Performance measure for quantum tomography

Z. Hradil, Yong Siah Teo, J. Řeháček, B. Stoklasa, L. Moťka,
Department of Optics, Palacký University, Olomouc, Czech Rep.
S. Wallentowitz, Instituto de Física, Pontificia Univesidad Católica de Chile
L.L. Sánchez-Soto, Departamento de Óptica, Universidad Complutense, Madrid, Spain
Erlangen group of prof G. Leuchs

Centra kompetence
Technology Agency of the Czech Republic
GAČR
GRANTOVÁ AGENTURA ČESKÉ REPUBLIKY

Outline

Quantum measurement and SVD decomposition

Fisher information matrix and quantum tomographic transfer function (qTTF)

Tomography in phase space

Optics: Image Processing to Wave Front Sensing

Measurement

Born rule for (normalized) measurement,
M- channels, D^2- 1 parameters

$$p_j = \mathrm{tr}(\rho\,\Pi_\mathrm{j}) \qquad \rho = \sum_i t_i \Omega_i \qquad p_j' = p_j - \mathrm{tr}\Pi_\mathrm{j}/\mathrm{D}$$

Linear structure of the measurement for
tomographically (over)complete measurement M> D^2- 1

$$\mathbf{p}' = \mathbf{C}\mathbf{t}$$

Dimensions: M x 1 = M x (D^2 - 1) (D^2 - 1)x 1

$$\mathbf{C} = \mathbf{O}\mathbf{S}\mathbf{O}'^{\mathbf{T}}$$

SVD and its meaning (S-diagonal, $O^TO = O'^TO' = OO'^T = 1$

$$(\mathbf{O}^{\mathbf{T}}\mathbf{p}') = \mathbf{S}(\mathbf{O}'^{\mathbf{T}}\mathbf{t})$$

Quest for Tomo Resolution Measure

Attempts to link the resolution with S-eigenvalues: conditional numbers of SVD – problems with channel duplication, comparison between SIC and MUB ...

Answer: Likelihood and Fisher information matrix

$$\mathcal{L}\left(\{n_j\};\rho\right) = \prod_k p_k^{n_k}$$

$$\boldsymbol{F}(\rho) = \frac{1}{N}\overline{\left(\sum_k \frac{n_k}{p_k}\frac{\partial p_k}{\partial \mathrm{t}}\right)\left(\sum_l \frac{n_l}{p_l}\frac{\partial p_l}{\partial \mathrm{t}}\right)}$$

$$\boldsymbol{F}(\rho) = \sum_l \mathrm{tr}(\boldsymbol{\Omega}\Pi_l)\frac{1}{p_l}\mathrm{tr}(\Pi_l\boldsymbol{\Omega}) = \boldsymbol{C}^T\boldsymbol{P}^{-1}\boldsymbol{C}$$

Fisher info as a measure

Quantum Tomografic Transfer Function (qTTF): the universal resolution measure quantifying the overall performance – depends on state, measurement, ...

Consider $\mathrm{Sp}[\boldsymbol{F}(\rho)^{-1}]$

Be aware that the inversion does not hold in general!

$$\left(\boldsymbol{C}^T \boldsymbol{P}^{-1} \boldsymbol{C}\right)^{-1} = \boldsymbol{C}^{-} \boldsymbol{P} \left(\boldsymbol{C}^T\right)^{-}$$

but using SVD

$$\mathrm{Sp}\boldsymbol{F}(\rho)^{-1} = \mathrm{Sp}(\boldsymbol{S}\boldsymbol{O}^T \boldsymbol{P}^{-1} \boldsymbol{O}\boldsymbol{S})^{-1}$$

Important: $\mathrm{Sp}F^{-1}$ does not depend on the state representation (O') but on the measurement through S and O matrices!

qTTF

SpF^{-1} gives the scaled optimal tomographic accuracy in the Hilbert-Schmidt norm for unbiased state estimators ρ in the limit of large sampling events, but is very difficult to handle!

Proposal: qTTF = < SpF^{-1} >$_{\text{all pure states}}$

Fischer Info Matrix is difficult to deal with in general!

Hint: expand SpF^{-1} around maximally mixed state and use the Matrix Inversion Lemma

$$(A+BCD)^{-1} = A^{-1} - A^{-1}BCDA^{-1}(1+BCDA^{-1})^{-1}$$

$$\Delta \boldsymbol{P} = \boldsymbol{P} - \overline{\boldsymbol{P}} \qquad \bar{p}_j = \mathrm{tr}(\Pi_j / D)$$

$$\mathrm{Sp}\boldsymbol{F}(\rho)^{-1} = \mathrm{Sp}\overline{\boldsymbol{F}}^{-1} + \mathrm{Sp}\frac{\boldsymbol{\mathcal{X}}\Delta\boldsymbol{P}}{1 - \boldsymbol{\mathcal{Y}}\Delta\boldsymbol{P}}$$

$$\boldsymbol{\mathcal{X}} = \overline{\boldsymbol{P}}^{-1}\boldsymbol{C}\overline{\boldsymbol{F}}^{-2}\boldsymbol{C}^T\overline{\boldsymbol{P}}^{-1} \qquad \boldsymbol{\mathcal{Y}} = \overline{\boldsymbol{P}}^{-1}\boldsymbol{C}\overline{\boldsymbol{F}}^{-1}\boldsymbol{C}^T\overline{\boldsymbol{P}}^{-1} - \overline{\boldsymbol{P}}^{-1}$$

