Fisher information and resolution beyond the Rayleigh limit

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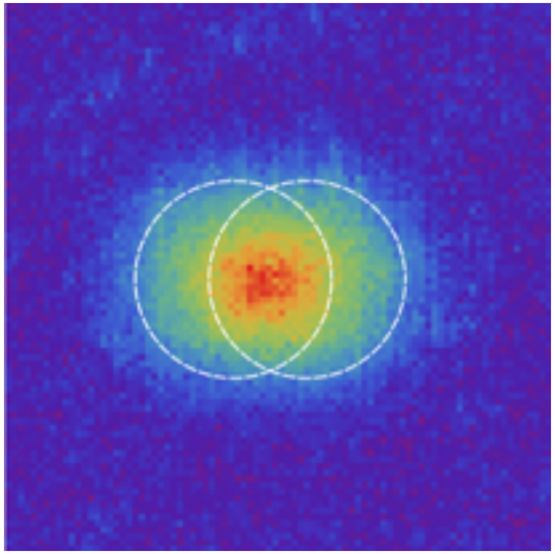
Outline

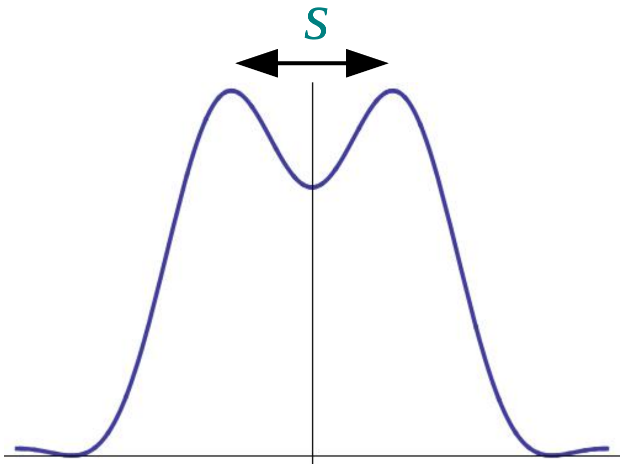
Background: Precision and Fisher information in optics

Quantum Fisher Information in general

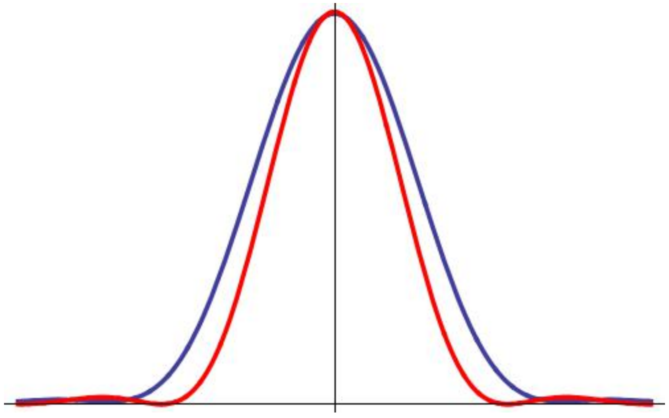
"Rayleigh limit" and two-point resolution

Optical resolution- Rayleigh criterion





standard resolution



super-resolution

Measurement and parameter estimation

Measurement: Born rule for (normalized) measurement on j-channel of transformed state

$$p_j(s) = \langle j|\rho(s)|j\rangle \qquad \rho(s) = U(s)^\dagger \rho U$$

$$A = \sum_j a_j |j\rangle\langle j| \qquad \Delta A = |\frac{\partial\langle A\rangle}{\partial s}|\Delta s$$

- Estimation: read-out of the parameter s from the registered values
- Variance of any unbiased estimation is limited by the Fisher Information (FI)
- Quantum Fisher Information (QFI) =Fisher information optimized over all possible detections

Fisher Information

$$\mathcal{F}_s = \mathbb{E}\left[\left(\frac{\partial \log p_n(s)}{\partial s}\right)^2\right] = \sum_n \frac{[p_n'(s)]^2}{p_n(s)}$$

Fisher information: limit for unbiased parameter estimation

$$\Delta s \geq 1\sqrt{nF}$$

Rayleigh curse

$$\mathcal{F}_s = \mathbb{E}\left[\left(\frac{\partial \log p_n(s)}{\partial s}\right)^2\right] = \sum_n \frac{[p_n'(s)]^2}{p_n(s)}$$

Fisher information for two point resolution: limit for unbiased parameter estimation

$$\Delta s \geq 1\sqrt{n\mathcal{F}}$$

$$p(x) = \frac{1}{2}[|\Psi(x+s)|^2 + |\Psi(x-s)|^2]$$
$$= I(x) + 1/2s^2 I''(x) + \ldots$$

$$\mathcal{F}_0 = s^2 \int dx \frac{I''(x)^2}{I(x)}$$

Quantum Fisher Information

For QFI, see the arguments of Helstrom 1975 …

Optimize over all the measurement!!!

