Palacký University
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Squeezing of quantum states

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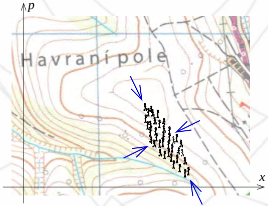
- Introduction
- Motivation: why squeezing?
- Basics of squeezing
- Hamilton canonical equations and squeezing rate
- Examples
- Conclusion

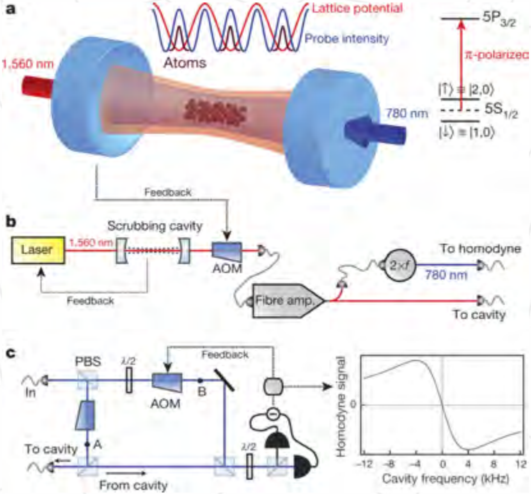


Why squeezing?

- Metrology: suppressed noise of interferometers
  - optics:detection of gravitational waves
  - squeezed atomic spin states: magnetometry
  - atomic clocks
- Quantum information processing: irreducble resource
  - quantum teleportation of continuous variables
  - quantum cryptography
  - quantum computation with continuous variables

Why squeezing?

Measurement noise 100 times lower than the quantum-projection limit using entangled atoms



[Hosten et al. (Kasevich group)]

Why squeezing?

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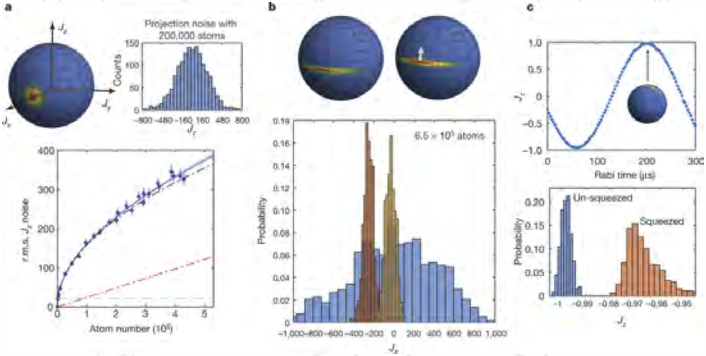

[Hosten et al. (Kasevich group)]

Why squeezing?

Gravitational wave detection, LIGO

[www.ligo.caltech.edu; PRL 116, 061102 (2016)]

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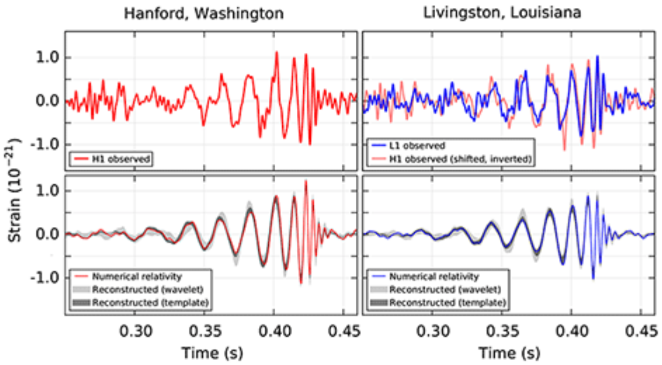
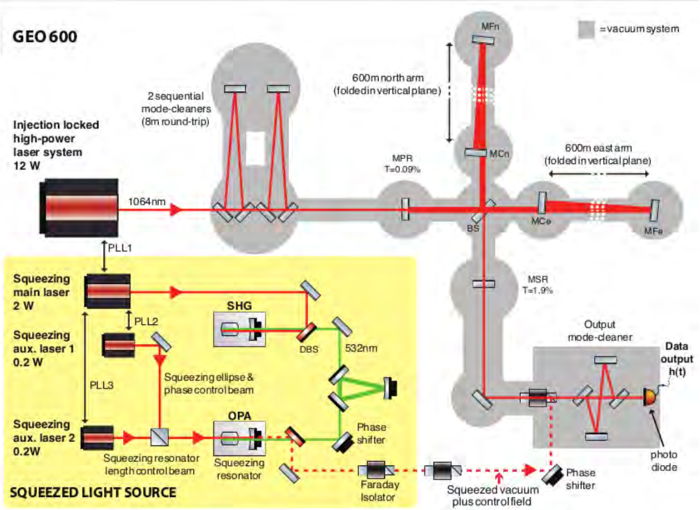


[www.ligo.caltech.edu; PRL 116, 061102 (2016)]

Why squeezing?