Mutual orthogonality $\quad \boldsymbol{\mathcal{X}}\overline{\boldsymbol{P}}\boldsymbol{\mathcal{Y}} = 0 = \boldsymbol{\mathcal{Y}}\overline{\boldsymbol{P}}\boldsymbol{\mathcal{X}}$

$-\boldsymbol{P}$ is generalized inverse of $\boldsymbol{\mathcal{Y}}$ $\quad \boldsymbol{\mathcal{Y}}\overline{\boldsymbol{P}}\boldsymbol{\mathcal{Y}} = -\boldsymbol{\mathcal{Y}}$

$$\boldsymbol{C}^T\boldsymbol{\mathcal{Y}} = 0 = \boldsymbol{\mathcal{Y}}\boldsymbol{C}$$

All these subtle relations are important for evaluation of qTTF in special cases!

State averaging for qTTF ...

$$qTTF\{\Pi_j\} = \underbrace{\mathrm{Sp}\overline{\boldsymbol{F}}^{-1}}_{\text{zeroth order}} + \underbrace{\frac{\alpha}{D(D+1)} \sum_{j_1,j_2=1}^{M} \boldsymbol{\mathcal{X}}_{j_2 j_1} \boldsymbol{\mathcal{Y}}_{j_1 j_2} \boldsymbol{\mathcal{G}}^{(2)}_{j_1 j_2}}_{\text{second order}} + \ldots$$

Gramm matrix of higher order $\boldsymbol{\mathcal{G}}^{(n)}_{j_1 j_2 \ldots j_n} = \mathrm{tr}(\Pi_{\mathrm{j_1}} \Pi_{\mathrm{j_2}} \ldots \Pi_{\mathrm{j_n}})$

For some Minimally Complete Tomo only two lowest terms exist!!!

Special cases SIC and MUB...

Minimaly complete tomo (SIC,..)

$$qTTF\{\Pi_j\}_{\mathrm{MIN}} = \mathrm{Sp}\overline{\boldsymbol{F}}_{\mathrm{MIN}}^{-1} - 1 + \frac{1}{\mathrm{D}}$$

$$qTTF\{\Pi_j\}_{\mathrm{SIC}} = D^2 + D - 2$$

Minimaly basis (over-complete) tomo (MUB,..)

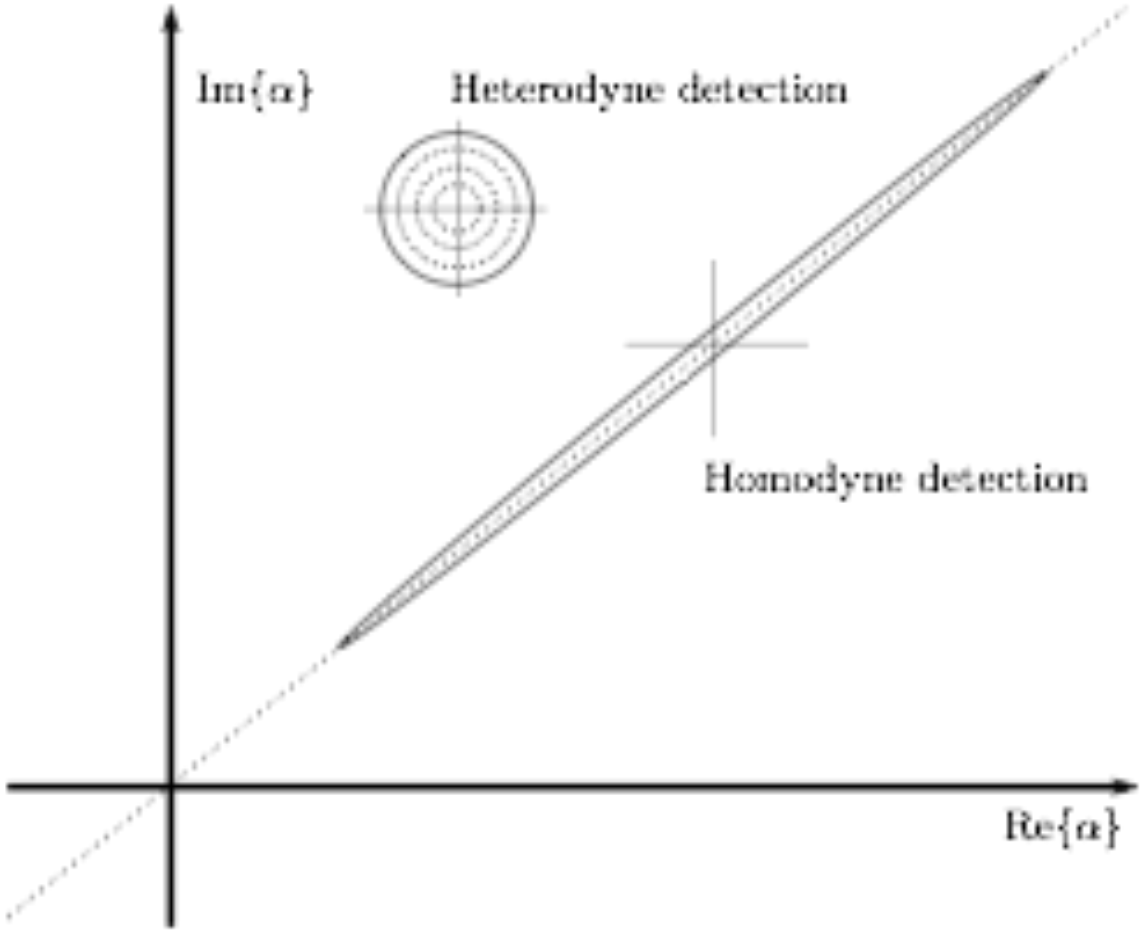
$$qTTF\{\Pi_j\}_{\mathrm{MIN\ BASES}} = \frac{\mathrm{Sp}(\boldsymbol{C}^T\boldsymbol{C})^{-1}}{(D+1)^2}$$

