Effective Dynamics for Open System: Time Average Approach

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Outline

1. Approaches to Effective Dynamics
2. Time Average Formalism for Closed Systems
3. Time Average Formalism for Open Systems
4. Examples
5. Summary and Outlook

Approaches to Effective Dynamics

Motivation

- Different time scales of dynamics
  - rapidly vs. slowly evolving components with large difference in time scales
  - Fast variables follow adiabatically slow degrees of freedom. (e.g. spinning top: While it is spinning at a high frequency, the rotation axis is usually precessing much slower.)
- Simpler (and more intuitive) description
  - smaller dimension of Hilbert space (e.g. (3-level) Λ system $\rightarrow$ effective 2-level system)
  - often possible to derive simple laws for slow variables when the scales are well separated
  - might be useful for gaining insight into gating operation or state engineering
- Less computational burden

Formalisms for Effective Dynamics

- Adiabatic elimination (AE)
  - Standard technique — Gardiner & Zoller (2000)
  - Sometimes tedious step of calculation in store
- Effective operator formalism — Reiter & Sørensen (2012)
  - Easy-to-use AE for open systems
  - Only ground state manifold is maintained.
- Flow equation (FE) approach — Kehrein (2006)
  - Aalogous to renormalization group procedure
  - Not yet developed for open systems
- Time average (TA) formalism — Gamel & James (2010)
  - Easy-to-use generalization of RWA
  - Analogous to FE in sense of not eliminating excited states
  - Developed for open systems in this work

Comparison of Formalisms

- In view of Hilbert space transformation

Kehrein (2006)

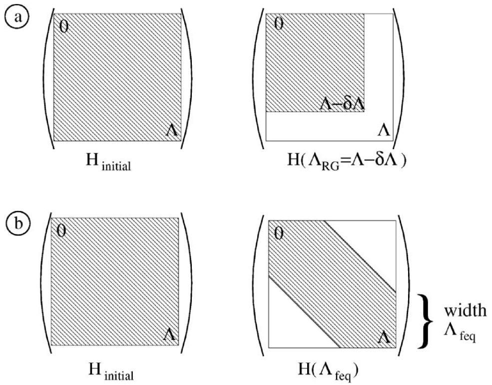


Fig. 1.1. Schematic view of different scaling approaches: (**a**) Conventional scaling methods successively reduce the high-energy cutoff Λ_{RG}. (**b**) Flow equations make the Hamiltonian successively more band-diagonal with an effective band width Λ_{feq}

- Classification

  ⓐ → Remove high energy sector (Renormalization group (RG), adiabatic elimination)

  ⓑ → Narrow energy band (Flow equation (FE), time average)

  (RG, FE → successive transformation)

Time Average Formalism for Closed Systems

Time Average Formalism

- Time average of an operator

$$\overline{O}(t) \equiv \int_{-\infty}^{\infty} dt' f(t-t') O(t') \quad [\text{or } \overline{O}(\omega) = f(\omega) O(\omega)]$$

- Gamel's time average: $f(\omega) \sim \Theta(\omega_c - |\omega|)$

$$\overline{e^{\pm i\omega_n t}} = 0, \quad \overline{e^{\pm i(\omega_m + \omega_n)t}} = 0, \quad \overline{e^{\pm i(\omega_m - \omega_n)t}} = e^{\pm i(\omega_m - \omega_n)t}$$

(set ω_c s.t. $|\omega_m - \omega_n| < \omega_c < \omega_n$ for any $\omega_m,\ \omega_n$)

- Rotating wave approximation (RWA)
  $\Longrightarrow$ Time average with sharp cutting-off

$$H_{\text{Rabi}} \sim (a e^{-i\omega t} + a^\dagger e^{i\omega t})(\sigma_- e^{-i\omega_0 t} + \sigma_+ e^{i\omega_0 t})$$

$$= a\sigma_+ e^{-i(\omega-\omega_0)t} + a^\dagger \sigma_- e^{i(\omega-\omega_0)t} + a\sigma_- e^{-i(\omega+\omega_0)t} + a^\dagger \sigma_+ e^{i(\omega+\omega_0)t}$$

$$\therefore \overline{H_{\text{Rabi}}} \sim a\sigma_+ e^{-i(\omega-\omega_0)t} + a^\dagger \sigma_- e^{i(\omega-\omega_0)t} \quad (\longrightarrow H_{\text{JC}})$$

