THE GAUSSIAN MINIMUM ENTROPY CONJECTURE
&
ITS IMPORTANCE FOR THE INFORMATION CAPACITY OF GAUSSIAN BOSONIC CHANNELS

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Outline

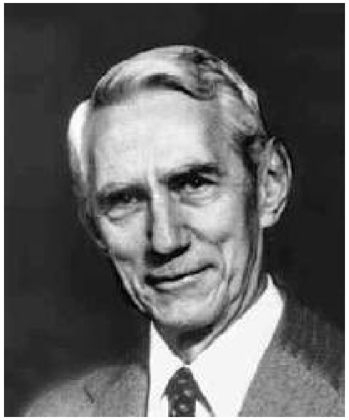
- Gaussian bosonic channels
  Classical capacity of quantum channels
  Gaussian minimum (output) entropy conjecture

- Link with entanglement of a 2-mode squeezer
  Gaussian minimum (output) entanglement conjecture

- Connection with majorization theory
  (Stronger) Gaussian majorization conjecture
  Incomplete proof of the conjecture (Fock state inputs)

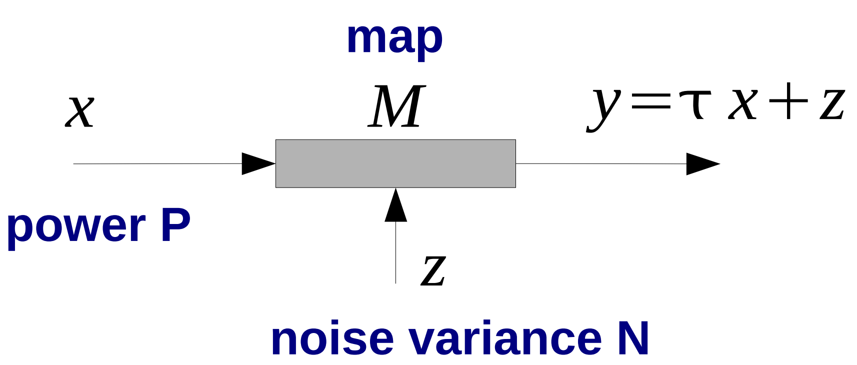
- Conclusions — importance for physics problems !
  … new path to the ultimate proof ?

Motivation: perfect (noiseless) channel



Shannon theory

Numerous communication links modeled by
a classical **Gaussian additive-noise channel**



$$C(M)=\frac{1}{2}\log\left(1+\frac{\tau P}{N}\right)$$

$$\tau=1 \quad N=0$$

Infinite capacity for finite input power P !

..... we need quantum mechanics to calculate the ultimate limits of communication !

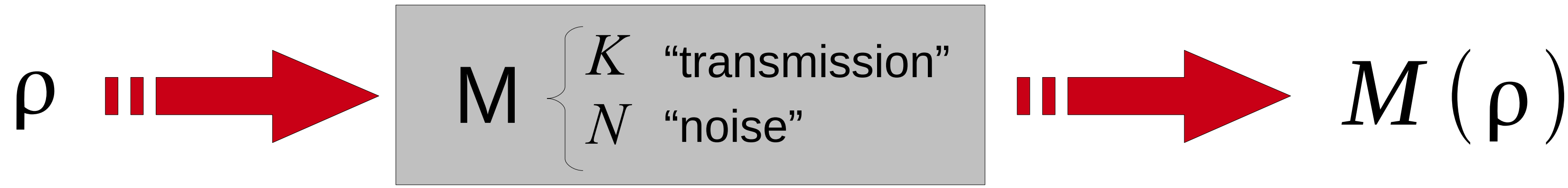
Quantum theory

Gaussian quantum channels
H. P. Yuen, and M. Ozawa, PRL 1993

$$C(M)=max_{\rho}S(\rho)=(\nu+1)\log(\nu+1)-\nu\log(\nu)$$ for identity channel

Finite capacity for finite input energy, achieved by a thermal state of ν photons

Gaussian (quantum) Bosonic channels



- Corresponds to linear CP maps $\rho \rightarrow M[\rho]$

  s.t. $M[\rho]$ Gaussian if ρ Gaussian

- M fully characterized by two matrices K, N

  $\vec{r} \rightarrow K\vec{r}$ $\qquad\vec{r}$ = coherent vector

  $\gamma \rightarrow K\gamma K^T + N$ $\qquad\gamma$ = covariance matrix

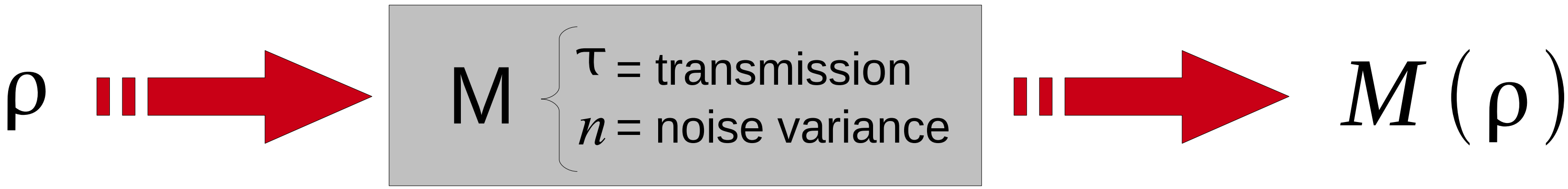
  (K) real; (N) real & symmetric

- M completely positive $\Longleftrightarrow$ one-mode case: $N \geq 0 \qquad det\, N \geq (det\, K - 1)^2$