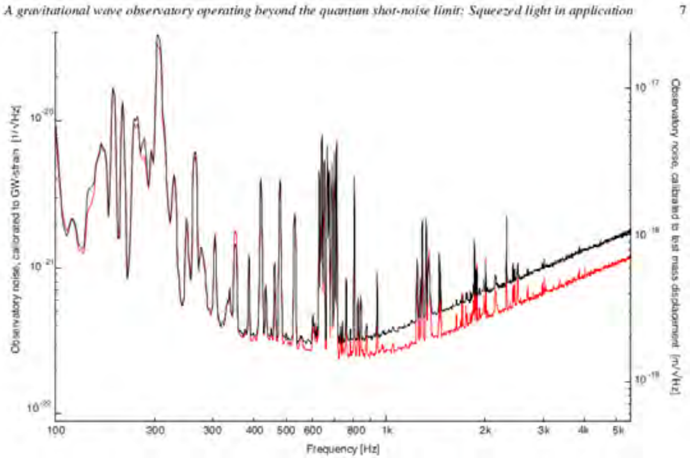
Gravitational wave detection: GEO600 (LIGO collaboration), R. Schnabel



[Abadie et al., Nature Physics, 7, 962-965 (2011)]

Why squeezing?

Gravitational wave detection: GEO600 (LIGO collaboration), R. Schnabel

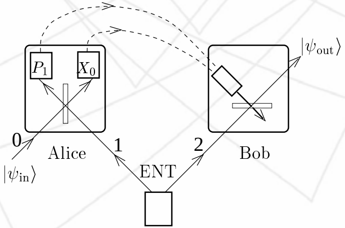


[Abadie et al., Nature Physics, 7, 962-965 (2011)]

Why squeezing?

Quantum information processing with squeezed states

- Quantum teleportation of continuous variables [Vaidman, PRA 49, 1473 (1994)].
- Quantum cryptography with continuous variables (e.g., [Hillery, PRA 61, 022309 (2000)]
- Quantum computation with continuous variables (e.g., [Lloyd & Braunstein, PRL 82, 1784 (1999)])
    - Analogue computation,
    - quantum simulators,
    - to have universal computer, necessary to have Hamiltonian of higher than quadratic nonlinearity in x and p.



Basics of squeezing

Squeezing

Example: harmonic oscillator

- Hamiltonian:

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

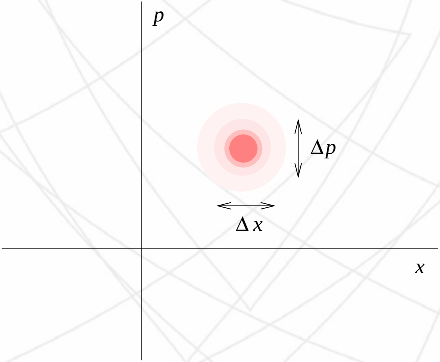
- Coherent states: saturate uncertainty relation

$$\begin{aligned} \Delta x &= \sqrt{\frac{\hbar}{2m\omega}} \\ \Delta p &= \sqrt{\frac{\hbar m\omega}{2}} \\ \Delta x \Delta p &= \frac{\hbar}{2} \end{aligned}$$

Squeezing

Example: harmonic oscillator

Coherent states: saturate uncertainty relation



Squeezing

Example: harmonic oscillator

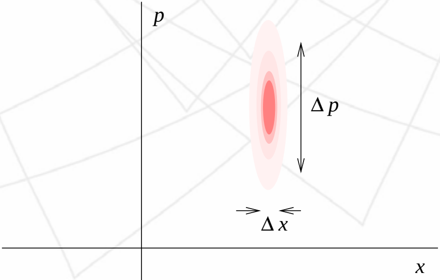
Creation and annihilation operators:

$$
\begin{aligned}
\hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \\
\hat{p} &= -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a} - \hat{a}^\dagger) \\
[\hat{x}, \hat{p}] &= i\hbar \\
[\hat{a}, \hat{a}^\dagger] &= 1 \\
H &= \hbar\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)
\end{aligned}
$$

Squeezing

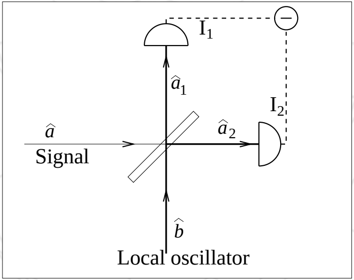
Example: harmonic oscillator Squeezed coherent states: can also saturate uncertainty relation, but, e.g.:

$$\begin{aligned}\Delta x &< \sqrt{\frac{\hbar}{2m\omega}}\\ \Delta p &> \sqrt{\frac{\hbar m\omega}{2}}\\ \Delta x\Delta p &= \frac{\hbar}{2}\end{aligned}$$



Basics of squeezing: detection

Homodyne detection

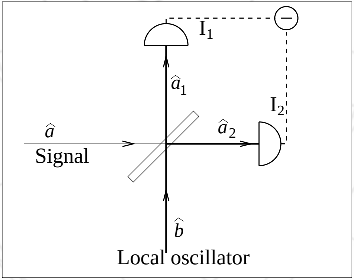


$$\hat{a}_1 = \frac{1}{\sqrt{2}}\left(\hat{a}+\hat{b}\right), \qquad \hat{a}_2 = \frac{1}{\sqrt{2}}\left(\hat{a}-\hat{b}\right)$$

$$I_1 \propto \hat{a}_1^\dagger \hat{a}_1 = \frac{1}{2}\hat{a}^\dagger \hat{a} + \frac{|b|}{2}\left(\hat{a}^\dagger e^{i\varphi} + \hat{a}e^{-i\varphi}\right) + \frac{|b|^2}{2}$$

$$I_2 \propto \hat{a}_1^\dagger \hat{a}_1 = \frac{1}{2}\hat{a}^\dagger \hat{a} - \frac{|b|}{2}\left(\hat{a}^\dagger e^{i\varphi} + \hat{a}e^{-i\varphi}\right) + \frac{|b|^2}{2}$$

