Running Boolean Matrix Factorization in Parallel

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Outline

1. the problem, our contribution
2. preliminaries in Boolean Matrix Factorization (BMF)
3. running BMF in parallel
   - base (sequential) algorithm
   - **general parallelization scheme**
4. experimental evaluation
5. conclusions

Boolean Matrix Factorization (BMF) in parallel?

- BMF also called **Boolean matrix decomposition**, **Boolean factor analysis**, ...

= (approximate) decomposition of Boolean matrix (entries 1 or 0) to (Boolean) matrix product of two Boolean matrices

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

- **optimization problems**:
  1. find a decomposition with inner matrix product dimension as low as possible for a given maximal decomposition approximation = **Approximate Factorization Problem (AFP)**
  2. find as exact decomposition as possible for a given maximal inner dimension = **Discrete Basis Problem (DBP)**
- least dimension of exact decomposition = Boolean (Schein) rank of matrix
- NP-hard problems → **approximation algorithms** for sub-optimal decompositions: GreConD, GreEss (both for AFP), Asso (for DBP) and other (PaNDa, Hyper)

Boolean Matrix Factorization (BMF) in parallel?

Algorithms

- **heuristic** = final decomposition constructed from partial (approximate) decompositions which are only **locally optimal**
- **sequential** = choice of optimal partial decomposition hardcoded in algorithm design, one cannot explore several most optimal (or even all) → parallel computation (preferred – multicore CPUs, GPGPU)
- no parallel algorithm for *Boolean* matrix factorization – there are for methods designed for real-valued matrices (SVD, NMF), but they lack interpretability when applied to Boolean matrices! – crucial for knowledge discovery ⇒ BMF more appropriate for Boolean matrices
- reasons? (most commonly used) greedy heuristic approach is inherently sequential, BMF is young compared to real-valued factorization methods (?)

Our contribution

- not a parallel BMF algorithm

→ **general parallelization scheme** to compute in parallel several locally optimal decompositions and select the most optimal one(s) hoping to find the globally optimal

- following several choices of locally most optimal partial decompositions in the heuristics, constructing several most optimal final decompositions – in more processes running simultaneously in parallel
- return the single most optimal decomposition or several top-k of them
- applicable to any sequential heuristic BMF algorithm – chosen GreConD for demonstration (simple, well-known, efficient)

Boolean Matrix Factorization (BMF) – preliminaries

= (approximate) decomposition of Boolean matrix I (entries 1 or 0) to (Boolean) matrix product of two Boolean matrices A and B

$$I_{ij} \approx (A \circ B)_{ij} = \max_{l=1}^{k} \min(A_{il}, B_{lj}) \qquad \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

- I … **object-attribute incidence relation**, A … **object-factor i. r.**, B … **factor-attribute i. r.**

= discovery of k **factors** (approximately) explaining I ≈ "new attributes":
$(A \circ B)_{ij}$ … "object i has attribute j ($I_{ij} = 1$) if and only if there exists a factor l that applies to i ($A_{il} = 1$) and j is one of the manifestations of l ($B_{lj} = 1$)"

- geometric view: factor ∼ **rectangle** full of 1s → decomposition of I ∼ **coverage** of 1s of I by rectangles

Boolean Matrix Factorization (BMF) – preliminaries

- optimization problem (AFP): find a decomposition with the number k of factors as small as possible such that $||I - A \circ B|| \leq \varepsilon$ ...explain a prescribed portion of data

$$E(I, A \circ B) = ||I - A \circ B|| = \sum_{i,j=1}^{m,n} |I_{ij} - (A \circ B)_{ij}|$$

- quality of decomposition $\rightarrow$ **coverage quality** of the first l factors:

$$c(l) = 1 - E(I, A \circ B)/||I||$$

Belohlavek R., Vychodil V.: Discovery of optimal factors in binary data via a novel method of matrix decomposition. *Journal of Computer and System Sciences* 76(1)(2010), 3–20.

