Algebras assigned to ternary relations

Miroslav Kolařík

DEPARTMENT OF COMPUTER SCIENCE
PALACKÝ UNIVERSITY, OLOMOUC

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Co-authors

Ivan Chajda
Palacký University, Olomouc, Czech Republic

Helmut Länger
Vienna University of Technology, Austria

Contents

1 Introduction

2 Ternary operations assigned to ternary relations

3 Centred ternary relational system

4 Median-like algebras

5 Cyclic algebras

Introduction

In [2] and [3], there were shown that to certain relational systems $\mathcal{A} = (A; R)$, where R is a binary relation on $A \neq \emptyset$, there can be assigned a certain groupoid $\mathcal{G}(A) = (A; \circ)$ which captures the properties of R. Namely, $x \circ y = y$ if and only if $(x, y) \in R$.

Hence, there arises the natural question if a similar way can be used for ternary relational systems and algebras with one ternary relation.

In the following let A denote a fixed arbitrary non-empty set.

Basic notions

Definition

Let T be a ternary relation on A and $a, b \in A$. The set

$$Z_T(a,b) := \{x \in A \,|\, (a,x,b) \in T\}$$

is called the **centre of** (a,b) **with respect to** T. The ternary relation T on A is called **centred** if $Z_T(a,b) \neq \emptyset$ for all elements $a, b \in A$.

Definition

Let T be a ternary relation on A and $a, b, c \in A$. The set

$$M_T(a,b,c) := Z_T(a,b) \cap Z_T(b,c) \cap Z_T(c,a)$$

will be called the **median of** (a,b,c) **with respect to** T.

Basic notions

Now we show that to every centred ternary relation there can be assigned ternary operations.

Definition

Let T be a centred ternary relation on A and t a ternary operation on A satisfying

$$t(a,b,c) \begin{cases} = b & \text{if } (a,b,c) \in T \\ \in Z_T(a,c) & \text{otherwise.} \end{cases}$$

Such an operation t is called **assigned to** T.

Example

Example

Let $\mathcal{L} = (L; \vee, \wedge)$ be a lattice. Define a ternary relation T on L as follows:

$$(a, b, c) \in T \quad \text{if and only if} \quad a \wedge c \leq b \leq a \vee c.$$

Put $m(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ and $M(x, y, z) := (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$. Then

$$M_T(a, b, c) = [m(a, b, c), M(a, b, c)]$$

is the interval in $\mathcal{L}$. It is well-known that $m(x, y, z) = M(x, y, z)$ if and only if $\mathcal{L}$ is distributive. Hence, $\mathcal{L}$ is distributive if and only if $|M_T(a, b, c)| = 1$ for all $a, b, c \in L$.

This example was used in [5] for the definition of a median algebra. If $\mathcal{L}$ is a distributive lattice then the algebra $(L; m)$ is called the median algebra derived from $\mathcal{L}$. Note that, there exist median algebras which are not derived from a lattice.

Theorem

Now, we get a characterization of some important properties of ternary relations by means of identities of their assigned operations.

Theorem

A ternary operation t on A is assigned to some centred ternary relation T on A if and only if it satisfies the identity

$$t(x, t(x, y, z), z) = t(x, y, z).$$

Properties of ternary relations

Definition

Let T be a ternary relation on A. We call T

– **reflexive** if $|\{a, b, c\}| \leq 2$ implies $(a, b, c) \in T$;
– **symmetric** if $(a, b, c) \in T$ implies $(c, b, a) \in T$;
– **antisymmetric** if $(a, b, a) \in T$ implies $a = b$;
– **cyclic** if $(a, b, c) \in T$ implies $(b, c, a) \in T$;
– R-**transitive** if $(a, b, c), (b, d, e) \in T$ implies $(a, d, e) \in T$;
– t_1-**transitive** if $(a, b, c), (a, d, b) \in T$ implies $(d, b, c) \in T$;
– t_2-**transitive** if $(a, b, c), (a, d, b) \in T$ implies $(a, d, c) \in T$;
– R-**symmetric** if $(a, b, c) \in T$ implies $(b, a, c) \in T$;
– R-**antisymmetric** if $(a, b, c), (b, a, c) \in T$ implies $a = b$;
– **non-sharp** if $(a, a, b) \in T$ for all $a, b \in A$;
– **cyclically transitive** if $(a, b, c), (a, c, d) \in T$ implies $(a, b, d) \in T$.