  uncertainty principle

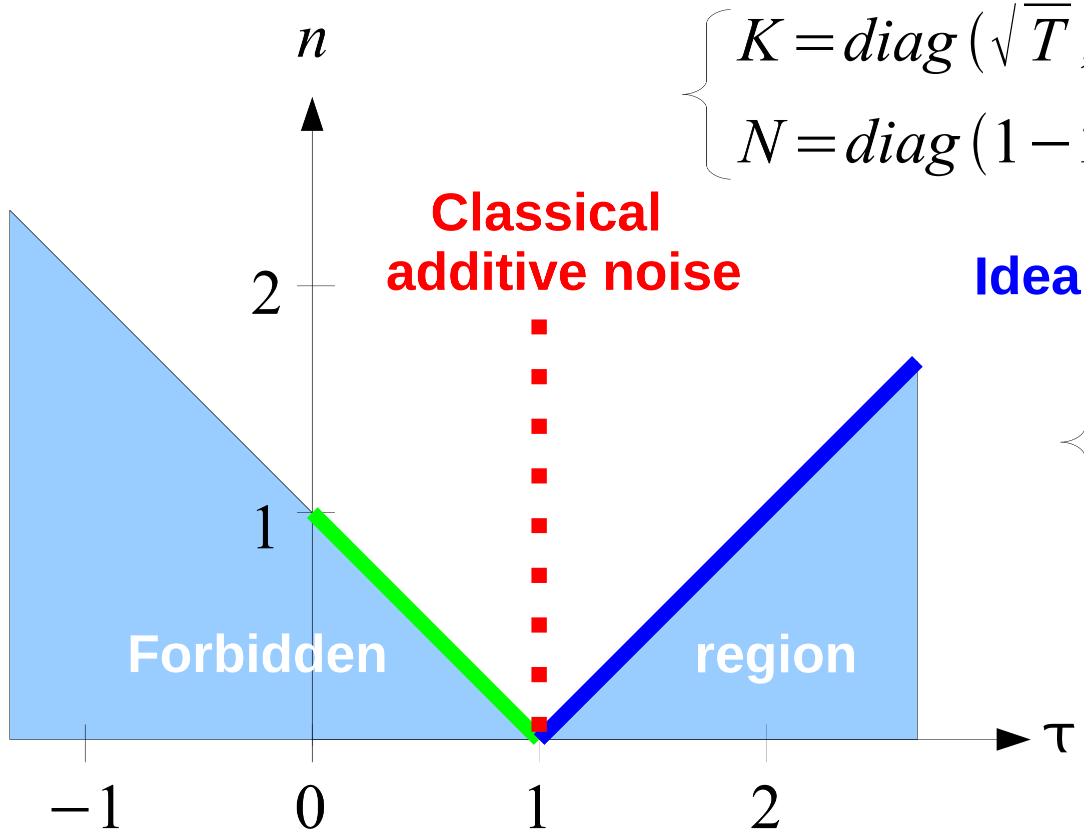
Phase-insensitive Gaussian Channels

$$\begin{cases} K = diag(\sqrt{\tau}, \sqrt{\tau}) \\ N = diag(n, n) \end{cases}$$



Purely lossy channel

$$\begin{cases} K = diag(\sqrt{T}, \sqrt{T}) \\ N = diag(1-T, 1-T) \end{cases} \qquad (T \leq 1)$$



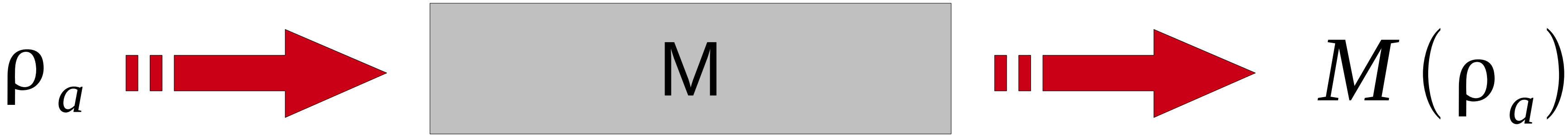
Ideal (quantum-limited) amplifier

$$\begin{cases} K = diag(\sqrt{G}, \sqrt{G}) \\ N = diag(G-1, G-1) \end{cases} \qquad (G \geq 1)$$

(our analysis is focused on phase-insensitive channels w.n.l.g.)

Classical Capacity of Quantum Channels

Holevo, Schumacher, Westmoreland, 1998

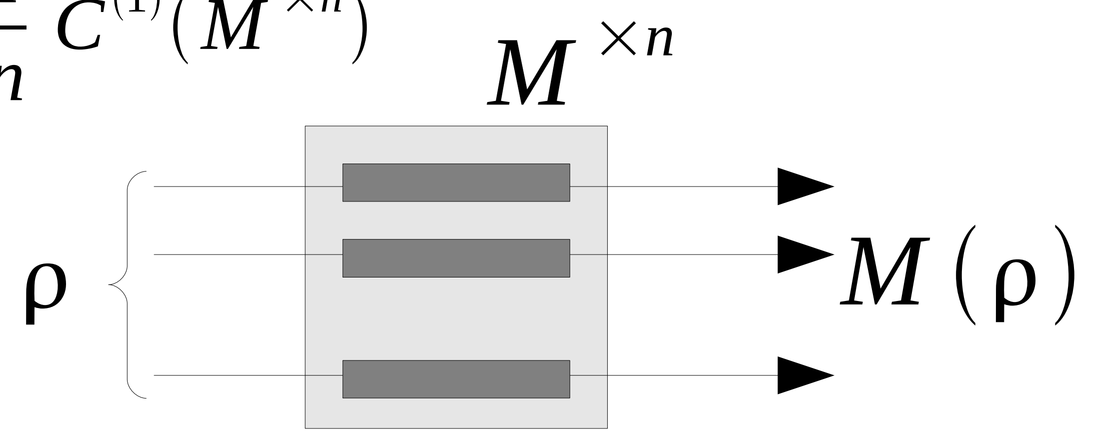


encoding $\{p_a, \rho_a\}_{a=1,\dots d}$ such that $\sum_a p_a \rho_a = \rho$

- **Holevo bound** $\chi(\{p_a, \rho_a\}, M) = S(M(\rho)) - \sum_a p_a S(M(\rho_a))$
- **Single-shot capacity** $C^{(1)}(M) = max_{\{p_a, \rho_a\}} \chi(\{p_a, \rho_a\}, M)$
- **Capacity** $C(M) = \lim_{n \to \infty} \frac{1}{n} C^{(1)}(M^{\times n})$

... in general, <u>not</u> additive !

(Hastings, Nature Phys. 2009)



Minimum Output Entropy

$$\rho_a \;\Rightarrow\; \boxed{M} \;\Rightarrow\; M(\rho_a)$$

encoding $\{p_a, \rho_a\}_{a=1,\dots d}$ such that $\sum_a p_a \rho_a = \rho$

$$C^{(1)}(M) = max_{\rho}\, \tilde{\chi}(\rho, M)$$

... maximization in 2 steps

with $\tilde{\chi}(\rho, M) \equiv S\big(M(\rho)\big) - min_{\substack{\{p_a,\rho_a\} \\ \sum_a p_a \rho_a = \rho}} \underbrace{\sum_a p_a S\big(M(\rho_a)\big)}_{\geq S(M(\Phi_0))}$

$$\leq S\big(M(\rho)\big) - S\big(M(\Phi_0)\big)$$

⇨ we need to find pure state Φ_0 minimizing output entropy

$$min_{\sigma} S\big(M(\sigma)\big) \equiv S\big(M(\Phi_0)\big)$$

Capacity of Gaussian Quantum Channels

Yuen and Ozawa, 1993

Holevo and Werner, 1998

- continuous encoding
- energy constraint

ρ_α → M → $M(\rho_a)$

encoding $\{p(\alpha), \rho_\alpha\}$ such that $\int d^2\alpha \; p(\alpha)\rho_\alpha = \rho$

$$C^{(1)}(M) = max_\rho \; \tilde{\chi}(\rho, M)$$

$$\le max_\rho \; S\left(M(\rho)\right) - S\left(M(\Phi_0)\right)$$

for fixed energy, achieved by a thermal state

$$S\left(M(\rho_{therm})\right)$$

minimum output entropy state ???