Homodyne detection



$$I_1 - I_2 \propto |b| \left(\hat{a}^\dagger e^{i\varphi} + \hat{a} e^{-i\varphi} \right)$$

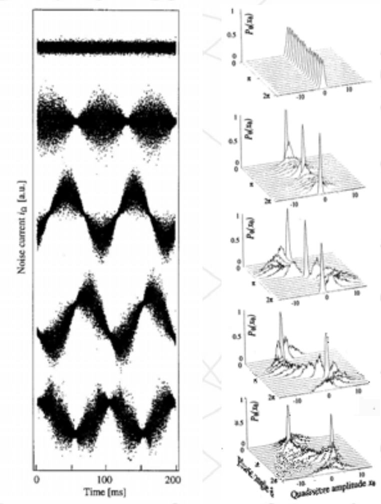
Example: $\varphi = 0$

$$I_1 - I_2 \propto |b| \left(\hat{a}^\dagger + \hat{a} \right) \propto \hat{x}$$

Example: $\varphi = \pi/2$

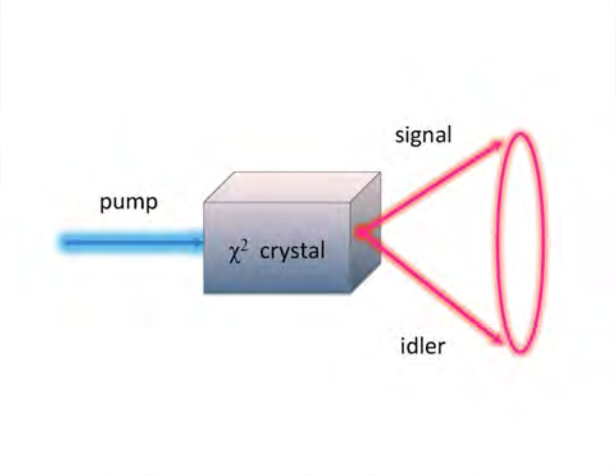
$$I_1 - I_2 \propto i|b| \left(\hat{a}^\dagger - \hat{a} \right) \propto \hat{p}$$

Homodyne detection



Noise squeezing [G. Breitenbach dissertation, 1998; Nature 387, 471 (1997)]

Parametric down-conversion



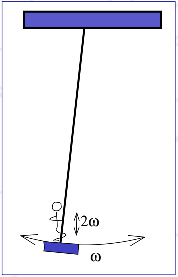
[Wikipedia, Denysbondar]

Parametric down-conversion

$$H = \chi(\hat{b}\ \hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{b}^\dagger \hat{a}_1 \hat{a}_2)$$

Parametric down-conversion, degenerate case

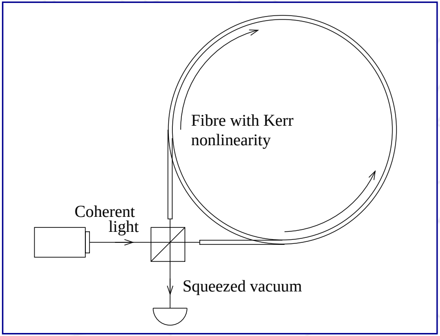
$$H = \chi(\hat{b}\ \hat{a}^{\dagger 2} + \hat{b}^\dagger \hat{a}^2)$$

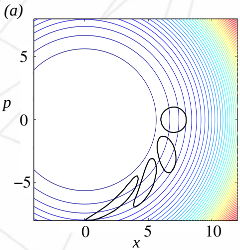


Kerr nonlinearity
(index of refraction proportional to light intensity)

$$H = \chi \hat{n}^2$$

Strong pulses propagating in optical fibres

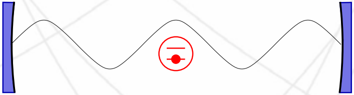


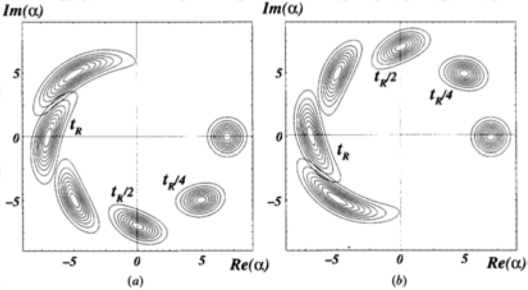


Jaynes-Cummings model

a two level atom and a single-mode field

$$\hat{H}_{JC} = g\left(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-\right)$$





[C. W. Woods and J. Gea-Banacloche, J. Mod. Opt. **40,** 2361-2379 (1993)]

Hamilton canonical equations and squeezing rate

Classical Hamiltonian $H(x,p)$, equations of motion

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p}, \\ \dot{p} &= -\frac{\partial H}{\partial x}. \end{aligned}$$