Time Average Formalism (Cont'd)

- Other time average method: $f(\omega) \sim \exp(-\omega^2/\omega_c^2)$
  $\Longrightarrow$ smooth frequency filtering [Wang & Haw (2015)]
  $\Longrightarrow$ The counter-rotating terms can survive but the dynamics is now dependent on ω_c.
- von Neumann equation and its solution

$$d\rho/dt = -i[\lambda H(t), \rho] \implies \rho(t) = U(t)\,\rho(0)U^\dagger(t)$$

  (λ: a (real) bookkeeping paramerter (later becomes unity))
- (Unitary) Time evolution operator

$$U(t) \equiv \mathbf{T}e^{-i\int_0^t H(t')dt'} \qquad (\mathbf{T}\text{: time-ordering operator})$$

$$= I - i\lambda \int_0^t dt_1 H(t_1) + (-i\lambda)^2 \int_0^t dt_1 \int_0^{t_1} dt_2\, H(t_1)H(t_2) + \cdots$$

$$\equiv I + \lambda U_1(t) + \lambda^2 U_2(t) + \cdots$$

Time Average Formalism (Cont'd)

- Inverse of the time evolution operator

$$U^{-1}(t) = U^{\dagger}(t) = \tilde{\mathbf{T}} e^{+i\int_0^t H(t')dt'} \quad (\tilde{\mathbf{T}}\text{: anti-time-ordering operator})$$

$$= I + i\lambda \int_0^t dt_1 H(t_1) + (i\lambda)^2 \int_0^t dt_1 \int_0^{t_1} dt_2\, H(t_2)H(t_1) + \cdots$$

$$= I + \lambda U_1^{\dagger}(t) + \lambda^2 U_2^{\dagger}(t) + \cdots$$

- Useful properties

$$U_1^{\dagger} = -U_1, \quad U_2^{\dagger} = U_1^2 - U_2, \quad i\dot{U}_n(t) = H(t)U_{n-1}(t), \cdots$$

- Time-averaged evolution operator

$$\overline{\rho(t)} = \overline{U(t)\,\rho_0 U^{\dagger}(t)} = \sum_{k=0}^{\infty} \lambda^k \sum_{j=0}^{k} \overline{U_{k-j}\rho_0 U_j^{\dagger}}$$

$$\equiv \sum_{k=0}^{\infty} \lambda^k \mathscr{E}_k[\rho_0] \equiv \mathscr{E}[\rho_0] \qquad (\rho_0 = \rho(0),\ \mathscr{E}_0 = I)$$

Time Average Formalism (Cont'd)

- Inverse of time-averaged evolution operator

$$\rho_0 = \mathscr{E}^{-1}[\overline{\rho}] \equiv \mathscr{F}[\overline{\rho}] \equiv \sum_k \lambda^k \mathscr{F}_k[\overline{\rho}]$$

- Identity relation: $\mathscr{F}[\mathscr{E}[\rho]] = \sum_{k=0}^{\infty} \lambda^k \sum_{j=0}^{k} \mathscr{F}_j[\mathscr{E}_{k-j}[\rho]] = \rho$

$$\implies \mathscr{F}_0 = \mathscr{E}_0 = I, \quad \mathscr{F}_1 = -\mathscr{E}_1, \quad \mathscr{F}_2 = \mathscr{E}_1^2 - \mathscr{E}_2, \quad \cdots .$$

- Time average of von Neumann eq.

$$i\dot{\bar{\rho}}(t) = i\dot{\mathscr{E}}[\rho_0] = i\dot{\mathscr{E}}[\mathscr{F}[\bar{\rho}(t)]] = i\sum_{k=0}^{\infty} \lambda^k \sum_{j=0}^{k} \dot{\mathscr{E}}_j[\mathscr{F}_{k-j}[\bar{\rho}(t)]]$$

$$\equiv \sum_{k=0}^{\infty} \lambda^k \mathscr{L}_k[\bar{\rho}(t)],$$

Time Average Formalism (Cont'd)