Gaussian minimum output entropy conjecture: $\Phi_0 = |0\rangle\langle 0|$

Conjectured Single-shot Capacity

ρ_α ⇒ **M** (τ = transmission, n = noise variance) ⇒ $M(\rho_a)$

encoding $\{p(\alpha), \rho_\alpha\}$ such that $\int d^2\alpha \; p(\alpha)\,\rho_\alpha = \rho$

$$C^{(1)}(M) \leq S\big(M(\rho_{therm})\big) - S\big(M(|0\rangle\langle 0|)\big)$$

($M(|0\rangle\langle 0|)$ → Φ_0)

✦ use encoding $\rho_\alpha = D(\alpha)|0\rangle\langle 0|D(\alpha)^{+}$ $\quad p(\alpha) = \frac{1}{\pi\nu}\exp\left(\frac{-|\alpha|^2}{\nu}\right)$

with ν = mean thermal photon number

$$C^{(1)}(M) = g[\tau\nu + n] - g[n] \quad \text{where} \quad g[x] = (x+1)\log(x+1) - x\log(x)$$

⇒ coherent states modulated with a Gaussian bivariate distribution do achieve the capacity (it is the *optimal encoding*)

... provided the **Gaussian minimum entropy conjecture** holds !

Gaussian Minimum Entropy Conjecture

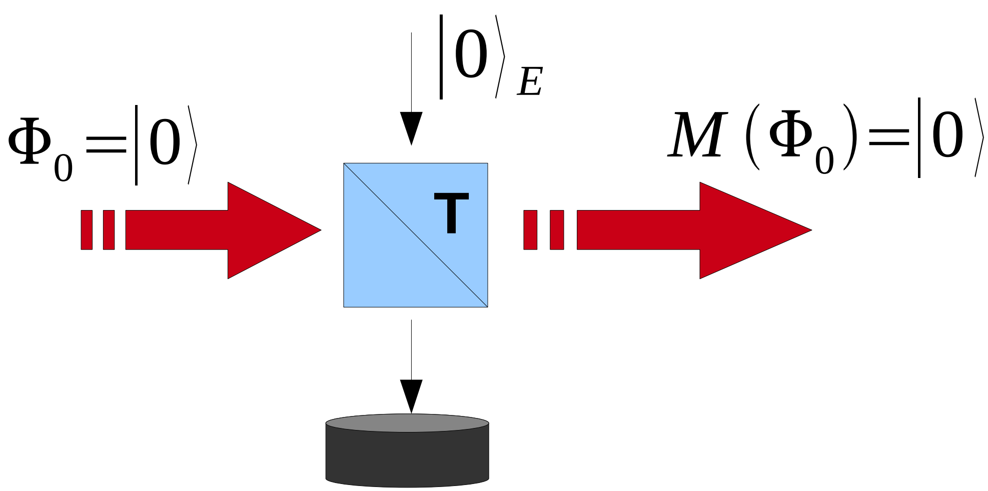
$$min_{\sigma} S\left(M(\sigma)\right) = S\left(M(|0\rangle\langle 0|)\right)$$

- The same conjecture is made for the joint channel $M^{\times n}$

then, $C(M) \equiv \lim_{n\to\infty} \frac{1}{n} C^{(1)}(M^{\times n}) = C^{(1)}(M)$

- All papers (> 30) on the topic of Gaussian bosonic channels
  rely on this widely admitted conjecture !!!

- Single exception: <u>pure lossy</u> channel (environment *E* in vacuum state)