$$qTTF\{\Pi_j\}_{\mathrm{MUB}} = D^2 - 1 < qTTF\{\Pi_j\}_{\mathrm{SIC}}$$

Conceptual vs practical issues...

- qTTF has good meaning as statistical measure of accuracy
- Dependence on state, (transformation), measurement, data processing and other aspect should be considered
- Tomography in phase space is good example: optical implementations available

Phase space

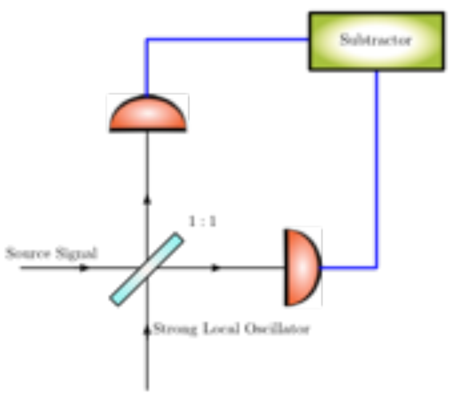


Homodyne vs heterodyne

Rotated quadrature operators

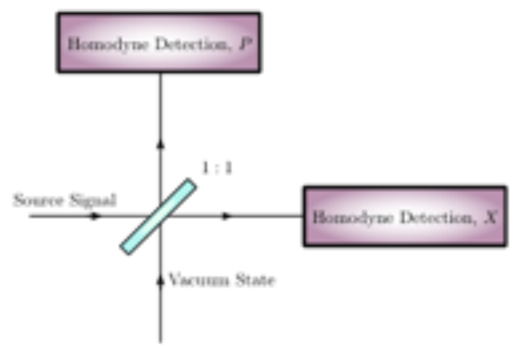
$$x_\theta = \frac{1}{2}\left[ae^{-i\theta} + a^\dagger e^{i\theta}\right] \quad p_\theta = \frac{1}{2i}\left[ae^{-i\theta} - a^\dagger e^{i\theta}\right]$$