- Time average of von Neumann eq. (Cont'd)

$$
\begin{aligned}
\mathscr{L}_0[\bar\rho] &= i\dot{\mathscr{E}}_0[\mathscr{F}_0[\bar\rho]] = 0,\\
\mathscr{L}_1[\bar\rho] &= i\dot{\mathscr{E}}_0[\mathscr{F}_1[\bar\rho]] + i\dot{\mathscr{E}}_1[\mathscr{F}_0[\bar\rho]] = [\bar H, \bar\rho],\\
\mathscr{L}_2[\bar\rho] &= \overbracket{H U_1}\bar\rho + \overbracket{H\bar\rho U_1^\dagger} - \bar\rho\overbracket{U_1^\dagger H} - \overbracket{U_1\bar\rho H},\\
&\quad\cdots
\end{aligned}
$$

where $\overbracket{AB} \equiv \overline{AB} - \bar A\bar B, \quad \overbracket{A\bar\rho B} \equiv \overline{A\bar\rho B} - \bar A\bar\rho\bar B.$

- Master equation up to 2nd order

$$i\dot\rho = [H_{\text{eff}}, \rho] + \left\{\tfrac{1}{2}(A - A^\dagger), \rho\right\} + \overbracket{H\bar\rho U_1^\dagger} - \overbracket{U_1\bar\rho H},$$

$$[\ A \equiv \overbracket{HU_1},\ A^\dagger \equiv \overbracket{U_1^\dagger H}, \text{ and } H_{\text{eff}} \equiv \overline{H} + \tfrac{1}{2}(A + A^\dagger)\ ]$$

Time Average Formalism (Cont'd)

- Typical form of (interaction) Hamiltonian

$$H = H_0 + \sum_n h_n e^{i\omega_n t} + h_n^\dagger e^{-i\omega_n t} \quad (H_0 : \text{time-independent})$$

- Time-averaged master equation $\Longrightarrow$ *Lindblad form*!

$$i\dot{\overline{\rho}} = [H_0, \overline{\rho}] + \sum_{m,n} \left[\frac{l_m l_n^\dagger}{\omega_n} - \frac{l_n^\dagger l_m}{\omega_m}, \overline{\rho} \right] + \frac{2}{\omega_{mn}^-}(l_m \overline{\rho} l_n^\dagger + l_n^\dagger \overline{\rho} l_m)$$

$$\Longrightarrow [H_{\text{eff}}, \overline{\rho}] + \sum_{m,n} \frac{2}{\omega_{mn}^-} \left[\mathscr{D}_{l_m, l_n^\dagger} \overline{\rho} - \mathscr{D}_{l_m^\dagger, l_n} \overline{\rho} \right]$$

where $\mathscr{D}_{A,B}\rho \equiv A\rho B - \frac{1}{2}(AB\rho + \rho AB)$, $\frac{1}{\omega_{mn}^\pm} \equiv \frac{1}{2}\left(\frac{1}{\omega_m} \pm \frac{1}{\omega_n}\right)$,
$l_n \equiv h_n e^{i\omega_n t}$, and

$$H_{\text{eff}} \equiv H_0 + \sum_{m,n} \frac{1}{\omega_{mn}^+}[l_m, l_n^\dagger].$$

Time Average Formalism for Open Systems

Time Average Formalism for an Open System

- Lindblad form of master equation in an open system

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_i \left(\mathcal{J}_{L_i}\rho - \mathcal{K}_{L_i}\rho\right),$$

where $\mathcal{J}_L\,\rho \equiv L\,\rho L^\dagger, \quad \mathcal{K}_L\,\rho \equiv \frac{1}{2}\left(L^\dagger L\,\rho + \rho L^\dagger L\right)$