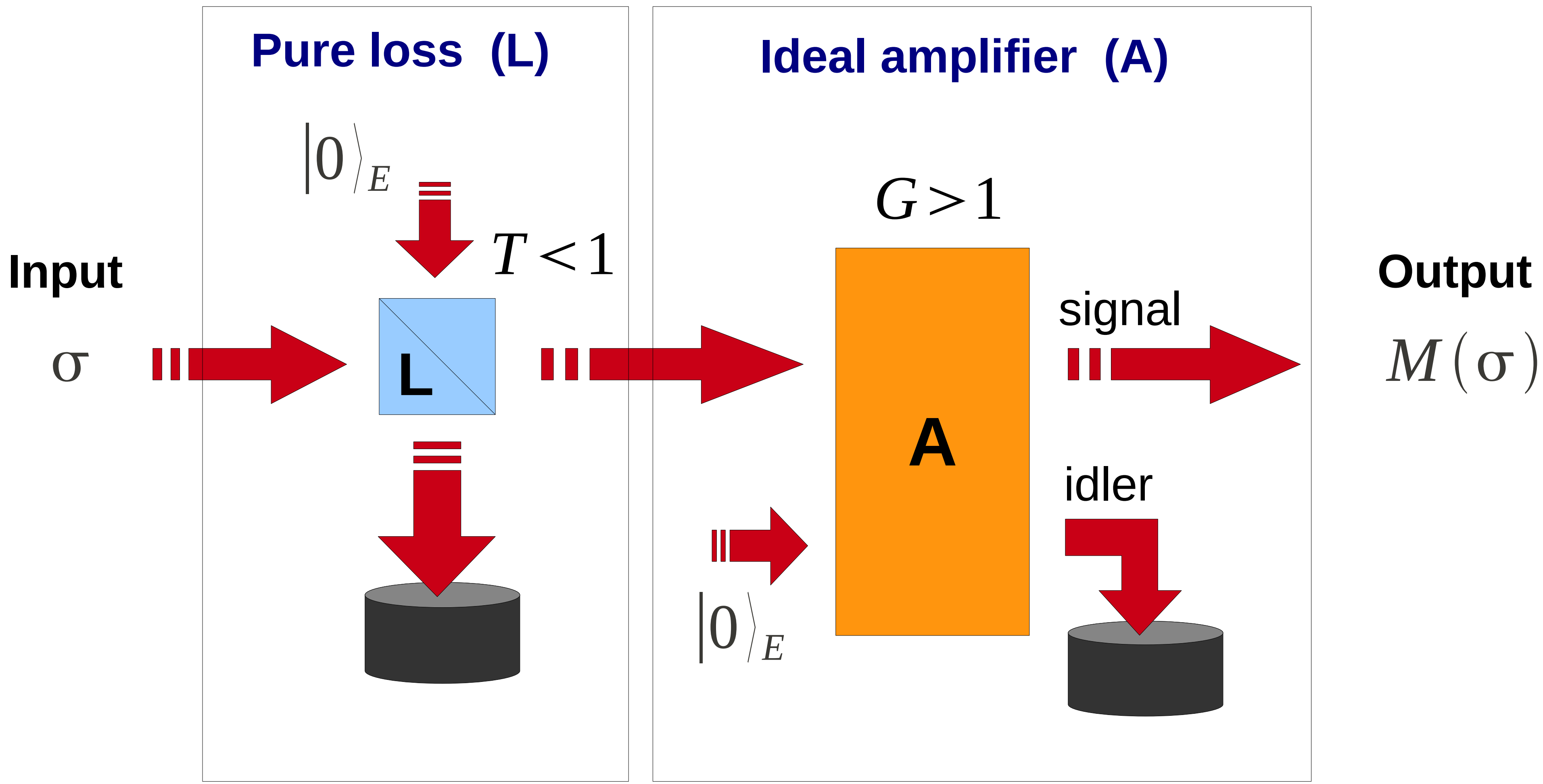


$$C^{(1)}(M) = g[\tau\nu]$$

V. Giovannetti *et al.*, PRL, 2004

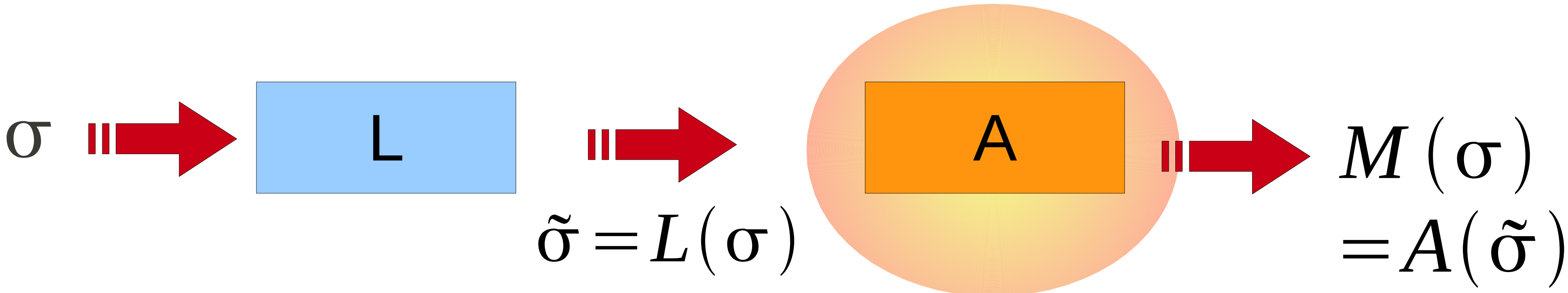
Generic Decomposition of Phase-insensitive Channels

- $\tau = T\,G < 1$ lossy fiber with thermal noise
- $\tau = T\,G = 1$ classical Gaussian additive noise
- $\tau = T\,G > 1$ (non-ideal) noisy amplifier



Reduction of the Conjecture

Conjecture I $\quad min_{\sigma} S(M(\sigma)) = S(M(|0\rangle\langle 0|))$

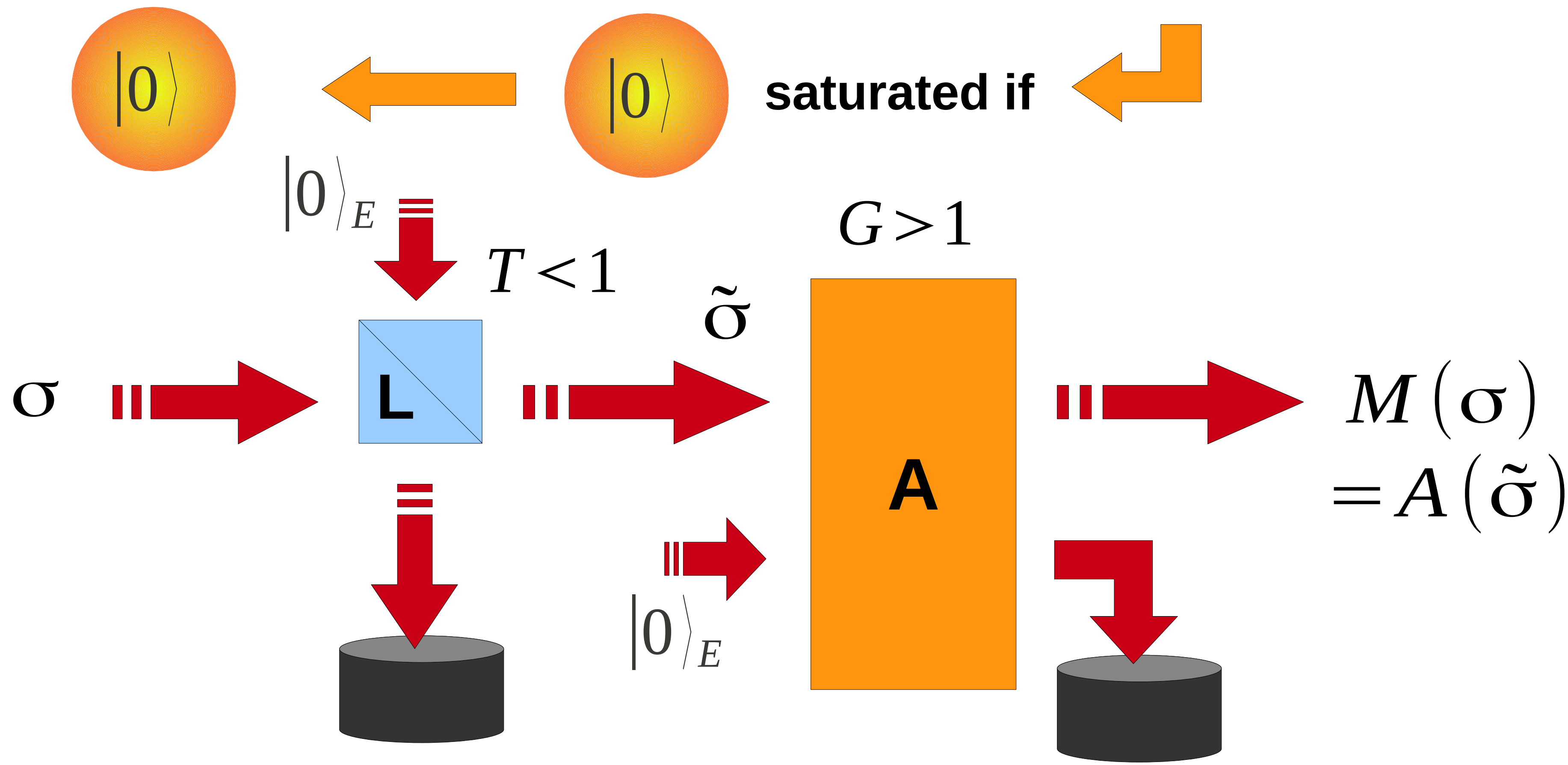


Conjecture II $\quad min_{\tilde{\sigma}} S(A(\tilde{\sigma})) = S(A(|0\rangle\langle 0|))$