The necessary ingredient are symmetric logarithmic derivation expressed in diagonalizing basis.

$$\frac{\partial \rho}{\partial s} = 1/2(\mathcal{L}\rho + \rho\mathcal{L}) \qquad \rho = \sum \lambda_i |\varphi_i\rangle\langle\varphi_i|$$

$$\mathcal{F}_Q = Tr(\rho\mathcal{L}^2) = 2\sum_{m,n} \frac{|\langle\varphi_n|\frac{\partial\rho}{\partial s}|\varphi_m\rangle|^2}{\lambda_n + \lambda_m}$$

Example: QFI for pure state

$$\rho(s) = |\Psi(s)\rangle\langle\Psi(s)|$$

$$\mathcal{F}_Q = 4\langle\Psi(s)|(\frac{\partial\rho}{\partial s})^2|\Psi(s)\rangle$$

Zero eigenvalues cannot be neglected but eliminated !

Problems of QFI: large ambiguity as far measurement is concerned, optimality many aspects…

Two-point resolution

$$\varrho_s = q\,|\Psi_+\rangle\langle\Psi_+| + (1-q)\,|\Psi_-\rangle\langle\Psi_-|$$

$$|\Psi_\pm\rangle = e^{\pm isP/2}|\Psi\rangle$$

- FI and QFI for two-point resolution: Tsang 2016
- Here: optical arguments and symmetry arguments" for optimal measurement achieving QFI

Symmetry for achieving QFI

Assume symmetry of the point-spread- function as well as the symmetry of the measurement

$$\Psi(x) = \Psi(-x) \qquad\qquad \langle x|n\rangle = \pm\langle -x|n\rangle$$

The measurement does not feel the two-component structure of the signal! The original two-point resolution problem has been effectively transformed to localization of a single point source.

$$p_n \equiv |a_n|^2 = |\langle n|\Psi_\pm\rangle|^2$$

QFI can be obtained from FI just by expressing probabilities by complex amplitudes …

