GAUSSIAN INTRINSIC ENTANGLEMENT

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M. B. Plenio *et al.*, Quant. Inf. Comput. 7, 1 (2007).

Applications of entanglement measures

- Description of monogamy of entanglement.
- Bounds on hardly computable quantities.
- Proofs of impossibility or limitations of some Gaussian protocols.
- Quantify quality of prepared entangled states and entangling gates.
- Bounds in some quantum-information experiments.

Axioms of entanglement measures

Entanglement monotone:

- $E(\rho) \geq 0$.
- $E(\rho) = 0$ if ρ is separable.
- E does not increase on average under LOCC,

$$E(\rho) \geq \sum_i p_i E\left(\frac{A_i \rho A_i^\dagger}{p_i}\right), \quad \{A_i\} - \text{LOCC operation}, \quad p_i = \mathrm{Tr}(A_i \rho A_i^\dagger).$$

Other axioms:

- Equality to marginal von Neumann entropy on pure states, convexity, additivity, asymptotic continuity.

M. B. Plenio *et al.*, Quant. Inf. Comput. 7, 1 (2007).

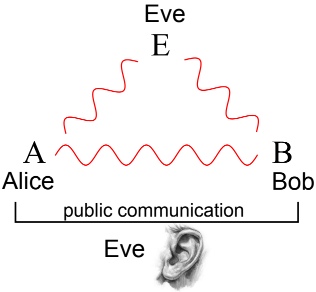
Principles of entanglement quantification

- Negativities: logarithmic negativity.
- Operational measures: distillable entanglement, entanglement of formation.
- Geometric measures: relative entropy of entanglement.
- Information-theoretic measures: squashed entanglement.

Practically each of the measures is either computable or physically meaningful but not both.

U. Maurer, IEEE TIT 39, 733 (1993).
U. Maurer *et al.*, IEEE TIT 45, 499 (1999).
N. Gisin *et al.*, CRYPTO 2000, 482 (2000).

Secret key agreement protocol



- A, B, E obey $P(A, B, E)$.
- Alice and Bob want a secret key.
- They can use only local operations and public communication (LOPC).

SKA is possible only if P cannot be created by LOPC – secret correlations.

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U. Maurer *et al.*, IEEE TIT 45, 499 (1999).
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Intrinsic information

$$I\,(A:B\downarrow E) := \inf_{E\to\tilde{E}} [I(A:B|\tilde{E})].$$

- Quantifies how much Bob learns about Alice's data by looking at his own data after Eve announces a function of her data.
- $P(A,B,E)$ contains secret correlations $\Longleftrightarrow I\,(A:B\downarrow E) > 0$.
- Upper bound on secret key rate in SKA:

$$I\,(A:B\downarrow E) \geq S\,(A;B\|E)\,.$$

U. Maurer, IEEE TIT 39, 733 (1993).
U. Maurer *et al.*, IEEE TIT 45, 499 (1999).
N. Gisin *et al.*, CRYPTO 2000, 482 (2000).

Quantifying entanglement by secret correlations

Mapping entanglement onto intrinsic information:

$$\rho_{AB} \to |\Psi\rangle_{ABE} \to P(A,B,E) = \mathrm{Tr}(\Pi_A \otimes \Pi_B \otimes \Pi_E |\Psi\rangle_{ABE}\langle\Psi|).$$

Faithful mapping requires optimization:

$$\mu\left(\rho_{AB}\right) := \inf_{\{\Pi_E, |\Psi\rangle_{ABE}\}} \sup_{\{\Pi_A, \Pi_B\}} \left[I\left(A;B\downarrow E\right)\right].$$

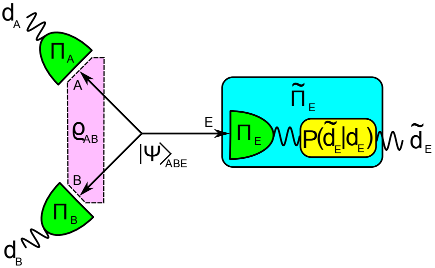
Faithful quantity, equal to vN entropy on pure states, hardly computable.