Reduction to Ideal Amplifier

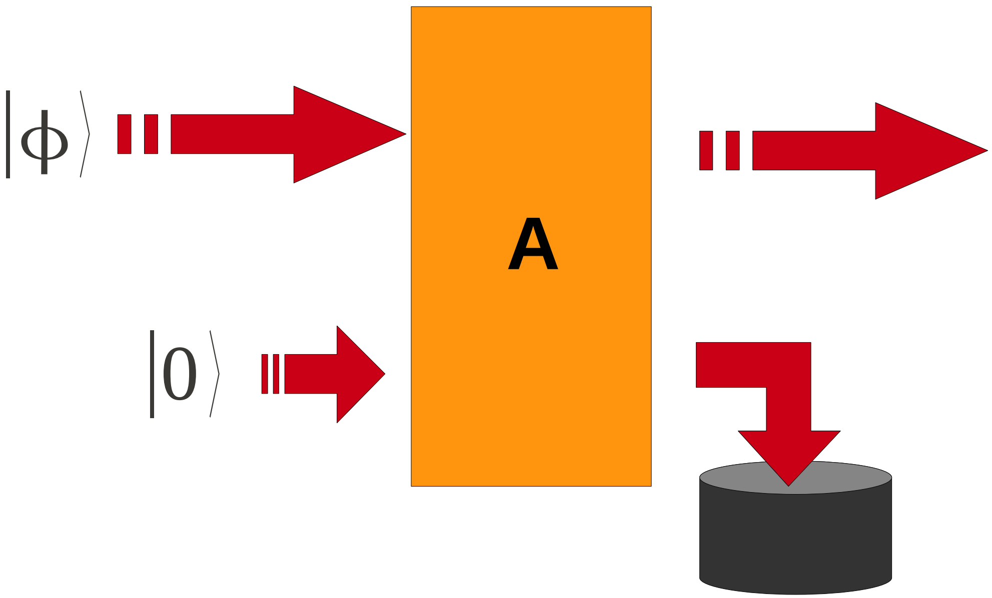
assume conjecture II holds

$$S\left(M\left(\sigma\right)\right) = S\left(A\left(\tilde{\sigma}\right)\right) \geq S\left(A\left(\left|0\right\rangle\left\langle 0\right|\right)\right)$$



it is (necessary and) sufficient to prove the reduced conjecture II

Link with Output Entanglement of a Two-Mode Squeezer

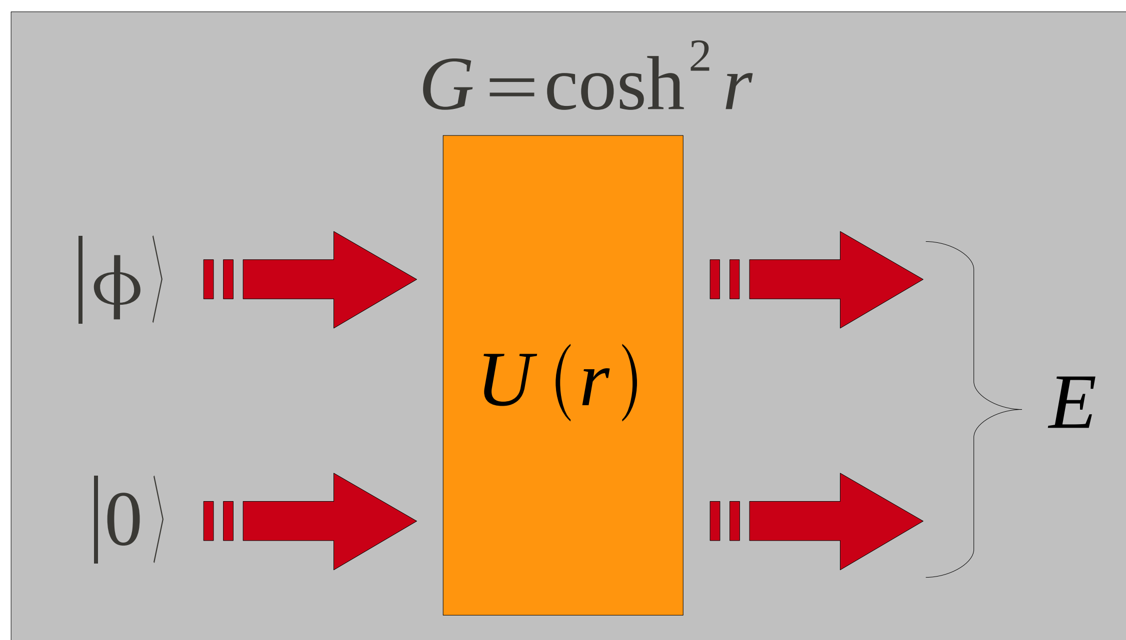


$$S\left(A\left(|\phi\rangle\langle\phi|\right)\right) = E\left(U(r)|\phi\rangle|0\rangle\right)$$

entanglement entropy of a pure state

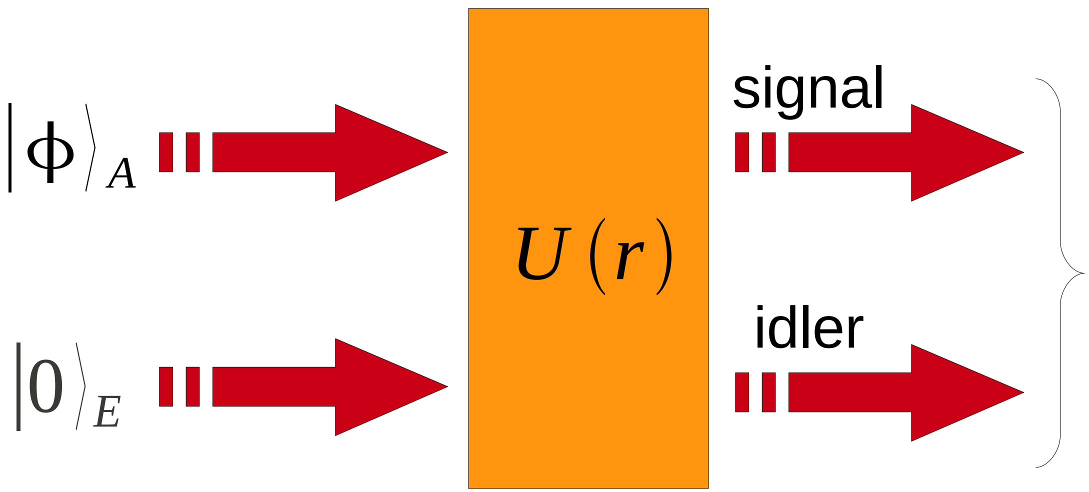
We are now dealing with the output entanglement of a **two-mode squeezer**