- Let $\mathcal{K}_{\text{tot}} = \sum_i \mathcal{K}_{L_i}$ then

$$\begin{aligned} e^{\mathcal{K}_{\text{tot}} t}\rho &= \left[1 + \mathcal{K}_{\text{tot}} t + \tfrac{1}{2}(\mathcal{K}_{\text{tot}} t)^2 + \cdots\right]\rho \\ &= \rho + (K\rho + \rho K)t + \tfrac{1}{2}(K^2\rho + 2K\rho K + \rho K^2)t^2 + \cdots \\ &= e^{Kt}\rho\, e^{Kt} \qquad\qquad (K \equiv \tfrac{1}{2}\sum_i L_i^\dagger L_i) \end{aligned}$$

$$\Longrightarrow e^{Kt}\dot{\rho}e^{Kt} = -i\left(H_K\rho_K - \rho_K H_K^\dagger\right) + \sum_i \mathcal{J}_{L_{i,K}}\rho - e^{Kt}(K\rho + \rho K)e^{Kt}$$

with $\rho_K \equiv e^{Kt}\rho e^{Kt},\ H_K \equiv e^{Kt}He^{-Kt},\ L_{i,K} \equiv e^{Kt}L_i e^{-Kt}.$

Time Average Formalism for an Open System (Cont'd)

- Transformation 1 $\longrightarrow$ decaying frame

$$\frac{d\rho_K}{dt} = -i\left(H_K\rho_K - \rho_K H_K^\dagger\right) + \sum_i \mathcal{J}_{L_{i,K}}\rho_K.$$

- Observation: in many practical cases

$$[K,\ L_i] = -\gamma_{i,K}L_i/2 \qquad \text{for some constant } \gamma_{i,K}$$

then

$$L_{i,K} \equiv e^{Kt}L_i e^{-Kt} = L_i + [K, L_i]t + \frac{1}{2!}[K,[K,L_i]]t^2 + \cdots = e^{-\gamma_{i,K}t/2}L_i.$$

- *Pseudo*-time-evolution operator

$$U \equiv \mathbf{T}e^{-i\int_0^t H_K(t')dt'}$$

Also note that

$$U^{-1} = \tilde{\mathbf{T}}e^{+i\int_0^t H_K dt'},\ U^\dagger = \tilde{\mathbf{T}}e^{+i\int_0^t H_K^\dagger dt'},\ (U^{-1})^\dagger = \mathbf{T}e^{-i\int_0^t H_K^\dagger dt'}.$$

Time Average Formalism for an Open System (Cont'd)

- Let $\rho_U \equiv U^{-1}\rho_K(U^{-1})^\dagger$ then

$$\frac{d\rho_U}{dt} = U^{-1}\dot{\rho}_K(U^{-1})^\dagger + U^{-1}i(H_K\rho_K - \rho_K H_K)(U^{-1})^\dagger,$$

- Transformation 2 $\longrightarrow$ pseudo-rotating frame

$$\frac{d\rho_U}{dt} = \sum_i U^{-1}\mathcal{J}_{L_{i,K}}\rho_K(U^{-1})^\dagger = \sum_i L_{i,U}\rho_U L_{i,U}^\dagger$$

$$= \sum_i \mathcal{J}_{L_{i,U}}\rho_U = \mathcal{J}_{\text{tot}}\rho_U \quad (L_{i,U} \equiv U^{-1}L_{i,K}U,\ \mathcal{J}_{\text{tot}} \equiv \sum_i \mathcal{J}_{L_{i,U}})$$

$$\rho_U(t) = e^{\int_0^t \mathcal{J}_{\text{tot}}dt}\rho_0$$

$$= \rho_0 + \sum_i \int_0^t dt_1 L_{i,U}(t_1)\rho_0 L_{i,U}^\dagger(t_1) + \frac{1}{2!}\sum_{i,j}\int_0^t dt_1 \int_0^{t_1} dt_2$$

$$\times\ L_{i,U}(t_1)L_{j,U}(t_2)\rho_0 L_{j,U}^\dagger(t_2)L_{i,U}^\dagger(t_1) + \cdots .$$

Time Average Formalism for an Open System (Cont'd)