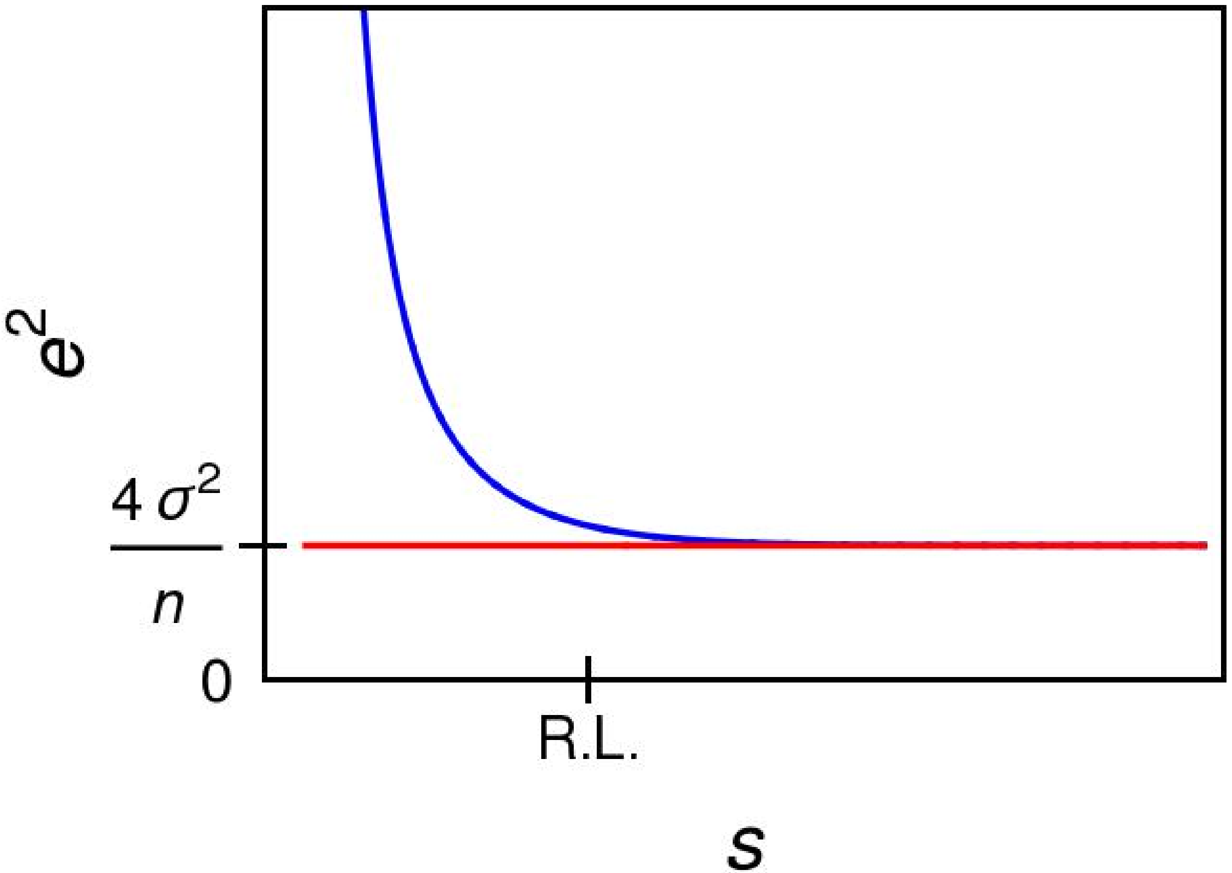
$$\mathcal{F} = \sum_n \frac{[p_n'(s)]^2}{p_n(s)}$$

$$= 4 \sum_n |\frac{\partial a_n}{\partial s}|^2 + \sum_n \frac{1}{p_n}[a_n^* \frac{\partial a_n}{\partial s} - a_n \frac{\partial a_n^*}{\partial s}]^2$$

Optimality conditions:

$$\mathrm{Im}\left(a_n \frac{\partial a_n^*}{\partial s} \right) = 0$$

FI vs FQI



Measurement achieving FQI

There is an ambiguity how to fulfill the optimality conditions. The ultimate resolution should not be considered as a rarity, but rather as a feature shared by many permissible detection schemes.

Efficiency vs. universality

How to do the detection efficiently?

Suggestion: Project the signal on a set of orthonormalized derivatives of $\Psi(x)$-PSF adapted schemes

$$\Phi_n(p) \equiv \langle p|n\rangle = Q_n(p)\Psi(p)$$

$$\Phi_n(x) \equiv \langle x|n\rangle = \frac{1}{\sqrt{2\pi}}\int Q_n(p)\Psi(p)e^{ipx}$$

Example 1: Gaussian PSF

$$\Psi(x) = (2\pi)^{-1/4} \exp(-x^2/4), \quad \sigma = 1$$

The optimal PSF-adapted set :

Hermite-Gauss modes

$$\mathcal{F}_s = 1/4$$

Example 2: Sinc PSF

$$\Psi(x) = \frac{1}{\sqrt{\pi}}\mathrm{sinc}(x), \;\; \Psi(p) = \frac{1}{\sqrt{2}}\mathrm{rect}(p/2)$$

The optimal PSF-adapted set is linked with

Legendre polynomials orthogonal on (-1/2,1/2)

$$a_n = \langle n|\Psi_\pm\rangle = \frac{\sqrt{2n+1}}{2}\int_{-1}^{1} L_n(p)\, e^{-isp/2}\, dp$$