$$U(r) = \exp\left(\frac{r}{2}\left(ab - a^{+} b^{+}\right)\right)$$



Gaussian Minimum Entanglement Conjecture

Conjecture I $\quad min_{\sigma} S(M(\sigma)) = S(M(|0\rangle\langle 0|))$



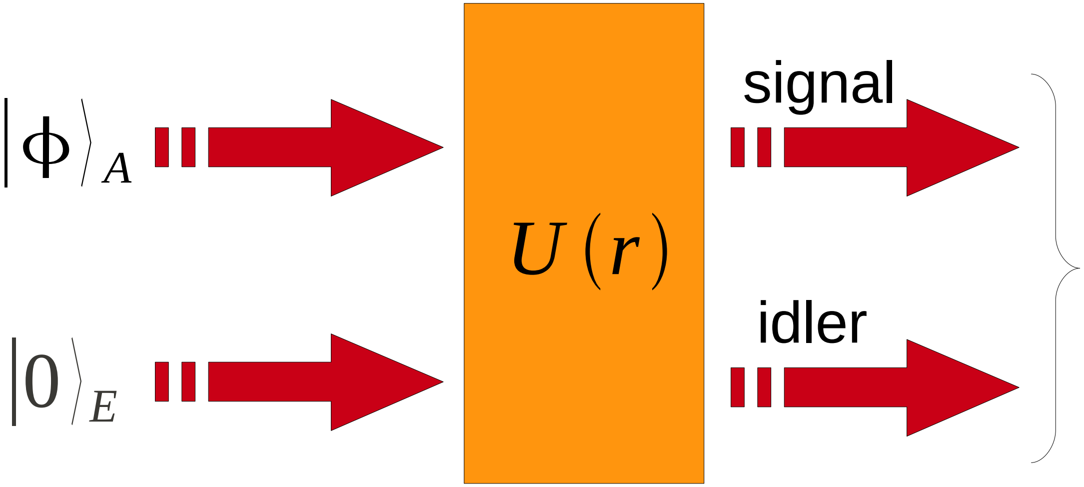
$$|\Phi\rangle_{AB} = U(r)|\phi\rangle_A |0\rangle_E$$

compare with vacuum input :

$$|V\rangle_{AB} = U(r)|0\rangle_A |0\rangle_E$$

Conjecture II (bis) $\quad min_{\phi} E(|\Phi\rangle_{AB}) = E(|V\rangle_{AB})$

Proof for Gaussian vs non-Gaussian states



$$|\Phi\rangle_{AB} = U(r)|\phi\rangle_A|0\rangle_E$$

actual output state

$$|V\rangle_{AB} = U(r)|0\rangle_A|0\rangle_E$$

2-mode vacuum squeezed state

- easy to prove for **Gaussian** states

$$E\left(|\Phi_{Gauss}\rangle_{AB}\right) \geq E\left(|V\rangle_{AB}\right)$$

- expansion of **non-Gaussian** states in Fock basis

$$|\phi\rangle_A = \sum_k c_k |k\rangle$$

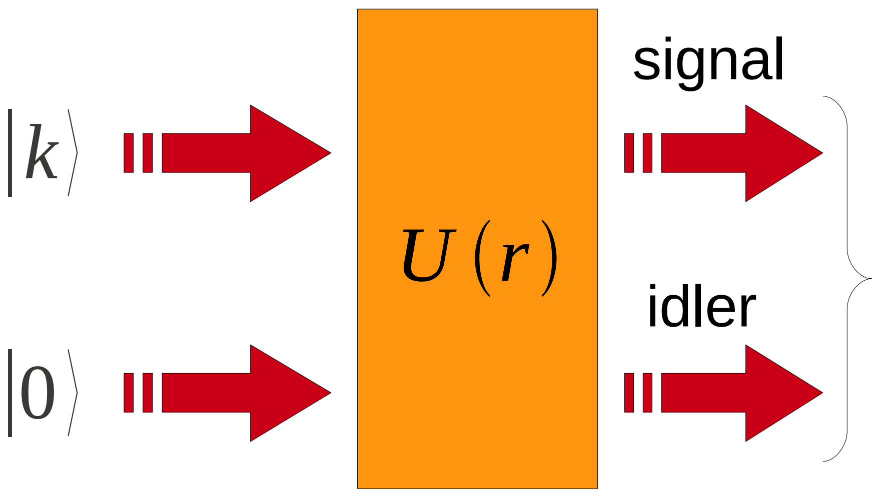
Fock states

... only an incomplete proof for Fock states !

Fock State Inputs

$$E(k) = H\left[P_n(k)\right] \quad \text{Shannon entropy}$$

$$P_n(k) = \frac{1}{\cosh^{2(k+1)} r} \binom{n+k}{n} \tanh^{2n} r$$



$$|\Psi_k\rangle = \sum_{n=0}^{\infty} \sqrt{P_n(k)}\; |n+k\rangle |n\rangle$$

Using Pascal identity $$\binom{n+k+1}{n} = \binom{n+k}{n} + \binom{n+k}{n-1}$$

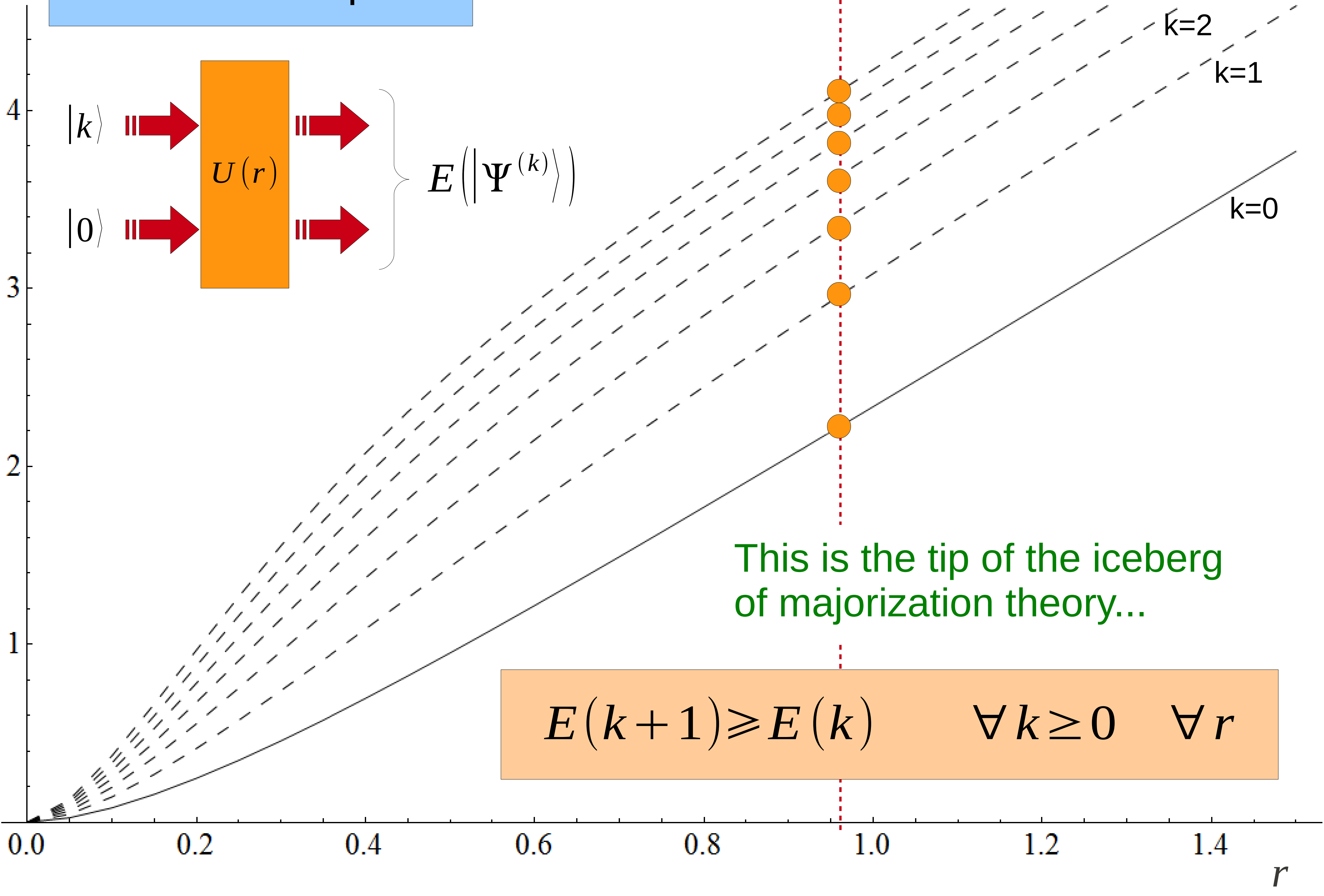
$$P_n(k+1) = (1-\lambda^2) P_n(k) + \lambda^2 P_{n-1}(k+1)$$ with $\lambda = \tanh r$