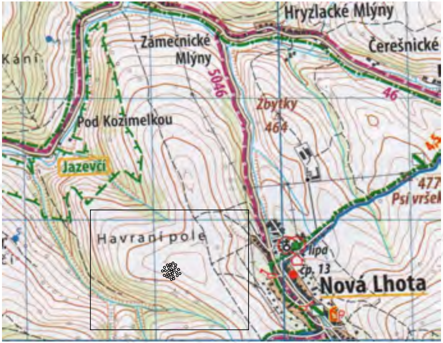
Continuity equation

$$\begin{aligned} \frac{\partial \varrho}{\partial t} &= -\sum_k \frac{\partial j_k}{\partial q_k}, \\ j_k &= \varrho \dot{q}_k. \end{aligned}$$

Liouville theorem

$$\frac{d\varrho}{dt} = 0.$$

Phase space is the countryside, Hamiltonian is the elevation

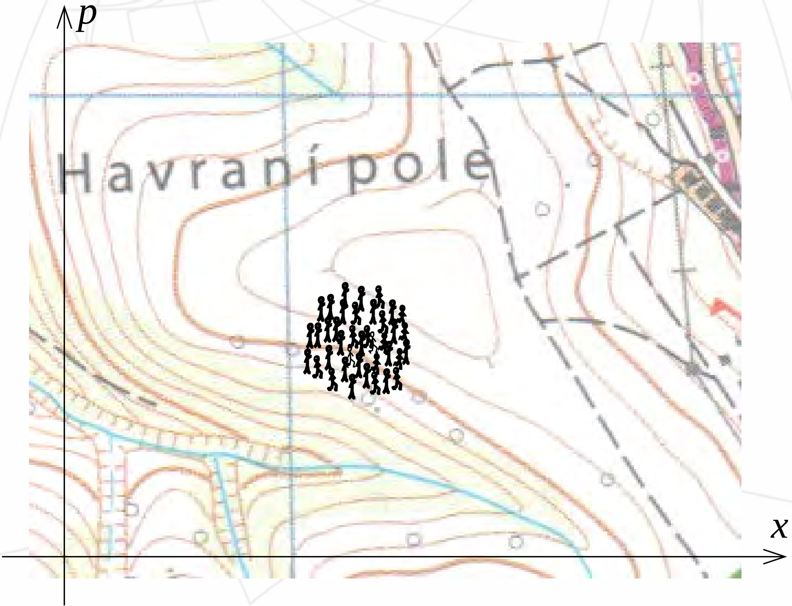


Rules of motion:

- Follow the countour line (constant elevation), hill on your left, valley on your right,
- speed is proportional to the slope magnitude.

Hamilton canonical equations and squeezing rate

Phase space is the countryside, Hamiltonian is the elevation



Hamilton canonical equations and squeezing rate

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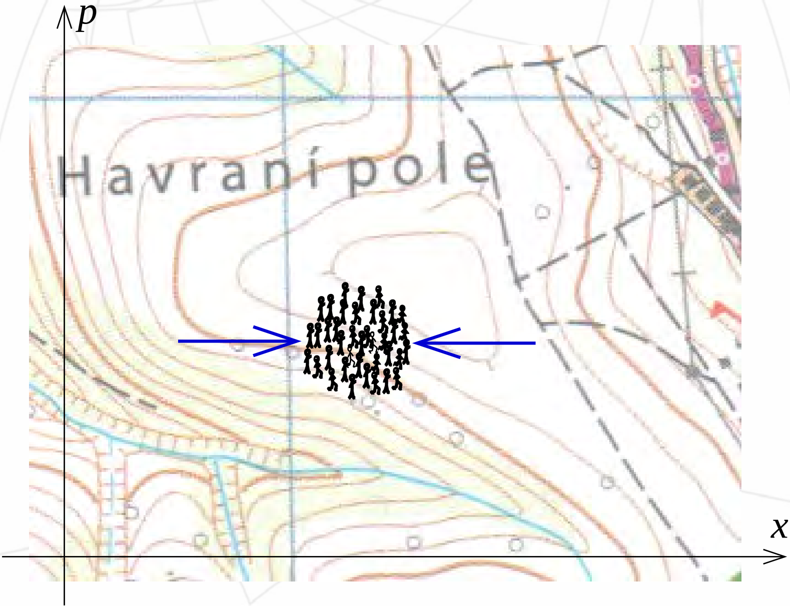
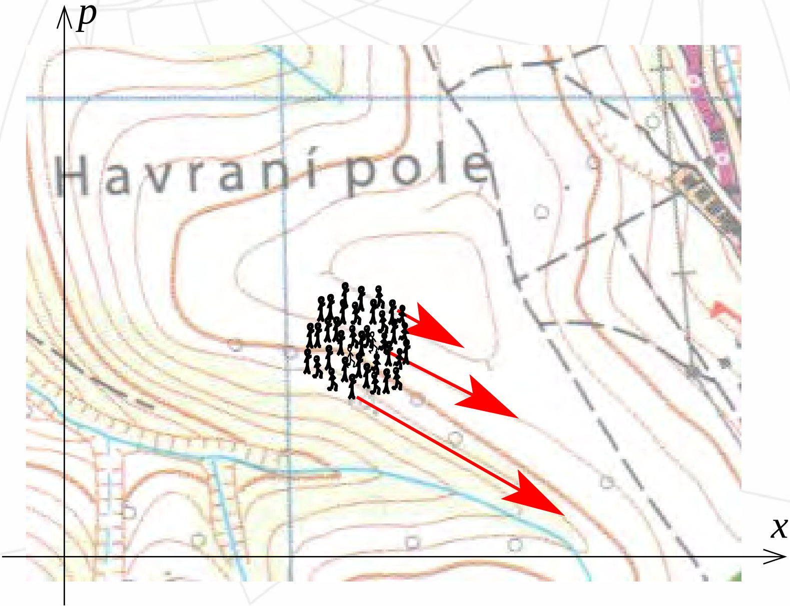