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GreConD – base (sequential) algorithm

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- chosen base (sequential) BMF algorithm for demonstration of our general parallelization scheme

= **greedy search** for factors – each factor explains as much of input matrix as possible, until the prescribed number of 1s is covered (i.e designed for the AFP)

- factor = **maximal rectangle** – maximal numbers of objects and attributes, stems **Formal concept analysis (FCA)** (maximal rectangle $\sim$ formal concept) $\rightarrow$ "Greedy Concepts on Demand"

GRECOND – base (sequential) algorithm

- greedy "on demand" factor/rectangle computation, not selection among candidates
  - starting empty set of attributes is repeatedly grown by a selected attribute – such that the rectangle grown by the attribute covers as many still uncovered 1s in input matrix as possible, as long as the number of 1s increases
  - other attributes may be added with the selected one – due to construction as maximal rectangle = **closure** (with all attributes shared by all objects having the attributes, see the paper)
- char. vectors of object sets of rectangles = columns of object-factor matrix A
- char. vectors of attribute sets of rectangles = rows of factor-attribute matrix B
- in details commented pseudocode in the paper (Algorithm 1)

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ & \\ & \\ & \end{pmatrix} \circ \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \\ & & \end{pmatrix}$$

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Running GreConD in parallel?

- alone factors computed by GreConD optimal (in explaining as much of input matrix as possible), but several (or all) together may be not $\Rightarrow$ partial decompositions (factor + previous factors) only locally optimal
- can be more equally optimal factors $\rightarrow$ different final decompositions – will be important in experiments later

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- alone selected attributes in factor/rectangle computation optimal (in covering as many still uncovered 1s in input matrix as possible), but several together may be not $\Rightarrow$ partial factor (attribute + previous attributes) also only locally optimal
- can also be more equally optimal attributes to select $\rightarrow$ different factors

Running a BMF algorithm in parallel

= construct, simultaneously in parallel, several (locally) most optimal partial decompositions and select among them several most optimal final decompositions, in hope to find the globally optimal one

For GRECOND

- in factor search (= decomposition construction), compute several factors explaining most of the input matrix = several locally optimal partial decompositions – **parallel computation**
- in factor computation, select several attributes so that the corresponding rectangle covers most still uncovered 1s in the input matrix = several locally optimal partial factors – **serial computation**
- in details commented pseudocode in the paper (Algorithms 2 and 3):
  - several instances of (modified) GRECOND running simultaneously in parallel processes – each (serially) computing several most optimal distinct (partial) factors

  → **GRECONDP = GRECOND in Parallel runs**
  - joint construction of several most optimal decompositions of input matrix – sorted from the most optimal one

Experimental evaluation

- comparison of GreConDP with base GreConD: quality of decomposition $\to$ **coverage quality** – most important in evaluation of performance of BMF algorithms
  1. numbers of factors for large coverage $\rightsquigarrow$ low, slowly increasing (AFP view)
  2. values of coverge for few factors $\rightsquigarrow$ high, quickly increasing to 1 (DBP view)
- comparison of GreConD with other BMF algorithms in e.g.
  - Belohlavek R., Trnecka M.: From-Below Approximations in Boolean Matrix Factorization: Geometry and New Algorithm, *Journal of Computer and System Sciences* 81(8)(2015), 1678–1697.
- in the paper examined also similarities of several (most optimal) decompositions delivered by GreConDP – they are rather similar but starting from different
- running time?: time complexity not a primary concern in BMF, GreConDP $p/2$ times slower than GreConD for p times more processes than processor units

Experimental evaluation

Datasets

1. synthetic: matrix product of randomly generated matrices with known characteristics (density, inner product dimension), enables average case evaluation
2. real: real factors, well known from various BMF papers and UCI Machine Learning Repository[1]