- Up to 1st order of decay constants

$$\rho_U(t) \approx \left(1 + \int_0^t \mathcal{J}_{\text{tot}} dt\right) \rho_0 \equiv (1 + \mathcal{J}_{\text{int}})\, \rho_0$$

- Return to ρ_K and take time average

$$\overline{\rho}_K(t) = \overline{U(1 + \mathcal{J}_{\text{int}})\rho_0 U^\dagger} \equiv \mathcal{E}[\rho_0] = \sum_k \mathcal{E}_k[\rho_0]$$

- Expansion of $L_{i,U}$ and $\mathcal{J}_{\text{int}}$ according to the order of H_K

$$\begin{aligned} L_{i,U} &= U^{-1} L_{i,K} U \\ &= e^{-\gamma_{i,K} t/2}\{L_i + [L_i, U_1] + ([L_i, U_2] - U_1[L_i, U_1]) + \cdots\} \\ &\equiv L_{i,0} + L_{i,1} + L_{i,2} + \cdots \end{aligned}$$

$$\begin{aligned} \mathcal{J}_{\text{int}} &= \int_0^t \mathcal{J}_{\text{tot}} dt \equiv \int_0^t \mathcal{J}_0^U dt + \int_0^t \mathcal{J}_1^U dt + \int_0^t \mathcal{J}_2^U dt + \cdots \\ &\equiv \mathcal{J}_0 + \mathcal{J}_1 + \mathcal{J}_2 + \cdots \end{aligned}$$

Time Average Formalism for an Open System (Cont'd)

$$\mathcal{J}_0^U \rho \equiv \sum_i L_{i,0}\, \rho L_{i,0}^\dagger = \sum_i e^{-\gamma_{i,K} t} L_i\, \rho L_i$$

$$\mathcal{J}_1^U \rho \equiv \sum_i \left(L_{i,0}\, \rho L_{i,1}^\dagger + L_{i,1}\, \rho L_{i,0}^\dagger \right)$$

$$\mathcal{J}_2^U \rho \equiv \sum_i \left(L_{i,0}\, \rho L_{i,2}^\dagger + L_{i,1}\, \rho L_{i,1}^\dagger + L_{i,2}\, \rho L_{i,0}^\dagger \right).$$

$$\mathcal{E}_0[\rho] = (1 + \overline{\mathcal{J}_0})\rho = (1 + \mathcal{J}_0)\rho,$$

$$\mathcal{E}_1[\rho] = \overline{\mathcal{J}_1}\rho + \overline{U_1}(1 + \mathcal{J}_0)\rho + [(1 + \mathcal{J}_0)\rho]\overline{U_1^\dagger},$$

$$\mathcal{E}_2[\rho] = \overline{\mathcal{J}_2}\rho + \overline{U_2}(1 + \mathcal{J}_0)\rho + \overline{U_1[(1 + \mathcal{J}_0)\rho]U_1^\dagger} + [(1 + \mathcal{J}_0)\rho]\overline{U_2^\dagger} + \overline{(\mathcal{J}_1\rho)U_1^\dagger} + \overline{U_1(\mathcal{J}_1\rho)},$$

$$\mathcal{F}_0[\rho] = (1 - \mathcal{J}_0)\rho,$$

$$\ldots$$

Time Average Formalism for an Open System (Cont'd)

- Time-averaged master equation for $\overline{\rho}_K$

$$
\begin{aligned}
i\,\dot{\overline{\rho}}_K(t) = i\dot{\mathscr{E}}[\rho_0] = i\dot{\mathscr{E}}[\mathscr{F}[\overline{\rho}_K(t)]] = i\sum_{k=0}^{\infty}\sum_{j=0}^{k}\dot{\mathscr{E}}_j[\mathscr{F}_{k-j}[\overline{\rho}_K(t)]] \\
\equiv \sum_{k=0}^{\infty}\mathscr{L}_k[\overline{\rho}_K(t)],
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{L}_0[\rho] &= i\mathscr{J}_0^{U}\rho \;\; (\neq 0)\,, \\
\mathscr{L}_1[\rho] &= \overline{H_K}\rho - \rho\overline{H_K^{\dagger}}, \\
\mathscr{L}_2[\rho] &= \overbracket{H_K U_1}\rho + \overbracket{H_K\rho U_1^{\dagger}} - \overbracket{U_1\rho H_K^{\dagger}} - \rho\overbracket{U_1^{\dagger}H_K^{\dagger}} \\
&\quad + \overbracket{H_K(\mathscr{J}_1}\rho) - \overbracket{(\mathscr{J}_1\rho)H_K^{\dagger}} \\
&\cdots.
\end{aligned}
$$