Hamilton canonical equations and squeezing rate

Phase space is the countryside, Hamiltonian is the elevation



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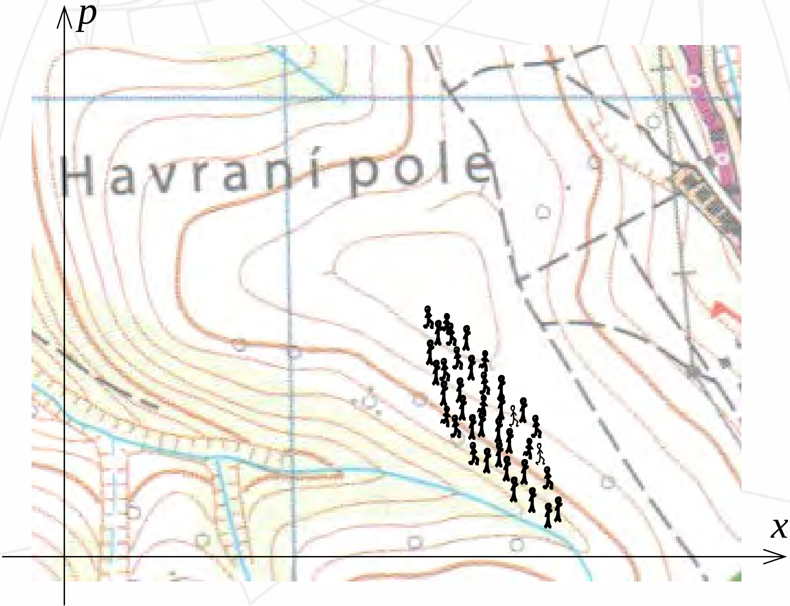

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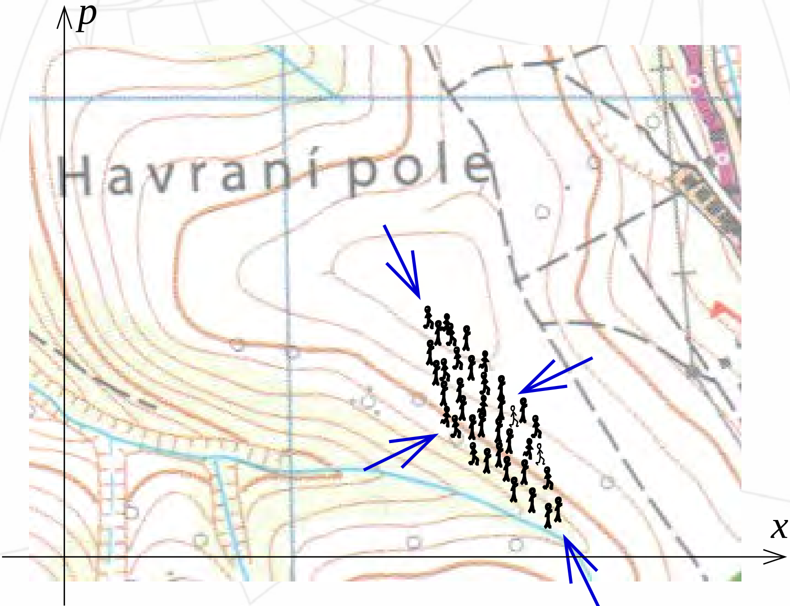

Derivation of the squeezing rate

Variation matrix

$$V = \begin{pmatrix} \langle \Delta x^2 \rangle & \langle \Delta x \Delta p \rangle \\ \langle \Delta x \Delta p \rangle & \langle \Delta p^2 \rangle \end{pmatrix} \equiv \begin{pmatrix} V_{xx} & V_{xp} \\ V_{xp} & V_{pp} \end{pmatrix},$$

for a state centered in (x_0, p_0).

Derivation of the squeezing rate

Change of position

$$\begin{aligned}
\tilde{x}_0 + \Delta\tilde{x} &\approx x_0 + \Delta x + \frac{d}{dt}\left(x_0 + \Delta x\right) dt \\
&= x_0 + \Delta x + \frac{\partial H(x_0 + \Delta x, p_0 + \Delta p)}{\partial p} dt \\
&\approx x_0 + \Delta x + \frac{\partial H(x_0, p_0)}{\partial p} dt \\
&\quad + \left(\frac{\partial^2 H(x_0, p_0)}{\partial p^2}\Delta p + \frac{\partial^2 H(x_0, p_0)}{\partial x \partial p}\Delta x\right) dt
\end{aligned}$$