Homodyne



(a) Homodyne detection

Heterodyne



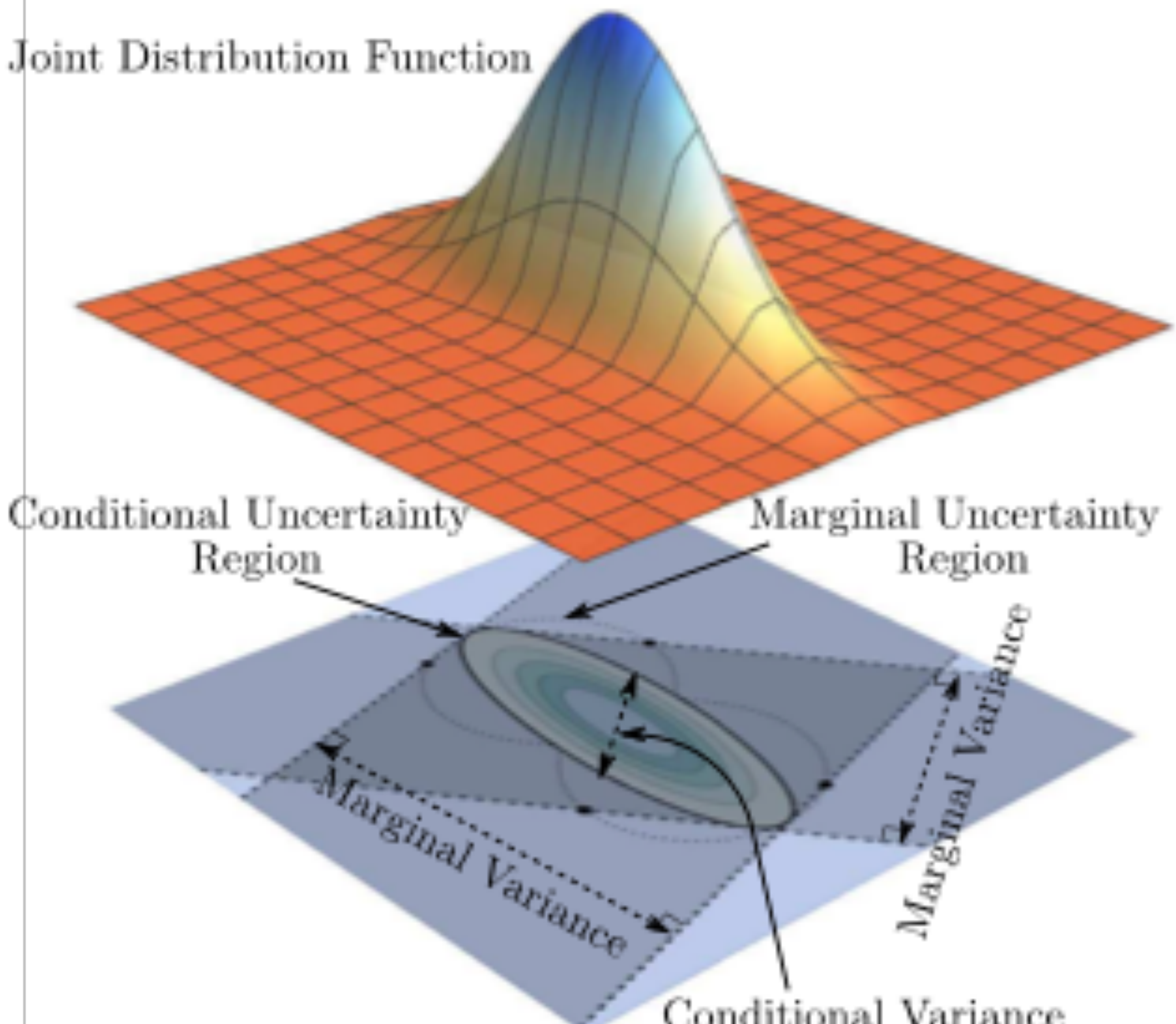
(b) Heterodyne (double-homodyne) detection

Tomography in phase space

Homodyne detection: Marginal distribution of Wigner function is sampled, the covariance must be reconstructed (more involved reconstruction)

Heterodyne detection: The Q function is sampled directly (direct reconstruction)

Tomography = Detected noise + error from inversion


Joint Distribution Function
Conditional Uncertainty Region
Marginal Uncertainty Region
Marginal Variance
Marginal Variance
Conditional Variance

Mathematical tools for reconstruction of covariance matrix

Estimated covariance

Hilbert-Schmidt distance

Cramer- Rao bound

Fisher information

$$\mathbf{G}_{\mathrm{W}} \widehat{=} \begin{pmatrix} g_1 & g_3/\sqrt{2} \\ g_3/\sqrt{2} & g_2 \end{pmatrix}$$

$$H = \overline{\left(\mathbf{G}_{\mathrm{W}} - \widehat{\mathbf{G}}_{\mathrm{W}}\right)^2} = \sum_k \overline{(g_k - \widehat{g}_k)^2}.$$

$$H \geq Sp\mathbf{F}^{-1}$$

$$\mathbf{F} = \frac{N}{2} Sp\, \mathbf{G}^{-1} \frac{\partial \mathbf{G}}{\partial \mathbf{g}} \mathbf{G}^{-1} \frac{\partial \mathbf{G}}{\partial \mathbf{g}}$$