Intrinsic entanglement

$$E_{\downarrow}\left(\rho_{AB}\right) := \sup_{\{\Pi_A,\Pi_B\}} \inf_{\{\Pi_E,|\Psi\rangle_{ABE}\}} \left[I\left(A;B\downarrow E\right)\right]$$

Restriction: ρ_{AB} – Gaussian $(N+M)$-mode state; covariance matrix γ

$$(\gamma_{ij} = \langle\{r_i, r_j\}\rangle,\ \mathbf{r} = (x_{A_1}, p_{A_1}, \ldots, x_{B_M}, p_{B_M})^T).$$



- $\Pi_{A,B,E}$ – Gaussian measurements, CMs $\Gamma_{A,B,E}$.
- $|\Psi\rangle_{ABE}$ – Gaussian purification, CM γ_{π}.
- $P(\tilde{d}_E|d_E)$ – Gaussian channel.

Simplification

1. $P(d_A, d_B, d_E)$ - Gaussian with CM

$$\gamma_\pi + \Gamma_A \oplus \Gamma_B \oplus \Gamma_E.$$

$$\begin{aligned} I(A:B|E) &= \langle I(A:B|E=d_E)\rangle \\ &= I(A:B|E=d_E) \end{aligned}$$

mutual inf. of $P(d_A, d_B|d_E)$ with CM

$$\sigma_{AB} = \Gamma_A \oplus \Gamma_B + \gamma_{AB} - \omega(\gamma_E + \Gamma_E)^{-1}\omega^T$$

$$\gamma_\pi = \begin{pmatrix} \gamma_{AB} & \omega \\ \omega^T & \gamma_E \end{pmatrix}$$

$$\boxed{I(A:B|E) = \frac{1}{2}\ln\left(\frac{\det\sigma_A \det\sigma_B}{\det\sigma_{AB}}\right)}$$

2. $P(\tilde{d}_E|d_E) \propto e^{-(\tilde{d}_E - Xd_E)^T Y^{-1}(\tilde{d}_E - Xd_E)}$,

$$\begin{aligned} \tilde{\sigma}_{AB} &= \ldots - \omega X^T[X(\gamma_E + \Gamma_E)X^T + Y]^{-1}X\omega^T \\ &= \ldots - \omega(\gamma_E + \tilde{\Gamma}_E)^{-1}\omega^T, \quad \tilde{\Gamma}_E - \text{CM}. \end{aligned}$$

$$\boxed{E \to \tilde{E} \text{ can be integrated into } \Gamma_E}$$

3. Purifications $|\Psi\rangle$ and $|\bar{\Psi}\rangle$ ($K \le \bar{K}$ modes E),

$$\bar{\gamma}_\pi = [I_{AB} \oplus S_E^{-1}][\gamma_\pi \oplus I_{(\bar{K}-K)}][I_{AB} \oplus (S_E^T)^{-1}],$$

$$\begin{aligned} \bar{\sigma}_{AB} &= \ldots \bar{\omega}(\bar{\gamma}_E + \bar{\Gamma}_E)^{-1}\bar{\omega}^T \\ &= \ldots \omega(\gamma_E + \Gamma_E)^{-1}\omega^T = \sigma_{AB} \end{aligned}$$

Holds also if $\bar{\Gamma}_E = S_E^{-1}[\Gamma_E \oplus I_{(\bar{K}-K)}](S_E^T)^{-1}$

$$\boxed{|\Psi\rangle \text{ can be taken fixed and arbitrary}}$$

Gaussian intrinsic entanglement

$$E_{\downarrow}^{G}\left(\rho_{AB}\right)=\sup_{\Gamma_A,\Gamma_B}\inf_{\Gamma_E}\left[\frac{1}{2}\ln\left(\frac{\det\sigma_A\det\sigma_B}{\det\sigma_{AB}}\right)\right]$$

$$\sigma_{AB}=\Gamma_A\oplus\Gamma_B+\gamma_{AB}-\omega(\gamma_E+\Gamma_E)^{-1}\omega^T.$$

Minimal purification:

$$\gamma_E=\bigoplus_{i=1}^{R}\nu_i I,\quad \omega=S^{-1}\begin{pmatrix}\bigoplus_{i=1}^{R}\sqrt{\nu_i^2-1}\sigma_z\\ \mathbb{O}\end{pmatrix},$$

$S\Omega S^T=\Omega$, $\Omega=\oplus_{i=1}^{N+M}i\sigma_y$, $S\gamma_{AB}S^T=\oplus_{i=1}^{N+M}\nu_i I$, $\nu_i\geq 1$-symplectic eigenvalues.

R – number of purifying modes E = number of $\nu_i>1$.

R. F. Werner *et al.*, Phys. Rev. Lett. 86, 3658 (2001).
A. Jamiołkowski, Rep. Math. Phys. 3, 275 (1972).
G. Giedke *et al.*, Phys. Rev. A 66, 032316 (2002).