Derivation of the squeezing rate

Change of momentum

$$\begin{aligned}
\tilde{p}_0 + \Delta\tilde{p} &\approx p_0 + \Delta p + \frac{d}{dt}\left(p_0 + \Delta p\right) dt \\
&= p_0 + \Delta p - \frac{\partial H(x_0 + \Delta x, p_0 + \Delta p)}{\partial x} dt \\
&\approx p_0 + \Delta p - \frac{\partial H(x_0, p_0)}{\partial x} dt \\
&\quad - \left(\frac{\partial^2 H(x_0, p_0)}{\partial x^2} \Delta x + \frac{\partial^2 H(x_0, p_0)}{\partial x \partial p} \Delta p \right) dt.
\end{aligned}$$

Hamilton canonical equations and squeezing rate

Derivation of the squeezing rate

New central position and momentum:

$$\begin{aligned} \tilde{x}_0 &\approx x_0 + H_p dt, \\ \tilde{p}_0 &\approx p_0 - H_x dt, \end{aligned}$$

new deviations:

$$\begin{aligned} \Delta\tilde{x} &\approx \Delta x + (H_{xp}\Delta x + H_{pp}\Delta p)\, dt, \\ \Delta\tilde{p} &\approx \Delta p - (H_{xp}\Delta p + H_{xx}\Delta x)\, dt. \end{aligned}$$

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Derivation of the squeezing rate

Assuming $\langle \Delta x \rangle = \langle \Delta p \rangle = 0$, the new variances are (up to the first order in dt)

$$\begin{aligned}
\langle \Delta \tilde{x}^2 \rangle &\approx \langle \Delta x^2 \rangle + 2 \left(H_{xp} \langle \Delta x^2 \rangle + H_{pp} \langle \Delta x \Delta p \rangle \right) dt, \\
\langle \Delta \tilde{p}^2 \rangle &\approx \langle \Delta p^2 \rangle - 2 \left(H_{xp} \langle \Delta p^2 \rangle + H_{xx} \langle \Delta x \Delta p \rangle \right) dt, \\
\langle \Delta \tilde{x} \Delta \tilde{p} \rangle &\approx \langle \Delta x \Delta p \rangle + \left(H_{pp} \langle \Delta p^2 \rangle - H_{xx} \langle \Delta x^2 \rangle \right) dt.
\end{aligned}$$

Derivation of the squeezing rate

In terms of the variation matrix:

$$\tilde{V} = SVS^T,$$

where

$$S = \begin{pmatrix} 1 + H_{xp}dt & H_{pp}dt \\ -H_{xx}dt & 1 - H_{xp}dt \end{pmatrix}. \qquad (1)$$

Derivation of the squeezing rate

For initially isotropic and uncorrelated fluctuations, i.e., $\langle \Delta x^2 \rangle = \langle \Delta p^2 \rangle = \sigma^2$, and $\langle \Delta x \Delta p \rangle = 0$:

$$
\begin{aligned}
\langle \Delta \tilde{x}^2 \rangle &\approx \sigma^2 \left(1 + 2H_{xp}\right) dt, \\
\langle \Delta \tilde{p}^2 \rangle &\approx \sigma^2 \left(1 - 2H_{xp}\right) dt, \\
\langle \Delta \tilde{x} \Delta \tilde{p} \rangle &\approx \sigma^2 \left(H_{pp} - H_{xx}\right) dt.
\end{aligned}
$$

Eigenvalues of the new variance matrix $\tilde{V}$:

$$
\tilde{V}_\pm = \frac{\langle \Delta \tilde{x}^2 \rangle + \langle \Delta \tilde{p}^2 \rangle}{2} \pm \frac{1}{2}\sqrt{\left(\langle \Delta \tilde{x}^2 \rangle - \langle \Delta \tilde{p}^2 \rangle\right)^2 + 4\langle \Delta \tilde{x} \Delta \tilde{p} \rangle^2}.
$$

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Hamilton canonical equations and squeezing rate

Derivation of the squeezing rate

Inserting the transformed variance:

$$\tilde{V}_{\pm} = \sigma^2 (1 \pm Q dt)$$

where

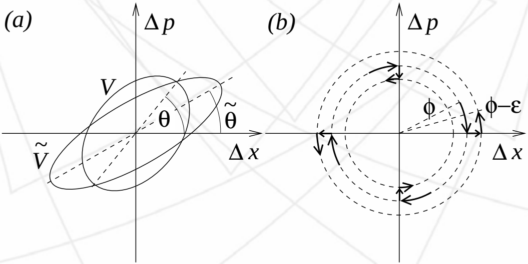
$$Q = \sqrt{(H_{pp} - H_{xx})^2 + 4H_{xp}^2} \tag{2}$$

is the **squeezing rate**.
[T.O., PRA **92,** 033801 (2015)]

Orientation and rotation of the squeezing ellipse

Transformation matrix S, general starting variance matrix:

$$S = \begin{pmatrix} \cos(\phi-\epsilon) & -\sin(\phi-\epsilon) \\ \sin(\phi-\epsilon) & \cos(\phi-\epsilon) \end{pmatrix} \begin{pmatrix} 1+\frac{Qdt}{2} & 0 \\ 0 & 1-\frac{Qdt}{2} \end{pmatrix} \times \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$



Orientation and rotation of the squeezing ellipse

Up to the first order in ϵ and dt, one finds

$$S = \begin{pmatrix} 1 + \frac{Qdt}{2}\cos 2\phi & \epsilon + \frac{Qdt}{2}\sin 2\phi \\ -\epsilon + \frac{Qdt}{2}\sin 2\phi & 1 - \frac{Qdt}{2}\cos 2\phi \end{pmatrix}. \tag{3}$$

Comparing this with Eq. (1) one finds

$$\begin{aligned} Q\cos 2\phi &= 2H_{xp}, \\ Q\sin 2\phi &= H_{pp} - H_{xx}, \\ 2\epsilon &= (H_{xx} + H_{pp})\,dt, \end{aligned}$$

with Q given by Eq. (2).