Rationale behind the noise analysis

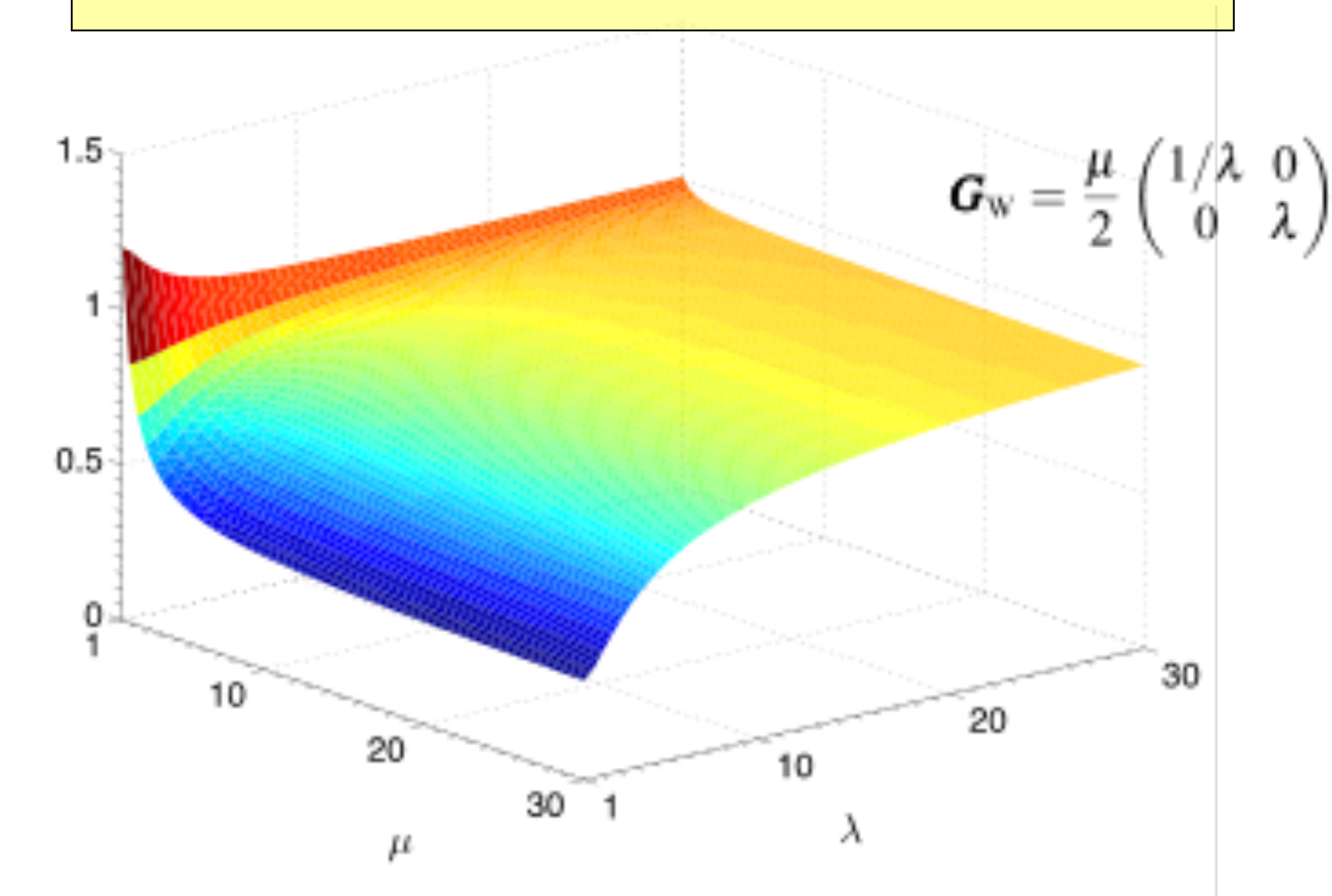
Performance of homodyne tomography

$$H_{\mathrm{hom}} = \frac{2\, Sp\ \mathbf{G}_{\mathrm{hom}} \left(Sp\ \mathbf{G}_{\mathrm{hom}} + 3\sqrt{Det\ \mathbf{G}_{\mathrm{hom}}} \right)}{N}$$

Performance of heterodyne detection

$$H_{\mathrm{het}} = \frac{2\left[(Sp\ \mathbf{G}_{\mathrm{het}})^2 - Det\ \mathbf{G}_{\mathrm{het}} \right]}{N}$$

Comparison Het and Homo Tomo strategies



Ratio<= 1 : Het is doing better !

Heterodyne vs heterodyne data

Covariance matrices for Gaussian states:

$$\mathbf{G}_{\mathrm{Q}} = \mathbf{G}_{\mathrm{W}} + 1/2$$

Variance of homodyne detection: marginal distribution of Wigner function

$$\sigma_\theta^2 = u_\theta^T \mathbf{G}_{\mathrm{W}} u_\theta + \delta_\eta^2$$

Conditional variance of Q function (along the line in the phase space)

$$\Sigma_\theta^2 = \left(u_\theta^T [\mathbf{G}_{\mathrm{Q}} + \delta_\eta^2]^{-1} u_\theta\right)^{-1}$$

Noise term:

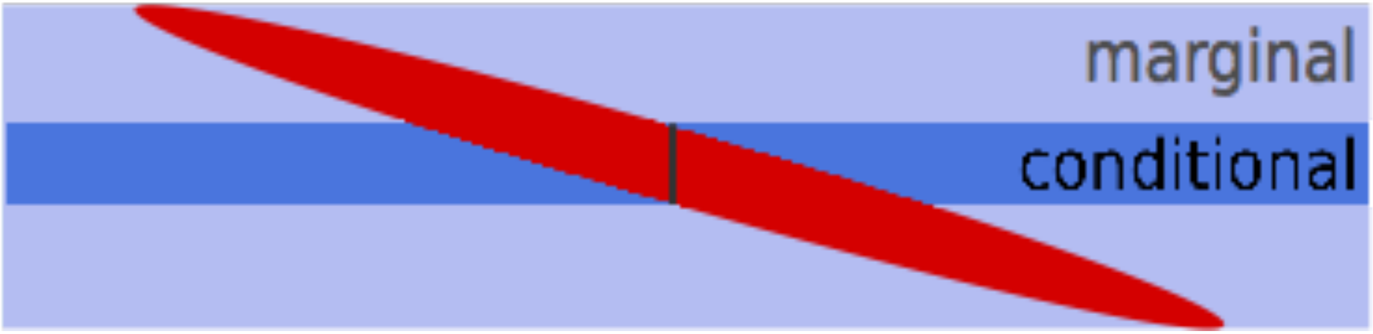
$$\delta_\eta^2 = (1-\eta)/2\eta$$

Geometrical relation between marginal and conditional distributions

Relation for generic covariance matrix G and orthogonal basis vectors u, v

$$\sigma_\theta^2 = u_\theta^T \mathbf{G} u_\theta \equiv G_{uu,\theta}$$

$$\Sigma_\theta^2 = \left(u_\theta^T \mathbf{G}^{-1} u_\theta\right)^{-1} = \frac{G_{uu,\theta} G_{vv,\theta} - G_{uv,\theta}^2}{G_{vv,\theta}} \leq G_{uu,\theta} = \sigma_\theta^2 \,.$$



Noise analysis for minimum uncertainty states

Wigner covariance matrix

$$\mathbf{G}_{\mathrm{W}} \widehat{=} \begin{pmatrix} \frac{1}{2\lambda} & 0 \\ 0 & \frac{\lambda}{2} \end{pmatrix}$$

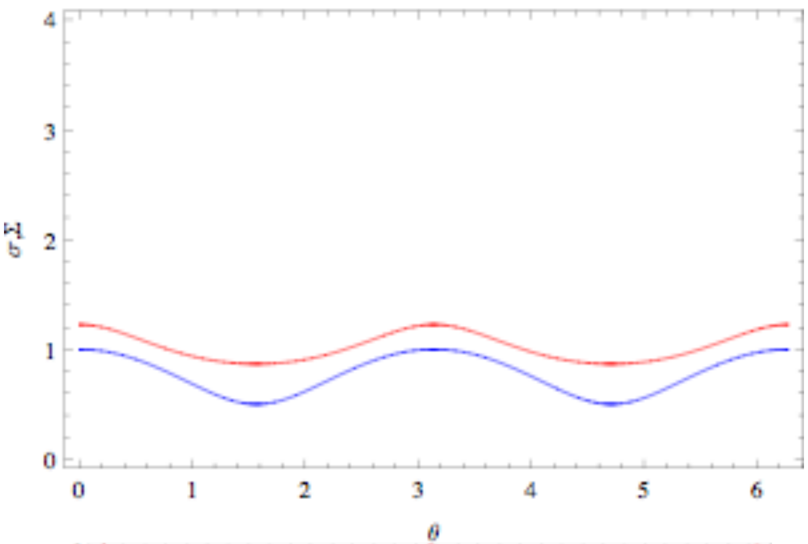
Marginal variance

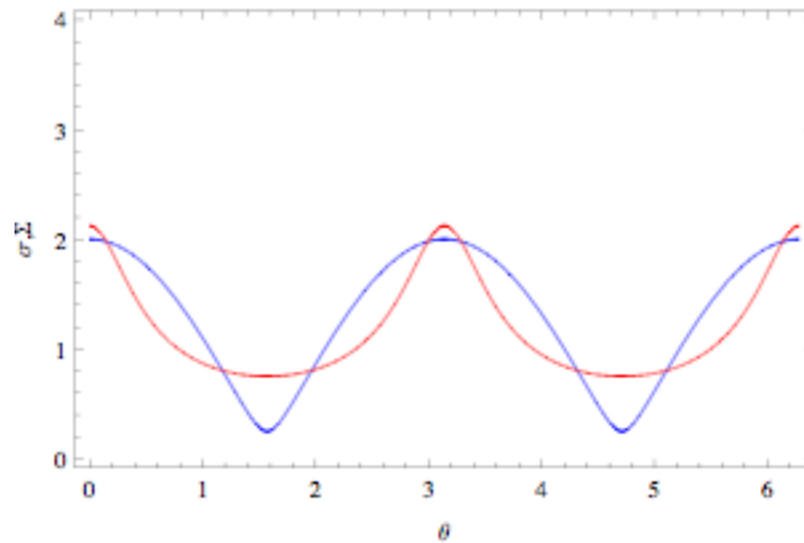
$$\sigma_\theta^2 = \frac{1}{2\lambda}(\cos\theta)^2 + \frac{\lambda}{2}(\sin\theta)^2$$