Faithfulness

Gaussian separable state:

$$\rho_{AB}^{\mathrm{sep}} = \int P_{\mathrm{Gauss}}(\mathbf{r}) D(\mathcal{V}\mathbf{r}) |\chi_A\rangle_A\langle\chi_A| \otimes |\chi_B\rangle_B\langle\chi_B| D^{\dagger}(\mathcal{V}\mathbf{r}) d\mathbf{r},$$

Purification:

$$|\tilde{\Psi}\rangle_{ABE} = \int \sqrt{P_{\mathrm{Gauss}}(\mathbf{r})} D(\mathcal{V}\mathbf{r}) |\chi_A\rangle_A |\chi_B\rangle_B |\mathbf{r}\rangle_E d\mathbf{r},$$

$|\mathbf{r}\rangle_E$ – product of position eigenvectors.

Measurement of $|\mathbf{r}'\rangle_E$:

$$D(\mathcal{V}\mathbf{r}')|\chi_A\rangle_A|\chi_B\rangle_B \Rightarrow \sigma_{AB} = \sigma_A \oplus \sigma_B \Rightarrow E_{\downarrow}^{G}\left(\rho_{AB}^{\mathrm{sep}}\right) = 0.$$

One can show by contradiction that $E_{\downarrow}^{G}(\rho_{AB}) = 0 \Rightarrow \rho_{AB}$ is separable.

$$\boxed{E_{\downarrow}^{G}(\rho_{AB}) = 0 \Leftrightarrow \rho_{AB} \text{ is separable}}$$

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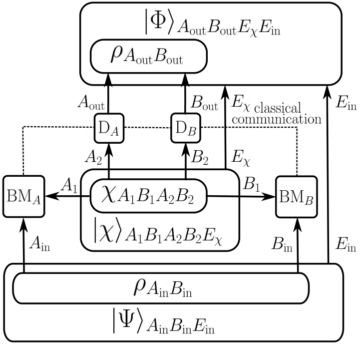
Monotonicity

Gaussian local trace-preserving operations and classical communication (GLTPOCC)

$$\mathcal{E}\colon\ \rho_{A_{\rm in}B_{\rm in}} \to \rho_{A_{\rm out}B_{\rm out}}.$$

Monotonicity: $E_{\downarrow}^{G}\left(\rho_{A_{\rm in}B_{\rm in}}\right) \geq E_{\downarrow}^{G}\left(\rho_{A_{\rm out}B_{\rm out}}\right)$.



- $\mathcal{E} \to$ quantum state $\chi_{A_1B_1A_2B_2}$
- $\mathcal{E}$ realized by teleportation with $\chi_{A_1B_1A_2B_2}$
- $|\Phi\rangle_{A_{\rm out}B_{\rm out}E_\chi E_{\rm in}} \propto {}_{A_1A_{\rm in}}\langle 0|_{B_1B_{\rm in}}\langle 0|\Psi\rangle_{A_{\rm in}B_{\rm in}E_{\rm in}}|\chi\rangle_{A_1B_1A_2B_2E_\chi}$

R. F. Werner *et al.*, Phys. Rev. Lett. 86, 3658 (2001).
A. Jamiołkowski, Rep. Math. Phys. 3, 275 (1972).
G. Giedke *et al.*, Phys. Rev. A 66, 032316 (2002).

Monotonicity

$\mathcal{E}$ is TPLOCC:

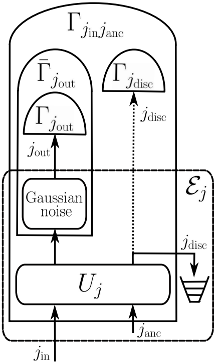
$$\chi_{A_1B_1A_2B_2} = \sum_i p_i \chi^{(i)}_{A_1A_2} \otimes \chi^{(i)}_{B_1B_1},$$

$$\chi^{(i)}_{j_1j_2} \leftrightarrow \text{TP operation } \mathcal{E}^{(i)}_j,\ j = A, B.$$

$\exists\ \Pi_{E_\chi}(\tilde{\Gamma}_{E_\chi})$ projecting $|\chi\rangle$ to $\chi^{(i)}_{A_1A_2} \otimes \chi^{(i)}_{B_1B_2}$.