Time Average Formalism for an Open System (Cont'd)

- Time-averaged master equation for ρ

$$
\begin{aligned}
i\dot{\overline{\rho}}(t) &= -i\sum_i \mathcal{K}_{L_i}\overline{\rho} + e^{-Kt}\Big(\mathcal{L}_0[\overline{\rho}_K] + \mathcal{L}_1[\overline{\rho}_K] + \mathcal{L}_2[\overline{\rho}_K]\Big)e^{-Kt} \\
&= i\sum_i \left(\mathcal{J}_{L_i}\rho - \mathcal{K}_{L_i}\rho\right) + [\overline{H},\overline{\rho}] + \overline{H\tilde{U}_1}\overline{\rho} + \overline{H\overline{\rho}\tilde{U}_1^\dagger} \\
&\quad - \overline{\tilde{U}_1\overline{\rho}H} - \overline{\overline{\rho}\tilde{U}_1^\dagger H} + \overline{H(\tilde{\mathcal{J}}_1\overline{\rho})} - \overline{(\tilde{\mathcal{J}}_1\overline{\rho})H} \\
&= [H_{\text{eff}},\overline{\rho}] + i\sum_i \left(\mathcal{J}_{L_i}\rho - \mathcal{K}_{L_i}\rho\right) + \left\{\tfrac{1}{2}(A - A^\dagger),\rho\right\} \\
&\quad + \overline{H\overline{\rho}\tilde{U}_1^\dagger} - \overline{\tilde{U}_1\overline{\rho}H^\dagger} + \overline{H(\tilde{\mathcal{J}}_1\overline{\rho})} - \overline{(\tilde{\mathcal{J}}_1\overline{\rho})H^\dagger}
\end{aligned}
$$

where H_{eff} is the same as before with $A \equiv \overline{H\tilde{U}_1}$, $A^\dagger \equiv \overline{\tilde{U}_1^\dagger H}$, $\tilde{U}_1 \equiv e^{-Kt}U_1\,e^{Kt}$, and $\tilde{\mathcal{J}}_1\overline{\rho} \equiv e^{-Kt}(\mathcal{J}_1\overline{\rho}_K)\,e^{-Kt}$.

Time Average Formalism for an Open System (Cont'd)

- Another assumption for time average process

$$\overline{O_K} = \overline{e^{Kt} O\, e^{Kt}} \approx e^{Kt} \overline{O}\, e^{Kt} \quad \text{for a generic operator } O$$

  (A decay process is relatively slower than any other oscillatory process.)

- Master equation up to 2nd order of Hamiltonian (and up to 1st order of decay constants)

$$i\,\dot{\overline{\rho}} = [H_{\text{eff}}, \overline{\rho}] + \sum_n \mathscr{D}_{L_n, L_n^\dagger} \rho + \sum_{m,n} \frac{2}{\tilde{\omega}^-_{mn}} \left[\mathscr{D}_{l_m, l_n^\dagger} \overline{\rho} - \mathscr{D}_{l_m^\dagger, l_n} \overline{\rho} \right]$$
$$+ \overline{H (\tilde{\mathcal{J}}_1} \overline{\rho}) - (\overline{\tilde{\mathcal{J}}_1 \overline{\rho}) H},$$

  with $e^{Kt} h_n e^{-Kt} = h_n e^{-\frac{\delta_n}{2} t}$, $\tilde{\omega}_n \equiv \omega_n + i\frac{\delta_n}{2}$, $\frac{1}{\tilde{\omega}^{\pm}_{mn}} \equiv \frac{1}{2}\left(\frac{1}{\tilde{\omega}_m} \pm \frac{1}{\tilde{\omega}^*_n}\right)$, and

