit, \diamondsuit$ be truth-stressing hedges on $\mathbf{L}$ *such that* $\mathrm{fix}(\blacktriangledown) \subseteq \mathrm{fix}(\heartsuit)$, $\mathrm{fix}(\blacklozenge) \subseteq \mathrm{fix}(\lozenge)$; *let* $\spadesuit, \spadesuit$ *be truth-depressing hedges on* $\mathbf{L}$ *s.t. and* $\mathrm{fix}(\spadesuit) \subseteq \mathrm{fix}(\spadesuit)$,

$$|\mathcal{B}^{\blacktriangle\blacktriangledown}_{\heartsuit,\blacklozenge,\spadesuit}(X, Y, \underline{I}, \overline{I})| \leqslant |\mathcal{B}^{\blacktriangle\blacktriangledown}_{\heartsuit,\lozenge,\spadesuit}(X, Y, \underline{I}, \overline{I})|$$

for all $\mathbf{L}$*-rough contexts* $\langle X, Y, \underline{I}, \overline{I}\rangle$.
In addition, if $\heartsuit = \heartsuit = \mathrm{id}$, *we have*

$$\mathrm{Ext}^{\blacktriangle\blacktriangledown}_{\heartsuit,\blacklozenge,\spadesuit}(X, Y, \underline{I}, \overline{I}) \subseteq \mathrm{Ext}^{\blacktriangle\blacktriangledown}_{\heartsuit,\lozenge,\spadesuit}(X, Y, \underline{I}, \overline{I}).$$

Similarly, if $\blacklozenge = \lozenge = \spadesuit = \spadesuit = \mathrm{id}$, *we have*

$$\mathrm{Int}^{\blacktriangle\blacktriangledown}_{\heartsuit,\blacklozenge,\spadesuit}(X, Y, \underline{I}, \overline{I}) \subseteq \mathrm{Int}^{\blacktriangle\blacktriangledown}_{\heartsuit,\lozenge,\spadesuit}(X, Y, \underline{I}, \overline{I}).$$

Summary

- We enrich concept-forming operators in $\mathbf{L}$-rough Concept Analysis with linguistic hedges which model semantics of logical connectives ‘very’ and ‘slightly’.
- Using hedges as parameters for the concept-forming operators we are allowed to modify our uncertainty when forming concepts.
- As a consequence, by selection of these hedges we can control the size of concept lattice.