$E^G_\downarrow(\rho_{A_kB_k}) = f(\gamma^k_\pi, \Gamma_{A_k}, \Gamma_{B_k}, \Gamma_{E_k}),\ j = \text{in}, \text{out}.$

$$E^G_\downarrow(\rho_{A_{\text{out}}B_{\text{out}}}) \leq f(\gamma^{\text{out}}_\pi, \Gamma_{A_{\text{out}}}, \Gamma_{B_{\text{out}}}, \tilde{\Gamma}_{E_\chi} \oplus \Gamma_{E_{\text{in}}})$$
$$\leq f(\gamma^{\text{in}}_\pi, \tilde{\Gamma}_{A_{\text{in}}}, \tilde{\Gamma}_{B_{\text{in}}}, \Gamma_{E_{\text{in}}}) \leq E^G_\downarrow(\rho_{A_{\text{in}}B_{\text{in}}}).$$



2nd inequality: LHS is MI for $(\mathcal{E}_A \otimes \mathcal{E}_B)(\rho_{A_{\text{in}}B_{\text{in}}|E_{\text{in}}})$,
integration of $\mathcal{E}_j$ into measurement with larger MI.

L. Mišta, Jr. *et al.*, PRA 83, 042325 (2011).
L. Mišta, Jr. *et al.*, PRA 91, 062313 (2015).

Two-mode states

Monotonicity ⇒ Invariance of GIE under Gaussian local unitaries ⇒

$$\gamma_{AB}=\begin{pmatrix} a & 0 & k_x & 0\\ 0 & a & 0 & -k_p\\ k_x & 0 & b & 0\\ 0 & -k_p & 0 & b\end{pmatrix},\quad k_x\geq k_p>0.$$

$\nu_{1,2}$ and S computable analytically via eigenvalues and eigenvectors of $i\Omega\gamma_{AB}$.

Upper bound for GIE:

$$U\left(\rho_{AB}\right):=\inf_{\Gamma_E}\sup_{\Gamma_A,\Gamma_B}\left[\frac{1}{2}\ln\left(\frac{\mathrm{det}\sigma_A\mathrm{det}\sigma_B}{\mathrm{det}\sigma_{AB}}\right)\right]=-\ln\sqrt{1-\inf_{\Gamma_E}\left(\frac{\tilde{c}_x^2}{\tilde{a}\tilde{b}}\right)}.$$

2nd equality holds if $2+\frac{1}{\sqrt{ab}}\geq\sqrt{\nu_1\nu_2}$ when homodyning of x_A and x_B is optimal for any Γ_E; $\tilde{a},\tilde{b},\tilde{c}_x$ - parameters of $\rho_{AB|E}$.

Lower bound for GIE:

$$L\left(\rho_{AB}\right):=\frac{1}{2}\ln\left\{\inf_{\Gamma_E}\left[\frac{\det(\Gamma_E+X_A)\det(\Gamma_E+X_B)}{\det(\Gamma_E+X_{AB})\det(\Gamma_E+\gamma_E)}\right]\right\}-\ln\sqrt{1-\frac{k_x^2}{ab}}.$$

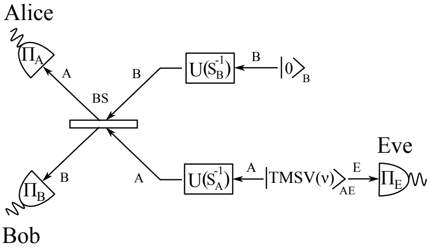
Symmetric states with a three-mode purification

States $\left(\equiv \rho_{AB}^{(1)}\right)$ satisfying:

$a = b, \quad \nu_2 = 1 \quad \Rightarrow \quad k_x = a - \frac{1}{a+k_p},$

$\nu_1 = \nu = \sqrt{\det\gamma_{AB}^{(1)}},$

S_A^{-1} (S_B^{-1}) – squeezing in p_A (x_B).

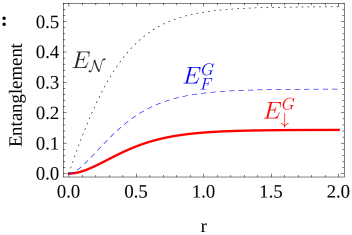


GIE for any $\rho_{AB}^{(1)}$:

$$E_{\downarrow}^{G}\left(\rho_{AB}^{(1)}\right) = \ln\left(\frac{a}{\sqrt{a^2-k_p^2}}\right)$$

Reached by homodyning of x_A, x_B and x_E.