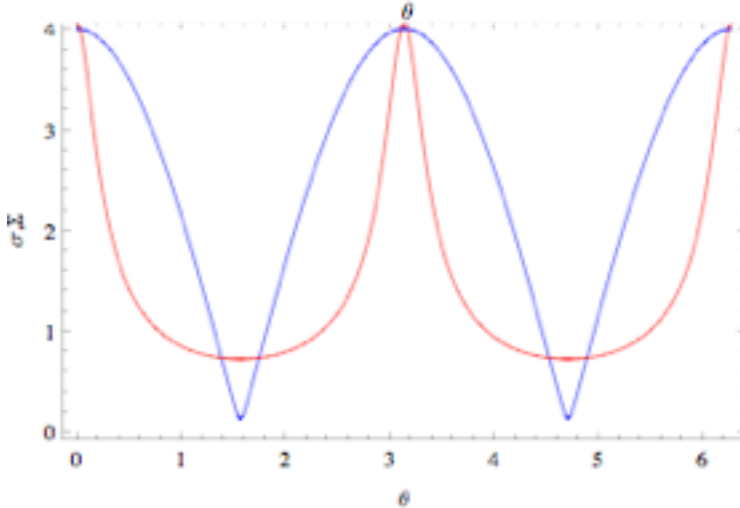
Conditional variance

$$\Sigma_\theta^2 = \left[1 + \frac{\lambda-1}{\lambda+1}\cos(2\theta)\right]^{-1}$$

Noise comparison







BLUE: marginal for homodyne detection

RED: conditional for heterodyne detection

Measurement in Phase Space

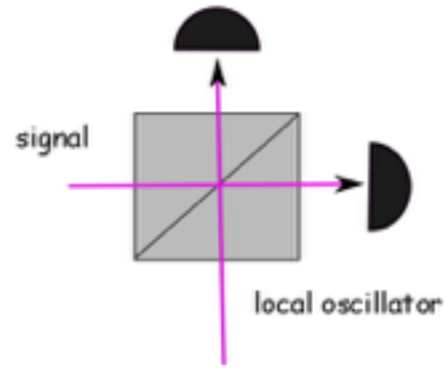
Homodyne detection - Projection into the Rotated Quadrature Eigenstates

Heterodyne detection - Projection into the coherent state basis with fluctuating position in phase space

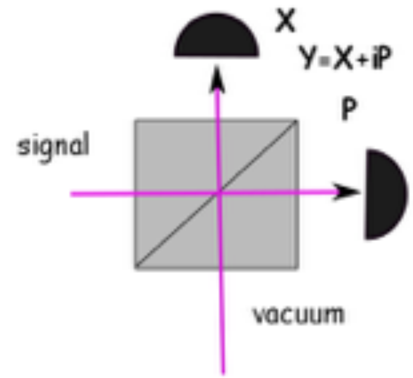
Unbalanced homodyning - Projection into the coherent state basis with prefixed position in phase space

Heterodyne detection

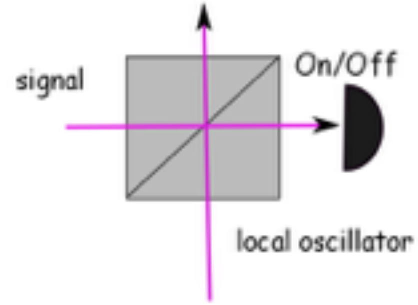
Frequency mismatch between signal and local oscillator



