Geometrical approach to theory of array detectors

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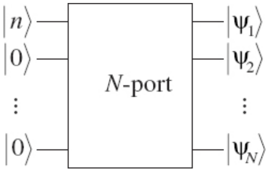
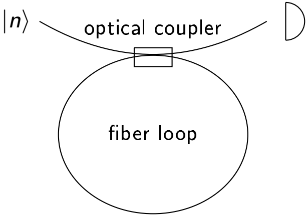
Array detectors



Array detection scheme



Loop detector

Light in state $|n\rangle$ is combined with $N-1$ vacuum states by N-port unitary operation. The final state is detected by N on/off-detectors

Binomial POVM:

$$\hat{\Pi}_k = \binom{N}{k} : \left(e^{-\left(\eta \frac{\hat{n}}{N} + \nu\right)}\right)^{N-k} \left(1 - e^{-\left(\eta \frac{\hat{n}}{N} + \nu\right)}\right)^{k} : \tag{1}$$

where N is the number of detectors in the array, η is the detection efficiency, ν corresponds to dark-count rate.

The corresponding operator of photocounts $\hat{c}$ is given by

$$\hat{c} = \sum_{n=0}^{+\infty} n \hat{\Pi}_n. \tag{2}$$

The eigenstate projectors for this operator are not equal to the POVM, cf. Eq. (1), and its eigenvalues are not equal to n

To estimate the mean value of a certain operator $\hat{A}$, which commutes with the photon-number operator з $\hat{n}$,

$$\left[\hat{A}, \hat{n}\right] = 0. \tag{3}$$

Examples of $\hat{A}$:

1. The phtoton-number operator $\hat{n}$;
2. The phtocounting operator $\hat{c}$;
3. The moments $\hat{n}^m$;
4. The normally-oredered moments $:\hat{n}^m:$;
5. The projectors $|n\rangle\langle n|$;
6. The POVM's with different values of parameters η, ν, N;
7. etc.

From the experiment we have an information about probabilities p_n given by (**??**).
Our aim is to find such coefficients C_n that

$$\left\langle \hat{A} \right\rangle = \sum_n C_n p_n + R, \tag{4}$$

where R describes a possible systematic error caused by incompleteness of the obtained information. This error should also be estimated.

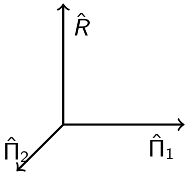
Set of POVMs

$$\hat{\Pi}_k = \binom{N}{k} : \left(e^{-\left(\eta \frac{\hat{n}}{N} + \nu\right)}\right)^{N-k} \left(1 - e^{-\left(\eta \frac{\hat{n}}{N} + \nu\right)}\right)^{k} :, \qquad k = 0, \dots N$$

forms a non-orthogonal basis in the Hilbert space of operator $\hat{A}$. One can expand $\hat{A}$ in this basis:

$$\hat{A} = \sum_n A^n \hat{\Pi}_n + \hat{R}, \tag{5}$$

The task is to find $A^n = \mathrm{Tr}\left[\hat{A}\,\hat{\Pi}^n\right]$ та $\left\langle \hat{R} \right\rangle$



$\hat{R}$ is the orthogonal completion of $\hat{A}$ for the basis $\hat{\Pi}_n$.

POVM $\hat{\Pi}_n$ form affine basis. For such a basis one can define the covariant metric tensor

$$g_{nm} = \mathrm{Tr}\left[\hat{\Pi}_n \hat{\Pi}_m\right]. \tag{6}$$

Inverse to g_{nm} is contravariant metric tensor g^{nm}

$$\sum_k g_{nk}\, g^{km} = \delta_n{}^m. \tag{7}$$

$$g_{nm} = \binom{N}{n}\binom{N}{m}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}\frac{(-1)^{k+l}}{1-\frac{(n-l)(m-k)}{N^2}} \tag{8}$$

$$\hat{g}(8) = \begin{pmatrix} 1.00 & 0 & 0 & \dots & 0 \\ 0 & 1.02 & 0.115 & \dots & 2.07 \times 10^{-9} \\ 0 & 0.115 & 0.883 & \dots & 3.53 \times 10^{-6} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 2.07 \times 10^{-9} & 3.53 \times 10^{-6} & \dots & \infty \end{pmatrix} \quad (9)$$

Regularisation

$$g_{nm}(\chi) = \binom{N}{n}\binom{N}{m} \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k}\binom{n}{l} \times \frac{(-1)^{k+l} N^2}{N^2 - (1-\chi)^2 (n-l)(m-k)}, \quad (10)$$

When $\chi \to 0$ $g_{nm}(\chi) \to g_{nm}$.