CV GHZ:



L. Mišta, Jr. *et al.*, PRA 83, 042325 (2011).
L. Mišta, Jr. *et al.*, PRA 91, 062313 (2015).

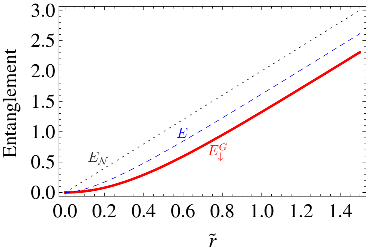
Pure states

For $k_x = k_p$ we get $a^2 - k_p^2 = 1$ and states $\rho_{AB}^{(1)}$ reduce to pure states ρ_{AB}^{p}.

GIE for pure states:

$$E_{\downarrow}^{G}\left(\rho_{AB}^{p}\right) = \ln\left(a\right)$$

where $a = \cosh(\tilde{r})$.



GIE is not equal to local von Neumann entropy on pure states

Equality is established by non-Gaussian photon counting on A and B.

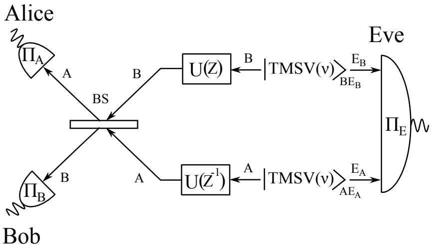
Symmetric squeezed thermal states

States $\left(\equiv \rho_{AB}^{(2)}\right)$ satisfying:

$a = b, \quad k_x = k_p \equiv k,$

$\nu_1 = \nu_2 \equiv \nu = \sqrt{a^2 - k^2},$

Z (Z^{-1}) – squeezing in x_B (p_A).



GIE for any $\rho_{AB}^{(2)}$ with $a \leq 2.41$:

$$E_{\downarrow}^{G}\left(\rho_{AB}^{(2)}\right) = \ln\left[\frac{(a-k)^2+1}{2(a-k)}\right]$$

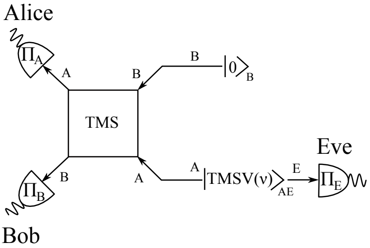
Reached by homodyning of x_A, x_B, x_{E_A} and p_{E_B}.

Asymmetric TMST states with a three-mode purification

States $\left(\equiv \rho_{AB}^{(3)}\right)$ with

$$\gamma_{AB} = \begin{pmatrix} a & 0 & k & 0 \\ 0 & a & 0 & -k \\ k & 0 & b & 0 \\ 0 & -k & 0 & b \end{pmatrix}.$$

$$\nu_2 = 1 \Rightarrow k = \begin{cases} \sqrt{(a+1)(b-1)}, & \text{if } a \geq b; \\ \sqrt{(a-1)(b+1)}, & \text{if } a < b. \end{cases}$$



Symplectic eigenvalue: $\nu \equiv \nu_1 = 1 + |a - b|$.

Symplectic matrix: $S = \begin{pmatrix} xI & -y\sigma_z \\ -y\sigma_z & xI \end{pmatrix}$, $x^2 - y^2 = 1$, two-mode squeezer.

GIE for any $\rho_{AB}^{(3)}$ with $\sqrt{ab} \leq 2.41$:

$$E_{\downarrow}^{G}\left(\rho_{AB}^{(3)}\right) = \ln\left(\frac{a+b}{|a-b|+2}\right)$$

Reached by homodyning of x_A, x_B and heterodyning on E.

G. Adesso *et al.*, PRL 109, 190502 (2012).

Relation to GR2EoF

Gaussian Rényi-2 entanglement of formation (GR2EoF):

$$E_{F,2}^{G}(\rho_{AB}) = \inf_{\substack{\theta_{AB}\leq\gamma_{AB} \\ \det\theta_{AB}=1}} \frac{1}{2}\ln\left(\det\theta_{A}\right).$$

Properties: faithfulness, monotonicity under all GLOCC, additivity on two-mode symmetric states, monogamy.

Computable for two-mode Gaussian states: analytically for all states with a three-mode purification, symmetric states and two-mode squeezed thermal states, and numerically for all other states.

For all considered states GIE coincides with GR2EoF!

Conjecture: GIE and GR2EoF are equal on all Gaussian states

Conclusion

- New quantifier of Gaussian entanglement.
- Operationally associated to secret key agreement protocol.
- Computable for several classes of two-mode Gaussian states.
- In all cases equal to GR2EoF $\rightarrow$ conjecture that the equality holds for all Gaussian states.

Thank you!