Double homodyne detection

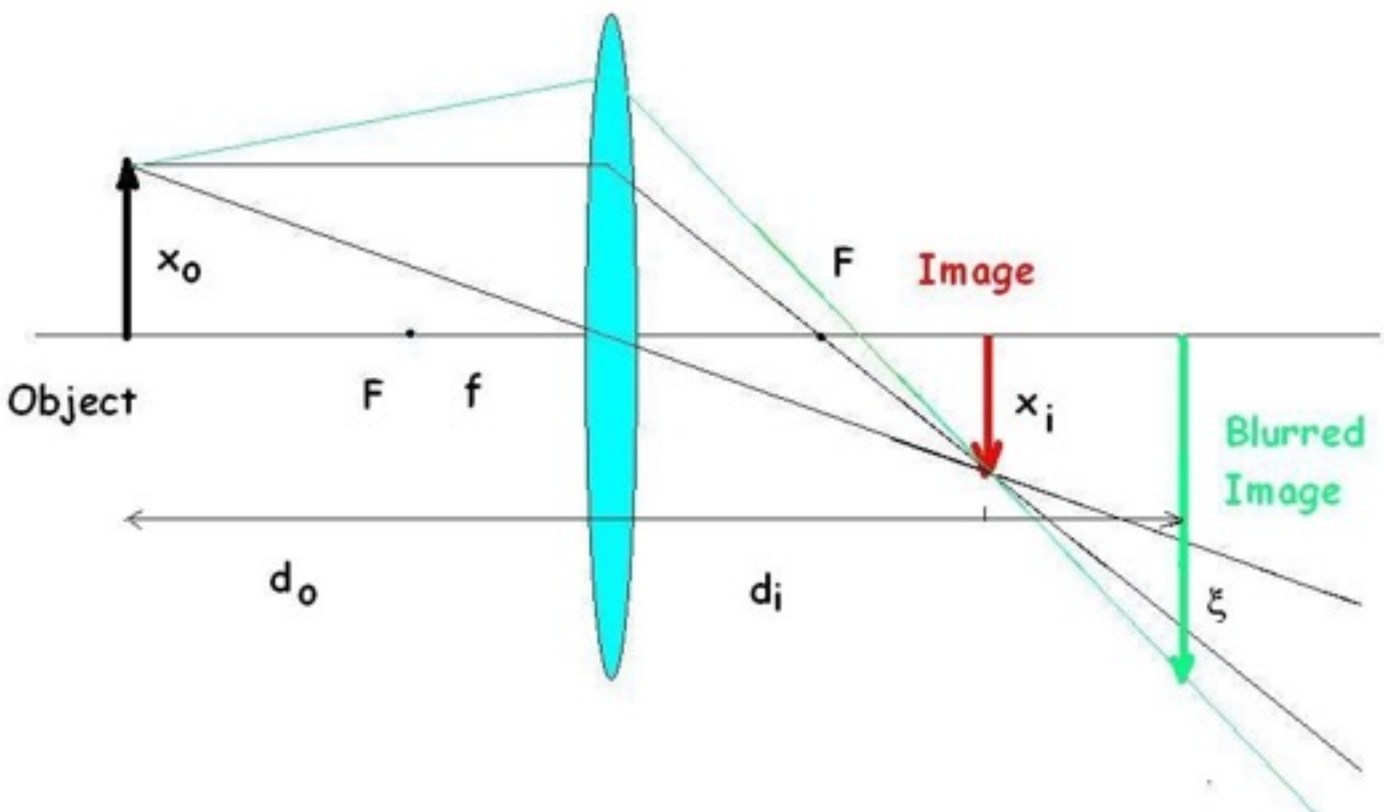


Unbalanced homodyning: low transmittivity for LO



Phase space in optics

Optical Imaging: Lens equation in geometrical optics



nature COMMUNICATIONS

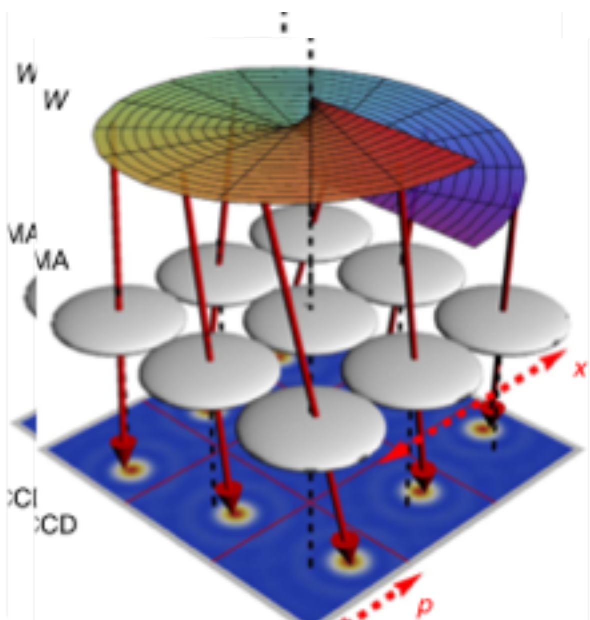
ARTICLE

Received 1 Sep 2013 | Accepted 17 Jan 2014 | Published 7 Feb 2014

DOI: 10.1038/ncomms4275

Wavefront sensing reveals optical coherence

B. Stoklasa[1], L. Motka[1], J. Rehacek[1], Z. Hradil[1] & L.L. Sánchez-Soto[2]



Conclusions

Fisher Info Matrix provides a useful tool for assessing the performance of reconstruction schemes

- Rehacek, J; Teo, YS; Hradil, Z.,Determining which quantum measurement performs better for state estimation PHYSICAL REVIEW A, 2015, 92, 012108
- Rehacek, J; Teo, YS; Hradil, Z; Wallentowitz, S, Surmounting intrinsic quantum-measurement uncertainties in Gaussian-state tomography with quadrature squeezing, SCIENTIFIC REPORTS, 2015, 5, 12289
- C. R. Muller, C. Peuntinger, T. Dirmeier, I. Khan, U. Vogl, Ch. Marquardt, G.Leuchs, L.L.Sanchez Soto,Y.S.Teo, Z.Hradil, and J.Rehacek, Evading Vacuum Noise: Wigner Projections or Husimi
- Samples? (unpublished).
- Yong Siah Teo, D. Koutny, Z. Hradil, and J. Rehacek, A fast universal performance certification of measurements for quantum tomography (unpublished)

Thanks for your attention!