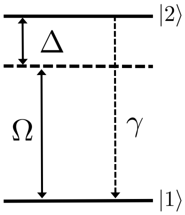
$$H_{\text{eff}} \equiv H_0 + \sum_{m,n} \frac{1}{\tilde{\omega}^+_{mn}} [l_m, l_n^\dagger].$$

Examples

Example: 2-level system

- Hamiltonian & decay operator

$$H = he^{i\Delta t} + h^\dagger e^{-i\Delta t},$$
$$h \;=\; \tfrac{\Omega}{2}\,|1\rangle\langle 2|\,,\; L = \sqrt{\gamma}\,|1\rangle\langle 2|$$



- Relation between decay operators

$$[\tfrac{1}{2}L^\dagger L, L] = \tfrac{\gamma}{2}L$$
$$\Longrightarrow H_K = he^{i\tilde{\Delta}t} + h^\dagger e^{-i\tilde{\Delta}t}, \quad \tilde{\Delta} \equiv \Delta + i\gamma/2$$
$$U_1 = \frac{1}{\tilde{\Delta}}\left(-he^{i\tilde{\Delta}t} + h^\dagger e^{-i\tilde{\Delta}t}\right) - \frac{1}{\tilde{\Delta}}\left(-h + h^\dagger\right),$$
$$\tilde{U}_1 = \frac{1}{\Delta}\left(-he^{i\Delta t} + h^\dagger e^{-i\tilde{\Delta}t}\right) - \frac{1}{\tilde{\Delta}}\left(-he^{\gamma t/2} + h^\dagger e^{-\gamma t/2}\right).$$

Example: 2-level system (Cont'd)

- Effective master equation (ME)

$$i\,\dot{\bar{\rho}} = [H_{\mathrm{eff}}, \bar{\rho}] + \mathscr{D}_{L,L^\dagger}\rho + \overbracket{H(\tilde{\mathcal{J}}}_1\bar{\rho}) - (\overbracket{\tilde{\mathcal{J}}_1\bar{\rho})H}$$

$$H_{\mathrm{eff}} = -\frac{\Omega^2\Delta}{4\Delta^2+\gamma^2}\sigma_z \xrightarrow{\gamma=0} -\frac{\Omega^2}{4\Delta}\sigma_z \text{ (ac-Stark shift)}$$

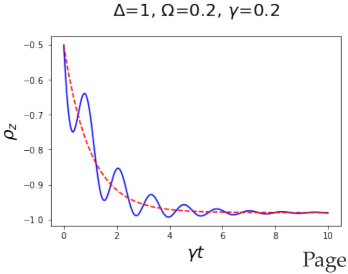
$$\overbracket{H(\tilde{\mathcal{J}}}_1\bar{\rho}) = \left[(\overbracket{\tilde{\mathcal{J}}_1\bar{\rho})H}\right]^\dagger = \frac{i\Omega^2\gamma}{4\Delta^2+\gamma^2}\left(|2\rangle\langle 2|\,\bar{\rho}\sigma_z - |1\rangle\langle 2|\,\bar{\rho}\,|2\rangle\langle 1|\right)$$

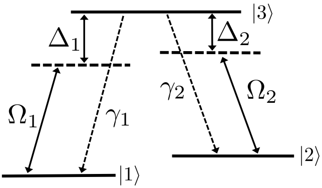
with $\sigma_z \equiv |2\rangle\langle 2| - |1\rangle\langle 1|$.