Theorem 1/3

Theorem

Let T be a centred ternary relation on A and t an assigned operation. Then (i) – (xi) hold:
(i) T is reflexive if and only if t satisfies the identities

$$t(x,x,y) = t(y,x,x) = t(y,x,y) = x.$$

(ii) T is symmetric if and only if t satisfies the identity

$$t(z,t(x,y,z),x) = t(x,y,z).$$

(iii) T is antisymmetric if and only if t satisfies the identity

$$t(x,y,x) = x.$$

(iv) T is cyclic if and only if t satisfies the identity

$$t(t(x,y,z),z,x) = z.$$

Theorem 2/3

Theorem

(v) T *is* R*-transitive if and only if* t *satisfies the identity*

$$t(x, t(t(x, y, z), u, v), v) = t(t(x, y, z), u, v).$$

(vi) T *is* t_1*-transitive if and only if* t *satisfies the identity*

$$t(t(x, u, t(x, y, z)), t(x, y, z), z) = t(x, y, z).$$

(vii) T *is* t_2*-transitive if and only if* t *satisfies the identity*

$$t(x, t(x, u, t(x, y, z)), z) = t(x, u, t(x, y, z)).$$

(viii) T *is* R*-symmetric if and only if* t *satisfies the identity*

$$t(t(x, y, z), x, z) = x.$$

Theorem 3/3

Theorem

(ix) If t satisfies the identity

$$t(t(x,y,z),x,z) = t(x,y,z)$$

then T is R-antisymmetric.
(x) T is non-sharp if and only if t satisfies the identity

$$t(x,x,y) = x.$$

(xi) T is cyclically transitive if and only if t satisfies the identity

$$t(x,t(x,y,t(x,z,u)),u) = t(x,y,t(x,z,u)).$$

Centred ternary relational system

By a **ternary relational system** is meant a couple $\mathcal{T} = (A; T)$ where T is a ternary relation on A. $\mathcal{T}$ is called **centred** if T is centred. As shown above, to every centred ternary relational system $\mathcal{T} = (A; T)$ there can be assigned an algebra $\mathcal{A}(T) = (A; t)$ with one ternary operation $t : A^3 \to A$ such that t is assigned to T. Now, we can introduce an inverse construction. It means that to every algebra $\mathcal{A} = (A; t)$ of type (3) there can be assigned a ternary relational system $\mathcal{T}(A) = (A; T_t)$ where T_t is defined by

$$T_t := \{(x, y, z) \in A^3 \,|\, t(x, y, z) = y\}. \tag{1}$$

Of course, an assigned ternary relational system $\mathcal{T}(A) = (A; T_t)$ need not be centred.

The best known correspondence between centred ternary relational systems and corresponding algebras of type (3) is the case of "betweenness"-relations and median algebras.

Median algebra

The concept of a median algebra was introduced by J. R. Isbell as follows: An algebra $\mathcal{A} = (A; t)$ of type (3) is called a **median algebra** if it satisfies the following identities:

(M1) $t(x, x, y) = x$;

(M2) $t(x, y, z) = t(y, x, z) = t(y, z, x)$;

(M3) $t(t(x, y, z), v, w) = t(x, t(y, v, w), t(z, v, w))$.

It is well-known (see e.g. [1], [5]) that the ternary relation T_t on A assigned to t via (1) is centred and, moreover, $|M_{T_t}(a, b, c)| = 1$ for all $a, b, c \in A$. In fact, $t(a, b, c) \in M_{T_t}(a, b, c)$. In particular, having a distributive lattice $\mathcal{L} = (L; \vee, \wedge)$ then $m(x, y, z) = M(x, y, z)$ and putting $t(x, y, z) := m(x, y, z)$, one obtains a median algebra. Conversely, every median algebra can be embedded into a distributive lattice. Moreover, the assigned ternary relation T_t is the so-called "betweenness", see [4] and [5].
In what follows, we focus on the case when $M_T(a, b, c) \neq \emptyset$ for all $a, b, c \in A$ and $t(a, b, c) \in M_T(a, b, c)$ also in case $|M_T(a, b, c)| \geq 1$.

Median-like algebra

Definition

A **median-like algebra** is an algebra $(A; t)$ of type (3) where t satisfies (M1) and (M2) and where there exists a centred ternary relation T on A such that $t(x, y, z) \in M_T(x, y, z)$ for all $x, y, z \in A$.