Error can be obtained by estimating the term $\left\langle \hat{R} \right\rangle$ in Eq. (5).

$$\left\langle \hat{R} \right\rangle^2 \leq \left\langle \hat{R}^2 \right\rangle \leq \mathrm{Tr}\left[\hat{R}^2\right]. \tag{11}$$

Utilizing the fact that $\hat{R}$ is orthogonal to the subspace spanned on $\hat{\Pi}_n$,

$$\left|\left\langle \hat{R} \right\rangle\right| \leq \left(\mathrm{Tr}\left[\hat{A}^2\right] - \sum_n A_n A^n\right)^{\frac{1}{2}}. \tag{12}$$

where

$$A_n = \mathrm{Tr}\left[\hat{A}\,\hat{\Pi}_n\right]. \tag{13}$$

The phtocounting operator
$\hat{c} = \sum_{n=0}^{N} n \hat{\Pi}_n = N \left(1 - : e^{-\frac{\hat{n}}{N}} \right) :$
Expansion coefficients for it:

$$\mathrm{Tr}\left\{ \hat{c}\ \hat{\Pi}^k(\chi) \right\} = \sum_{m=0}^{\mathrm{N}} \sum_{l=0}^{m} g^{mk} \binom{N}{m} \binom{m}{l} (-1)^l \sum_{n=0}^{\mathrm{N}} \sum_{j=0}^{m} n \binom{N}{n} \binom{n}{j} \frac{(-1)^j}{1 - \frac{(1-\chi)^2 (m-l)(n-j)}{N^2}}. \quad (14)$$

The components of $\hat{c}$ in POVM basis are the numbers of photocounts n:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$\mathrm{Tr}\left\{ \hat{c}\ \hat{\Pi}^k \right\}$	0	1	2	3	4	5	6	7	8

(15)

$$\left| \left\langle \hat{R} \right\rangle \right| = 0 \quad (16)$$

The truncated photon-number operator $\hat{n}_M$

$$:\hat{n}_M^l := \sum_{n=0}^{M} n^l \left|n\right\rangle \left\langle n\right| . \tag{17}$$

Expansion coefficients for $:\hat{n}_M^l:$, including $\hat{n}_M$:

$$\mathrm{Tr}\left[:\hat{n}_M^l: \ \hat{\Pi}^k\right] = \sum_{n=0}^{M} n^l \sum_{m=0}^{N} \sum_{l=0}^{m} (-1)^l g^{mk} \binom{N}{m} \binom{m}{l} \frac{((1-\chi)(m-l))^n}{N^n} \tag{18}$$

Assuming photocounts have Poisson distribution

$\langle\hat{c}\rangle$	1.0	2.0	3.0	4.0		$\left\|\left\langle\hat{R}\right\rangle\right\|_{max}$
$\langle\hat{n}_4\rangle$	1.002	2.061	2.943	3.154		0.661
$\langle\hat{n}_8\rangle$	1.079	2.260	3.155	3.378		4.99

(19)

Error grows as number of clicks gets closer to N

	$\hat{n}_0$	$\hat{n}_1$	$\hat{n}_2$	$\hat{n}_3$	$\hat{n}_4$	$\hat{n}_5$	$\hat{n}_6$	$\hat{n}_7$	$\hat{n}_8$
$\left\|\left\langle\hat{R}\right\rangle\right\| \leq$	0.0	0.00240	0.08062	0.661	1.89	2.35	3.532	4.34	4.99

(20)

Symmetrically-oredered moments

$$\hat{n}_M^l = \sum_{k=0}^{l} S(l,k) \ : \hat{n}_M^k : . \tag{21}$$

where $S(l,k)$ are Stirling numbers of the second kind

Moments of photons and photocounts

	photons	clicks		photons	clicks		photons	clicks
μ_1	1.08	1.0		2.26	2.0		3.15	3.0
μ_2	2.44	2.0		8.11	6.0		14.54	12.0
μ_3	7.22	5.0		36.23	22.0		78.46	57.0
μ_4	26.54	15.0		188.20	94.0		468.76	309.0
μ_5	115.96	52.0		1085.83	454.0		3001.54	1866.0
μ_6	580.56	203.0		6751.6	2430.0		20180.80	12351.0

- We have suggested an approach that allows to find mean values for operators that commute with the photon number operator, if statistics of photocounts is given. And to estimate systematic error occurring.
- As an example we apply this approach to find the photon statistics from the statistics of photocounts and to see dependency of error on number of detectors in the array.
- The approach seems to be reasonable at least in case of small detectors.

Thank you for your attention!