Example 2: Sinc PSF…

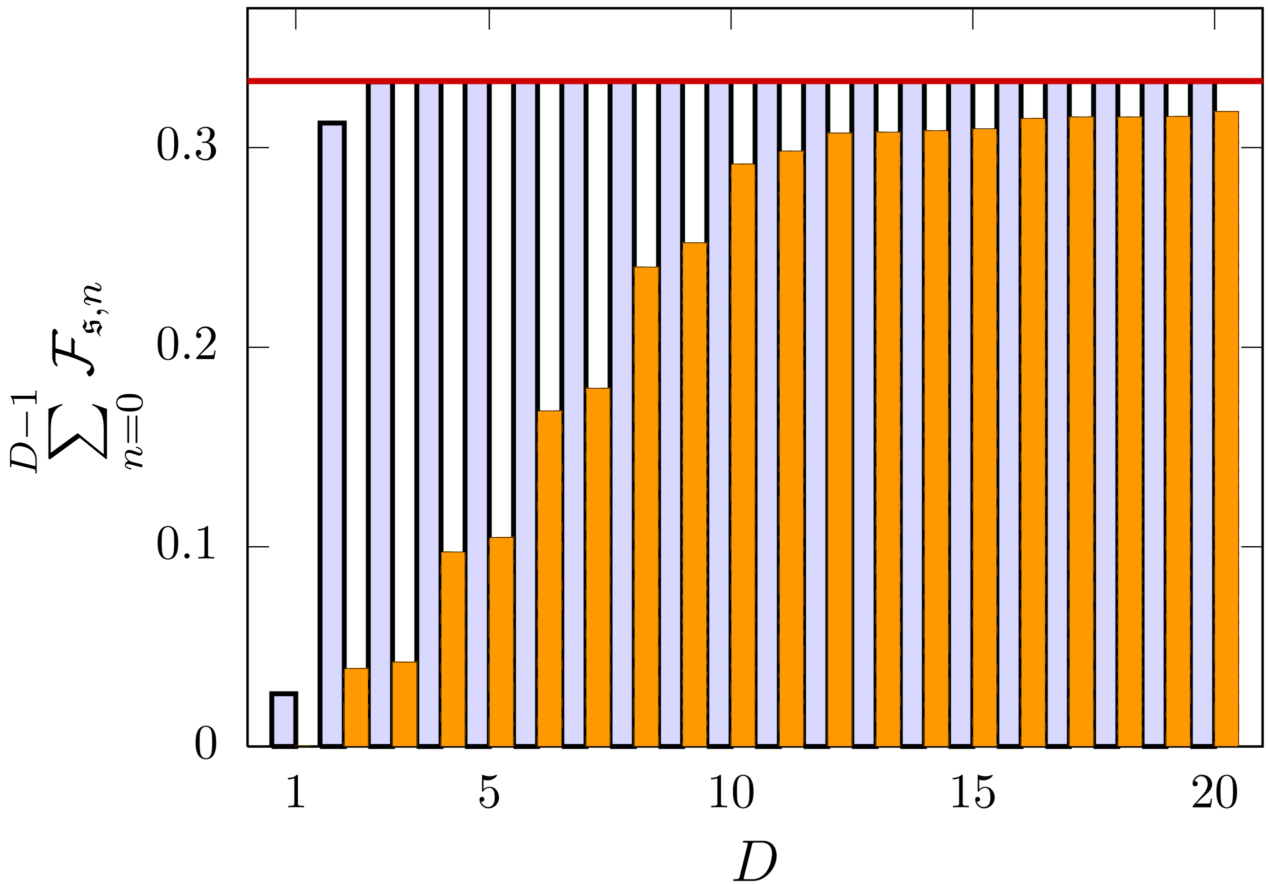
Efficient measurement modes:

$$\Phi_n(x) = \sqrt{n + 1/2}\,\frac{J_{n+\frac{1}{2}}(x)}{\sqrt{x}}$$

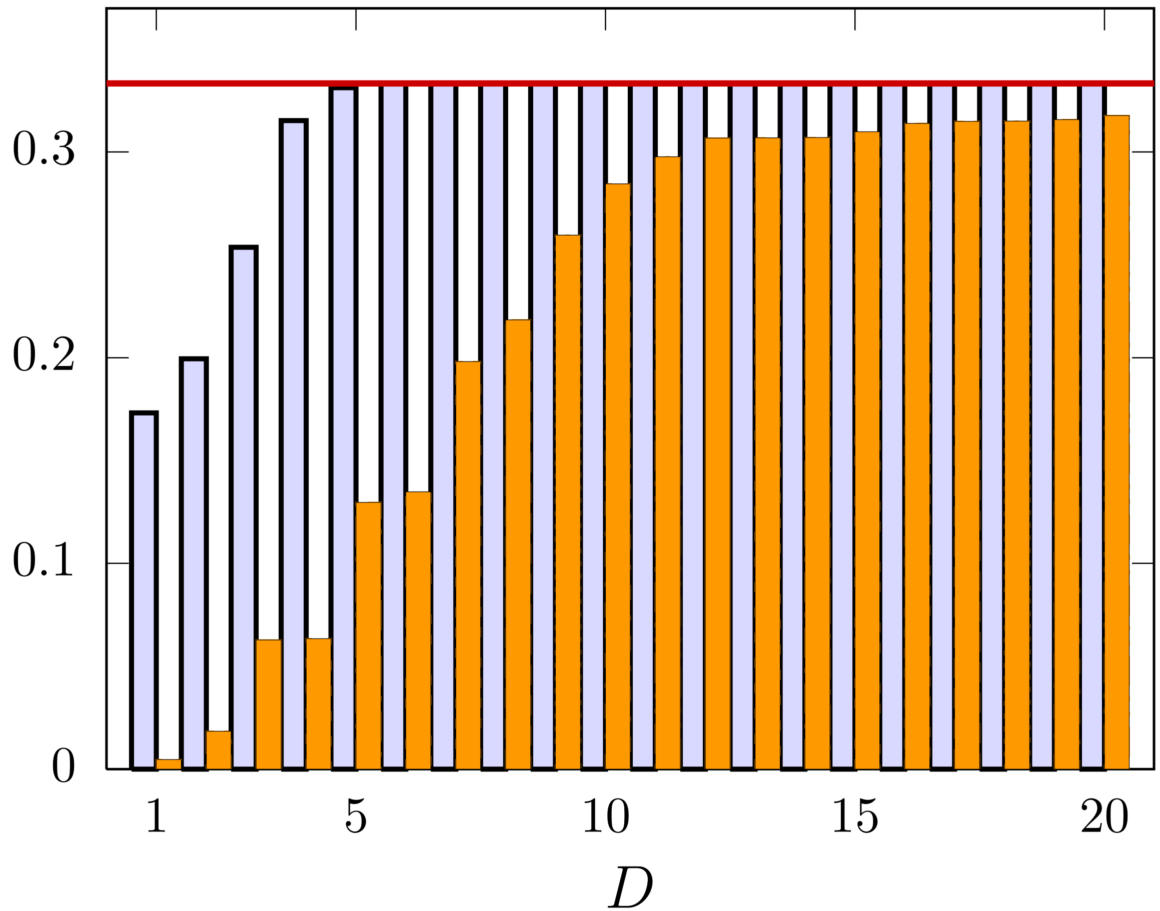
Fisher information consists of partial contributions:

$$\mathcal{F}_{s,n} = \frac{\pi\left[nJ_{n-\frac{1}{2}}\left(s/2\right) - (n+1)J_{n+\frac{3}{2}}\left(s/2\right)\right]^2}{(2n+1)s}$$