Concavity of entropy $$E(k+1) \geqslant (1-\lambda^2) E(k) + \lambda^2 E(k+1)$$

$$E(k+1) \geqslant E(k) \qquad \forall\, k \geq 0$$

vacuum state beats all other Fock states

Fock State Inputs



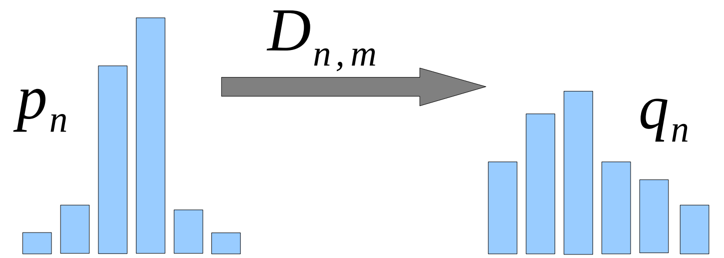
Majorization Theory

= (partial) order relation for probability distributions

$D_{n,m}$ →

p_n majorizes q_n

$p_n \succ q_n$



with p_n, q_n probability distributions

if and only if

- p_n can be converted to q_n by applying a *random permutation*

  $q_n = \sum_m D_{n,m} p_m$ $D_{n,m}$ is *doubly-stochastic* matrix

or
- $\sum_{n=0}^{m} p_n^{\downarrow} \geq \sum_{n=0}^{m} q_n^{\downarrow} \quad \forall\, m \geq 0$ (p is "*more peaked*" than q)

or
- $\sum_n h(p_n) \leq \sum_n h(q_n) \quad \forall\, h(x)$ concave function

  e.g. entropy: $h(x) = -x \log(x)$

  ⇒ $H(p_n) \leq H(q_n)$ entropy can only <u>increase</u>

Quantum case : Interconversion of pure bipartite states

M. Nielsen, G. Vidal, 2000

$D_{n,m}$ →

p_n majorizes q_n

p_n, q_n probability distributions

← **LOCC**

$|\psi\rangle$ majorizes $|\phi\rangle$

$$|\psi\rangle = \sum_n \sqrt{p_n}\,|e_n\rangle\,|f_n\rangle \qquad |\phi\rangle = \sum_n \sqrt{q_n}\,|e_n'\rangle\,|f_n'\rangle$$

with p_n majorizing q_n

- $|\phi\rangle$ can be converted to $|\psi\rangle$ by applying a deterministic LOCC
- $E(|\psi\rangle) \leq E(|\phi\rangle)$ entanglement can only <u>decrease</u>

<u>Trick</u>: $$\rho_A = tr_B(|\psi\rangle\langle\psi|) = \sum_n p_n \underbrace{|e_n\rangle\langle e_n|}_{\text{orthonormal}}$$ eigenbasis representation

$$= \sum_n q_n \underbrace{|\zeta_n\rangle\langle\zeta_n|}_{\text{not orthonormal}}$$ provided p_n majorizes q_n

Explicit conversion LOCC

LOCC ← $|\psi\rangle$ majorizes $|\phi\rangle$

$$|\psi\rangle = \sum_n \sqrt{p_n}\,|e_n\rangle\,|f_n\rangle$$

$$|\psi\rangle = \sum_n \sqrt{q_n}\,|\zeta_n\rangle\,|f_n''\rangle$$ above trick (provided p_n majorizes q_n)

LOCC ↑ ↕ U

$$|\phi\rangle = \sum_n \sqrt{q_n}\,|e_n'\rangle\,|f_n'\rangle$$

POVM: $A_m = \sum_n \omega^{nm} |\zeta_n\rangle\langle e_n'|$ with $\omega = e^{i2\pi/d}$ and $\sum_m A_m^+ A_m = I$

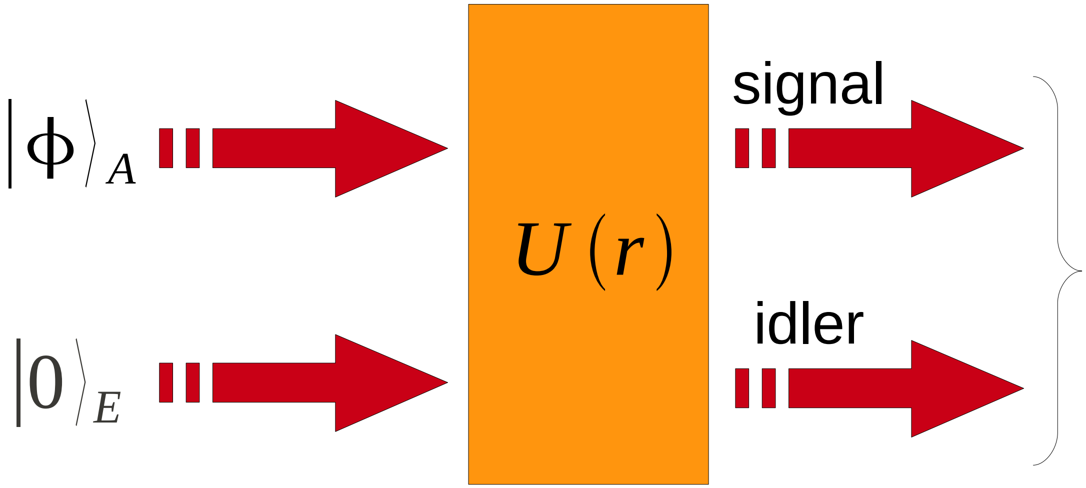
$$(A_m \times I)\,|\phi\rangle = \sum_n \sqrt{q_n}\,\omega^{nm}\,|\zeta_n\rangle\,|f_n'\rangle \equiv |\phi_m\rangle$$ depends on outcome m