Orientation and rotation of the squeezing ellipse

Rotation angle

$$\tan 2\phi = \frac{H_{pp} - H_{xx}}{2H_{xp}},$$

and assuming that ϵ evolves with time as $\epsilon = \omega_v dt$, one gets

$$\omega_v = \frac{H_{xx} + H_{pp}}{2} \tag{4}$$

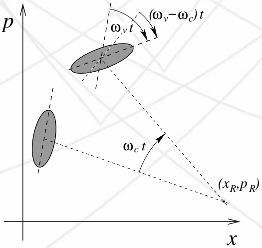
Compensation of motion of the uncertainty ellipse

Rotation center at $(x_R, p_R) = (x, p) + (R_x, R_p)$ with

$$R_x = -\frac{H_x}{\omega_c}, \qquad R_p = -\frac{H_p}{\omega_c},$$

angular frequency of the motion of the center

$$\omega_c = \frac{H_x^2 H_{pp} + H_p^2 H_{xx} - 2H_x H_p H_{xp}}{H_x^2 + H_p^2}. \tag{5}$$



Compensation of motion of the uncertainty ellipse

To compensate the drift, add the Hamiltonian

$$H_{\mathrm{ad1}} = -\frac{1}{2}\omega_c \left[(x - x_R)^2 + (p - p_R)^2\right].$$

To keep the optimal orientation, add

$$H_{\mathrm{ad2}} = -\frac{1}{2}(\omega_v - \omega_c)\left[(x - x_0)^2 + (p - p_0)^2\right]$$

Combined Hamiltonian $H_{\mathrm{ad}} = H_{\mathrm{ad1}} + H_{\mathrm{ad2}}$,

$$H_{\mathrm{ad}} = -\frac{1}{2}\omega_v \left[(x - x_r)^2 + (p - p_r)^2\right] + const., \tag{6}$$

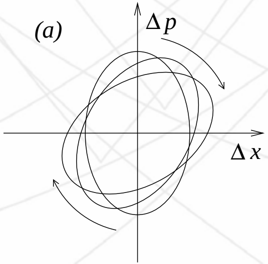
with the center localized at

$$(x_r, p_r) = (x_R, p_R) + \left(1 - \frac{\omega_c}{\omega_v}\right)(x_0 - x_R, p_0 - p_R).$$

Harmonic oscillator

$$H = \frac{1}{2}\omega(p^2 + x^2),$$

Eq. (2) yields $Q = 0$.
Rotation frequencies $\omega_v = \omega_c = \omega$.

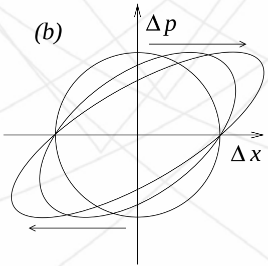


Examples

Free particle

$$H = \frac{1}{2m} p^2.$$

Squeezing rate $Q = 1/m$,
optimum orientation of the uncertainty ellipse $\theta = \pi/4$,
rotations $\omega_v = 1/(2m)$, and $\omega_c = 0$.



Free particle

In terms of quantum optical bosonic operators:

$$\begin{aligned}\hat{x} &= \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right),\\ \hat{p} &= \frac{i}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right),\end{aligned}$$

the Hamiltonian is

$$H = \frac{1}{2}\hat{p}^2 = -\frac{1}{4}\left(\hat{a}^{\dagger 2} + \hat{a}^2\right) + \frac{1}{2}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right),$$

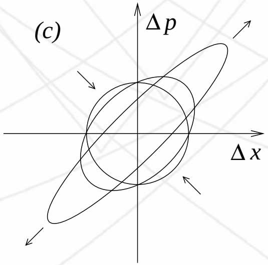
i.e., parametric down-conversion plus harmonic oscillator.

Examples

Inverted oscillator

$$\frac{1}{2}\zeta(p^2 - x^2)$$

Squeezing rate $Q = 2\zeta$,
optimum orientation $\theta = \pi/4$,
no rotation, $\omega_v = 0$.



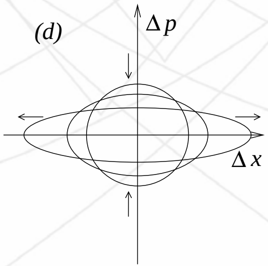
xp-Hamiltonian

$$H = \zeta xp$$

Classical counterpart of the quantum operator

$$\hat{H} = \frac{1}{2}\zeta\left(\hat{x}\hat{p} + \hat{p}\hat{x}\right) = \frac{i}{2}\zeta\left(\hat{a}^{\dagger 2} - \hat{a}^2\right),$$

i.e., parametric down-conversion.
Squeezing rate $Q = 2\zeta$,
optimum orientation $\theta = 0$,
no rotation, $\omega_v = 0$.