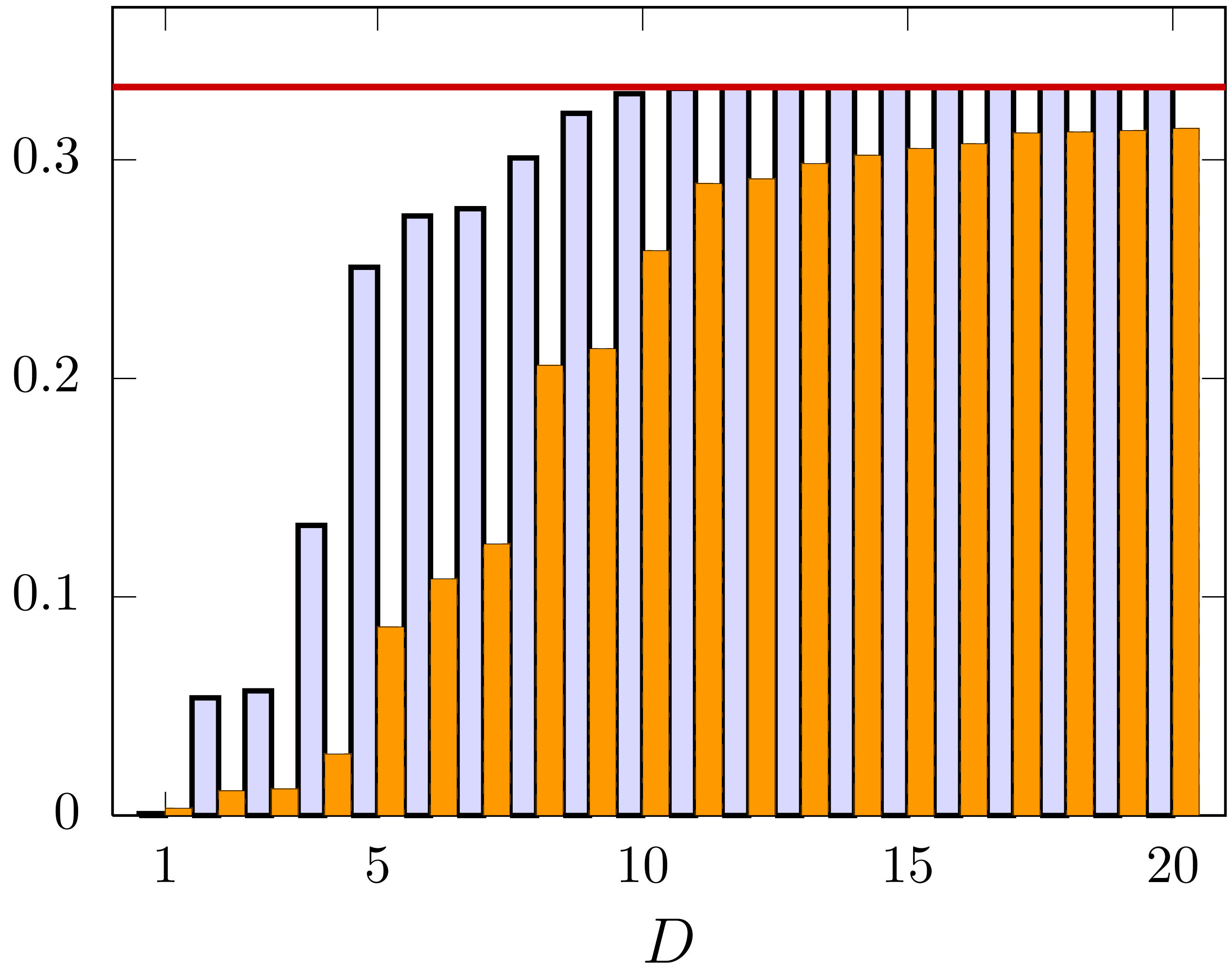
$$\mathcal{F}_s = 1/3$$



FI for the first D projections on the HG basis with arbitrarily chosen σ = π (orange bars) and the PSF Sinc adapted measurement, **Separation s= 1, Rayleigh limit = π**. More than a hundred of Hermite-Gauss projections must be measured to access 98.5% of the QFI (horizontal red line), whereas just three projections of the PSF-adapted measurement are sufficient.



As before, **Separation s= 2, Rayleigh limit = π**



As before, **Separation s= 15,Rayleigh limit = π**

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Wavefront sensing reveals optical coherence

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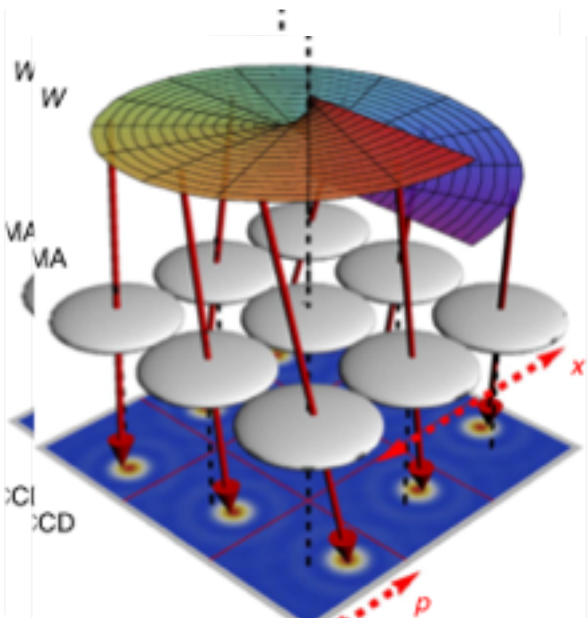

Fisher Info Matrix provides a useful tool for assessing the performance of reconstruction schemes

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Thanks for your attention!