- Plot of atomic polarization

$$\rho_z \equiv \langle\sigma_z\rangle$$

(blue solid: original ME,
red dashed: effective ME)





- Hamiltonian & decay operator

$$H = h_1 e^{i\Delta_1 t} + h_2 e^{i\Delta_2 t} + \text{h.c.},$$

$$h_i = \frac{\Omega_i}{2}\,|i\rangle\langle 3|\,, \quad L_i = \sqrt{\gamma_i}\,|i\rangle\langle 3|$$

  - Relation between decay operators

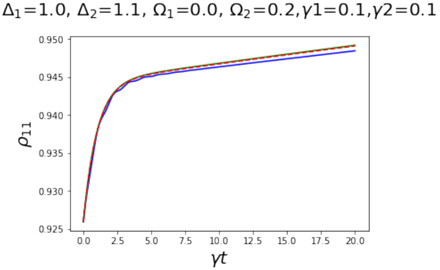
$$[(L_1^\dagger L_1 + L_2^\dagger L_2)/2, L_i] = (\gamma/2)L_i \ \ (\gamma \equiv \gamma_1 + \gamma_2)$$

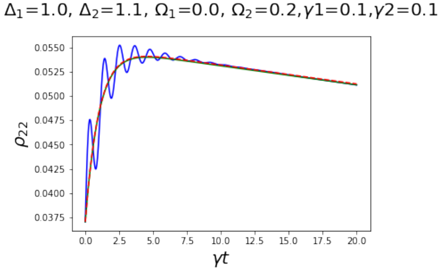
  - Effective Hamiltonian

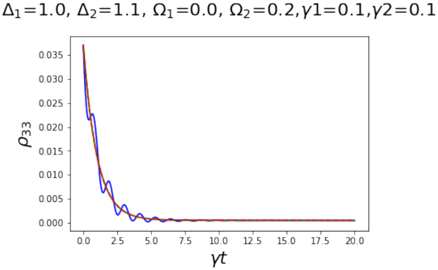
$$H_{\text{eff}} = \sum_i \frac{\Omega_i^2 \Delta_i}{4\Delta_i^2 + \gamma^2}\left(|i\rangle\langle i| - |3\rangle\langle 3|\right) + \left[\frac{(\Delta_1 + \Delta_2)\Omega_1\Omega_2}{8(\Delta_1 - i\gamma/2)(\Delta_2 + i\gamma/2)}\,|1\rangle\langle 2|\,e^{i\Delta_{12}t} + \text{h.c.}\right].$$

$\Longrightarrow$ Agrees with a different approach (e.g., Reiter & Sørenson).

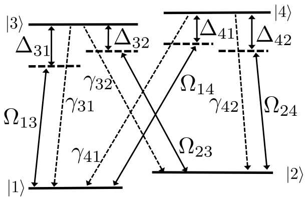
- Expectation values of populations: $\rho_{ii} = \langle i|\,\rho\,|i\rangle$

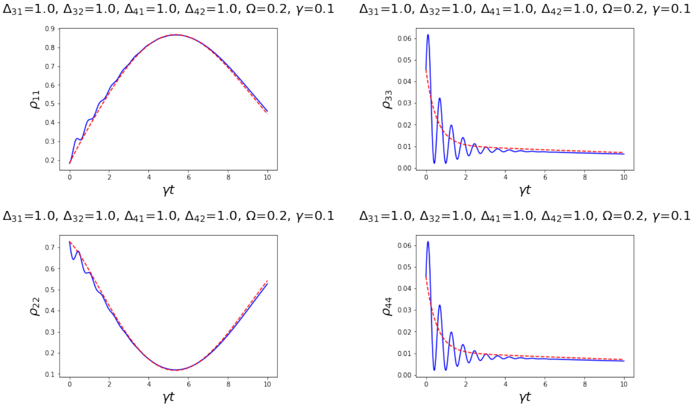






Example: 4-level double-Λ system





Summary and Outlook

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- Advantages over adiabatic elimination
  - easy and simple to use
  - need no knowledge of levels to be removed
  - provide infinite hierarchy in principle
  - An initial state can have excited state populations.
- Still lengthy and tedious algebra for *higher order* calculation
  $\longrightarrow$ Numerous terms appear in 4th order perturbation.
- Other approaches to be considered
  - Nonequilibrium funtional integral (insensitve to operator ordering) + RG (successive flow approximation)
  - Extending flow equation approach to an open system
  - Machine learning for numerical quantization