Examples

Pendulum

$$H = \frac{1}{2}p^2 - \cos x$$

Squeezing rate

$$Q = 1 - \cos x = 2\sin^2\frac{x}{2},$$

rotation $\omega_v = \cos^2\frac{x}{2}$

Examples

Kerr nonlinearity

$$H = \chi(p^2 + x^2)^2,$$

Squeezing rate

$$Q = 8\chi(p^2 + x^2),$$

rotation

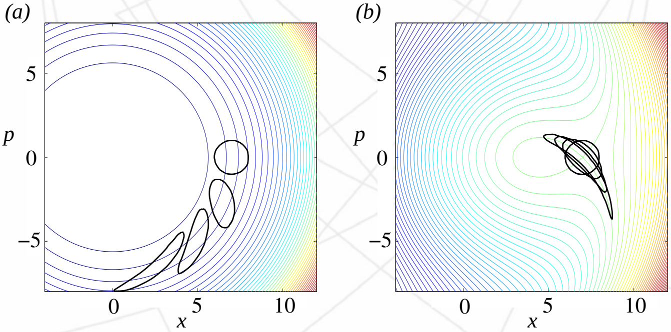
$$\begin{aligned} \omega_v &= 8\chi(p^2 + x^2), \\ \omega_c &= 4\chi(p^2 + x^2), \\ (x_r, p_r) &= \frac{1}{2}(x_0, p_0). \end{aligned}$$

Examples

Kerr nonlinearity

Compensating Hamiltonian

$$H_{\mathrm{ad}} = -4\chi(x_0^2 + p_0^2)\left[\left(x - \frac{x_0}{2}\right)^2 + \left(p - \frac{p_0}{2}\right)^2\right].$$



Examples

Jaynes-Cummings Hamiltonian

Classical Hamiltonian postulated as

$$H = \pm g\sqrt{\frac{p^2 + x^2}{2}}.$$

Stems from the quantum Hamiltonian

$$\hat{H}_{JC} = g\left(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-\right),$$

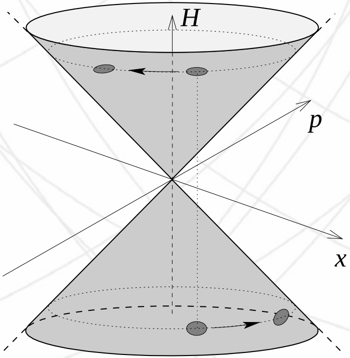
assume the initial quantum state prepared as

$$|\Phi_\pm\rangle = |\alpha\rangle \otimes \frac{1}{\sqrt{2}}\left(|g\rangle \pm e^{i\varphi}|e\rangle\right),$$

with $\alpha = \sqrt{n}e^{i\varphi} = 2^{-1/2}(x + ip)$.
The mean energy is $\langle\Phi_\pm|\hat{H}_{JC}|\Phi_\pm\rangle = \pm g\sqrt{n} = \pm 2^{-1/2} g\sqrt{x^2 + p^2}$.

Jaynes-Cummings Hamiltonian

$H = \pm g\sqrt{\frac{p^2+x^2}{2}}$, visualization:



Examples

Jaynes-Cummings Hamiltonian

Squeezing rate

$$Q = \frac{g}{\sqrt{2}} \frac{1}{\sqrt{x^2 + p^2}},$$

angular velocities

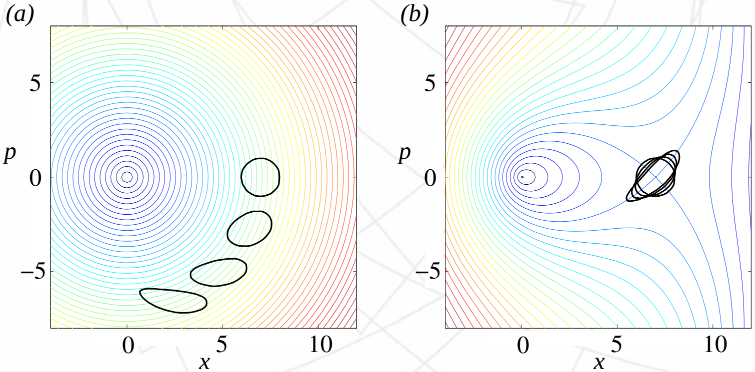
$$\begin{aligned} \omega_v &= \pm \frac{g}{\sqrt{2}} \frac{1}{2\sqrt{x^2 + p^2}}, \\ \omega_c &= \pm \frac{g}{\sqrt{2}} \frac{1}{\sqrt{x^2 + p^2}}, \\ (x_r, p_r) &= (-x_0, -p_0). \end{aligned}$$

Compensating Hamiltonian:

$$H_{\mathrm{ad}} = \mp \frac{g}{4\sqrt{2}\sqrt{x_0^2 + p_0^2}} \left[(x + x_0)^2 + (p + p_0)^2 \right].$$

Jaynes-Cummings Hamiltonian

Phase trajectories



Summary

Squeezing with classical Hamiltonians

- Squeezing rate for planar phase-space $Q = \sqrt{(H_{pp} - H_{xx})^2 + 4H_{xp}^2}$, $\dot{V}_\pm = \pm Q V_\pm$.
- Interpretation: for zero-gradient points, difference of principal curvatures of the Hamiltonian
- Add non-squeezing Hamiltonians to keep the state at right place and optimally oriented

For initial stages of squeezing, classical formulas perfectly agree with quantum predictions.