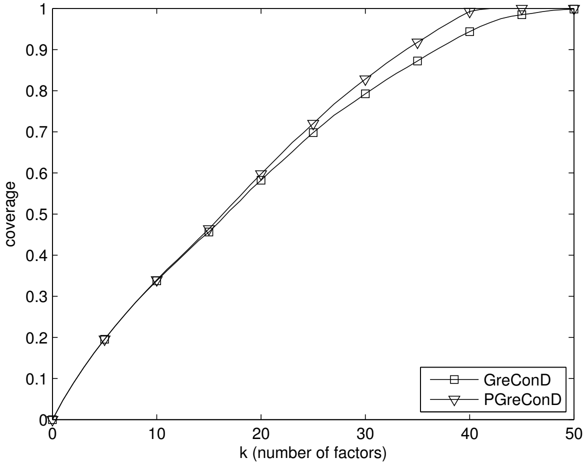
Dataset	Size	Dens. 1	Equal
Emea	3046×35	0.095	157.279
DBLP	19×6980	0.130	2.105
Firewall 1	365×709	0.124	31.168
Mushroom	8124×119	0.193	3.148
Paleo	501×139	0.051	5.868
Zoo	101×28	0.305	5.867

real datasets and their characteristics

- column **Equal** = average number of equally (locally) optimal factors per factor in GreConD – recall slide 18, **new characteristics** influencing results

[1] `archive.ics.uci.edu/ml`

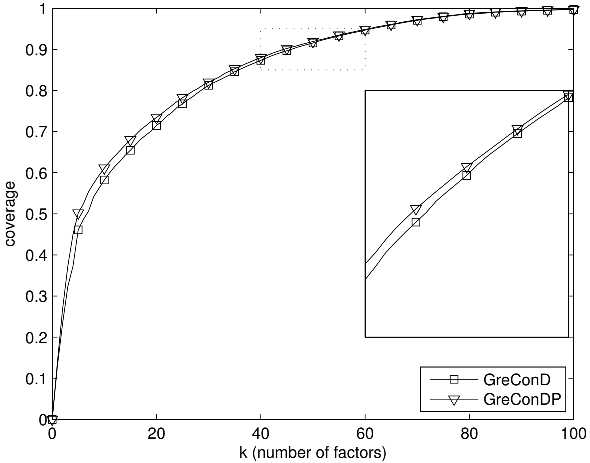
Experimental evaluation



synthetic datasets
16 parallel processes

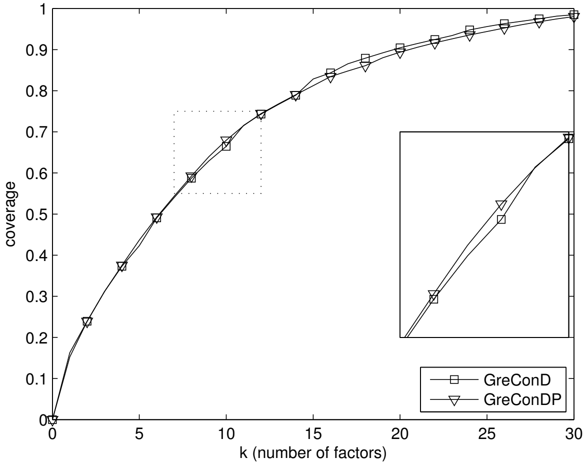
$k = 40$ . . . expected number of factors (inner matrix product dimension), delivered original factors

Experimental evaluation



Mushroom dataset
4 parallel processes

Experimental evaluation



Emea dataset,
4 parallel processes

GreConDP worse than GreConD (from AFP viewpoint) – extreme Equal characteristics, advantage of utilizing more equally optimal factors vanishes

Conclusions

- **general parallelization scheme for Boolean matrix factorization (BMF)** – applicable to any sequential heuristic BMF algorithm
- **new algorithm GreConDP** utilizing the scheme – based on simple, well-known and efficient GreConD
- in experiments GreConDP outperforms GreConD in quality of decomposition, at moderate computing time expenses
- **decomposition quality improvement** depends the number of parallel runs (higher = better) and the number of equally locally optimal factors in decomposition constructions (not much higher that the number of parallel runs)

Future research

- application to other BMF algorithms (GreEss, Asso)
- study of properties of the equally locally optimal factors – to factorize better