Theorem

An algebra $\mathcal{A} = (A; t)$ of type (3) is median-like if t satisfies (M1), (M2) and

$$t(x, t(x, y, z), y) = t(y, t(x, y, z), z) = t(z, t(x, y, z), x) = t(x, y, z).$$

Lemma

Every median algebra is a median-like algebra.

Median-like algebra

Example

Put $A := \{1, 2, 3, 4, 5\}$, let t denote the ternary operation on A defined by $t(x, x, y) = t(x, y, x) = t(y, x, x) := x$ for all $x, y \in A$ and $t(x, y, z) := \min(x, y, z)$ for all $x, y, z \in A$ with $x \neq y \neq z \neq x$ and put $T := \{(x, x, y) \mid x, y \in A\} \cup \{(y, x, x) \mid x, y \in A\} \cup \{(x, y, z) \in A^3 \mid y < x < z\} \cup \{(x, y, z) \in A^3 \mid y < z < x\}$. Then t satisfies (M1) and (M2) and $t(x, y, z) \in M_T(x, y, z)$ for all $x, y, z \in A$. This shows that $(A; t)$ is median-like. However, this algebra is not a median algebra since

$$t(t(1, 3, 4), 2, 5) = t(1, 2, 5) = 1 \neq 2 = t(1, 2, 2) = t(1, t(3, 2, 5), t(4, 2, 5))$$

and hence (M3) is not satisfied.

Median-like algebra

Theorem

Let $\mathcal{L} = (L; \vee, \wedge)$ *be a lattice. Define* $t_1(x, y, z) := m(x, y, z)$, $t_2(x, y, z) := M(x, y, z)$. *Then* $\mathcal{A}_1 := (L; t_1)$ *and* $\mathcal{A}_2 := (L; t_2)$ *are median-like algebras. Moreover, the following conditions are equivalent*

(a) $\mathcal{A}_1 = \mathcal{A}_2$;

(b) $\mathcal{A}_1$ *is a median algebra;*

(c) $\mathcal{L}$ *is distributive.*

Median-like algebra

Let us mention that median-like algebras form a variety because they are defined by identities. Moreover, this variety is congruence distributive, i. e. $\mathrm{Con}\mathcal{A}$ is distributive for every median-like algebra $\mathcal{A}$, because the operation t is a majority term, i. e. it satisfies by (M1) and (M2)

$$t(x, x, y) = t(x, y, x) = t(y, x, x) = x.$$

Theorem

Let $\mathcal{L} = (L; \vee, \wedge)$ be a lattice and t a ternary operation on L satisfying (M1) and (M2) and $t(x, y, z) \in [m(x, y, z), M(x, y, z)]$ for all $x, y, z \in A$. Then $\mathcal{A} := (L; t)$ is a median-like algebra.

Cyclic order

Apart from the "betweenness" relation, another ternary relation plays an important role in mathematics. It is the so-called **cyclic order**, see e.g. [4], [3].

Definition

A ternary relation T on A is called **asymmetric** if

$$(a,b,c) \in T \text{ for } a \neq b \neq c \quad \text{implies} \quad (c,b,a) \notin T. \tag{2}$$

A ternary relation C on A is called a **cyclic order** if it is cyclic, asymmetric, cyclically transitive and satisfies $(a,a,a) \in C$ for each $a \in A$.

Applying (2), we derive immediately

Cyclic algebra

Lemma

A centred ternary relation T on A is asymmetric if and only if any assigned ternary operation t satisfies the implication:

$$(t(x,y,z) = y \text{ and } x \neq y \neq z) \quad \Longrightarrow \quad t(z,y,x) \neq y. \tag{3}$$

Similarly as for "betweenness" relations, we can derive an algebra of type (3) for a centred cyclic order by means of its assigned operation.

Definition

A **cyclic algebra** is an algebra assigned to a cyclic relation.

Cyclic algebra

Theorem

An algebra $\mathcal{A} = (A; t)$ of type (3) is a cyclic algebra if and only if it satisfies (3) and

$$
\begin{aligned}
&t(x, t(x, y, z), z) = t(x, y, z),\\
&t(t(x, y, z), z, x) = z,\\
&t(x, t(x, y, t(x, z, u)), u) = t(x, y, t(x, z, u)),\\
&t(x, x, x) = x.
\end{aligned}
$$

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Thanks

Thank you for your attention!