Conditional U: $B_m = \sum_n \omega^{-nm} |f_n''\rangle\langle f_n'|$ conditional on m

$$(I \times B_m)\,|\phi_m\rangle = \sum_n \sqrt{q_n}\,|\zeta_n\rangle\,|f_n''\rangle \equiv |\psi\rangle$$

Gaussian Majorization Conjecture

For a given 2-mode squeezer (Bogoliubov transformation),
the 2-mode vacuum squeezed state **majorizes** all other output states !!!



$$|\Phi\rangle_{AB} = U(r)|\phi\rangle_A|0\rangle_E$$

compare with vacuum input

$$|V\rangle_{AB} = U(r)|0\rangle_A|0\rangle_E$$

Conjecture III $|V\rangle_{AB}$ majorizes $|\Phi\rangle_{AB}$ $\forall \; |\phi\rangle_A$

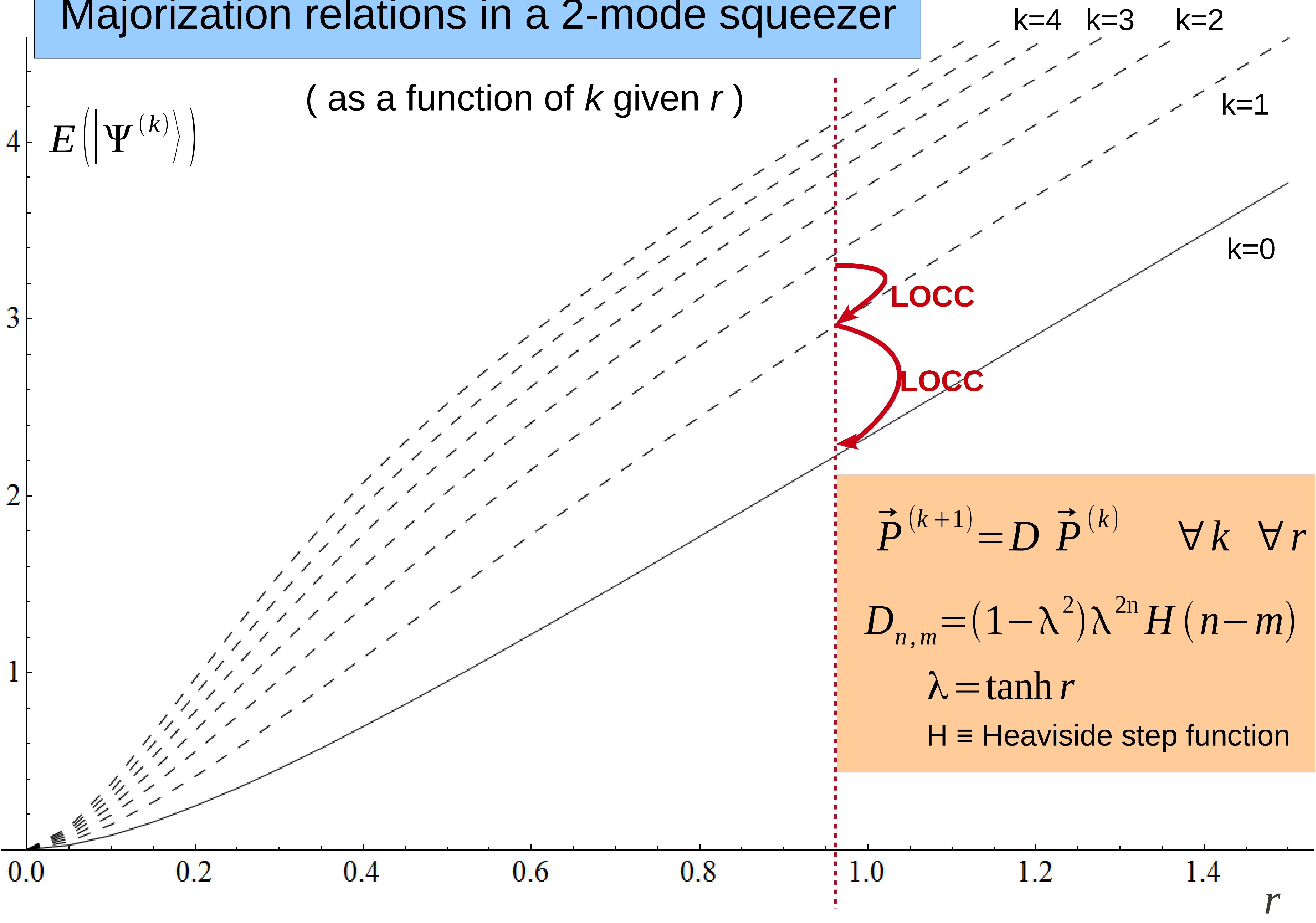
$$|V\rangle_{AB} \xleftarrow{\text{LOCC}} |\Phi\rangle_{AB}$$ implying $$E(|V\rangle_{AB}) \le E(|\Phi\rangle_{AB})$$

... stronger than minimum output entropy conjecture

... but perhaps easier to prove (?)

Majorization relations in a 2-mode squeezer

(as a function of *k* given *r*)



$$\vec{P}^{(k+1)} = D\,\vec{P}^{(k)} \quad \forall k \;\; \forall r$$

$$D_{n,m} = (1-\lambda^2)\lambda^{2n} H(n-m)$$

$$\lambda = \tanh r$$

H ≡ Heaviside step function

Explicit LOCC

$$|\Psi_k\rangle = \sum_{n=0}^{\infty} \sqrt{p_n(k)}\ |n+k\rangle|n\rangle$$

$$\uparrow$$

$$|\Psi_{k+1}\rangle = \sum_{n=0}^{\infty} \sqrt{p_n(k+1)}\ |n+k+1\rangle|n\rangle$$

Alice applies a POVM

$$\begin{cases} A_{YES} = \sum_{m=0}^{\infty} \sqrt{\dfrac{(1-\lambda^2)\,p_m(k)}{p_m(k+1)}}\ |m+k\rangle\langle m+k+1| \\ A_{NO} = \sum_{m=0}^{\infty} \sqrt{\dfrac{\lambda^2\,p_{m-1}(k+1)}{p_m(k+1)}}\ |m+k\rangle\langle m+k+1| \end{cases}$$

$$(A_{YES}\times 1)\,|\Psi_{k+1}\rangle = \sqrt{(1-\lambda^2)}\ \sum_{n=0}^{\infty}\sqrt{p_n(k)}\ |n+k\rangle|n\rangle = \sqrt{(1-\lambda^2)}\ |\Psi_k\rangle \quad \text{YES}$$

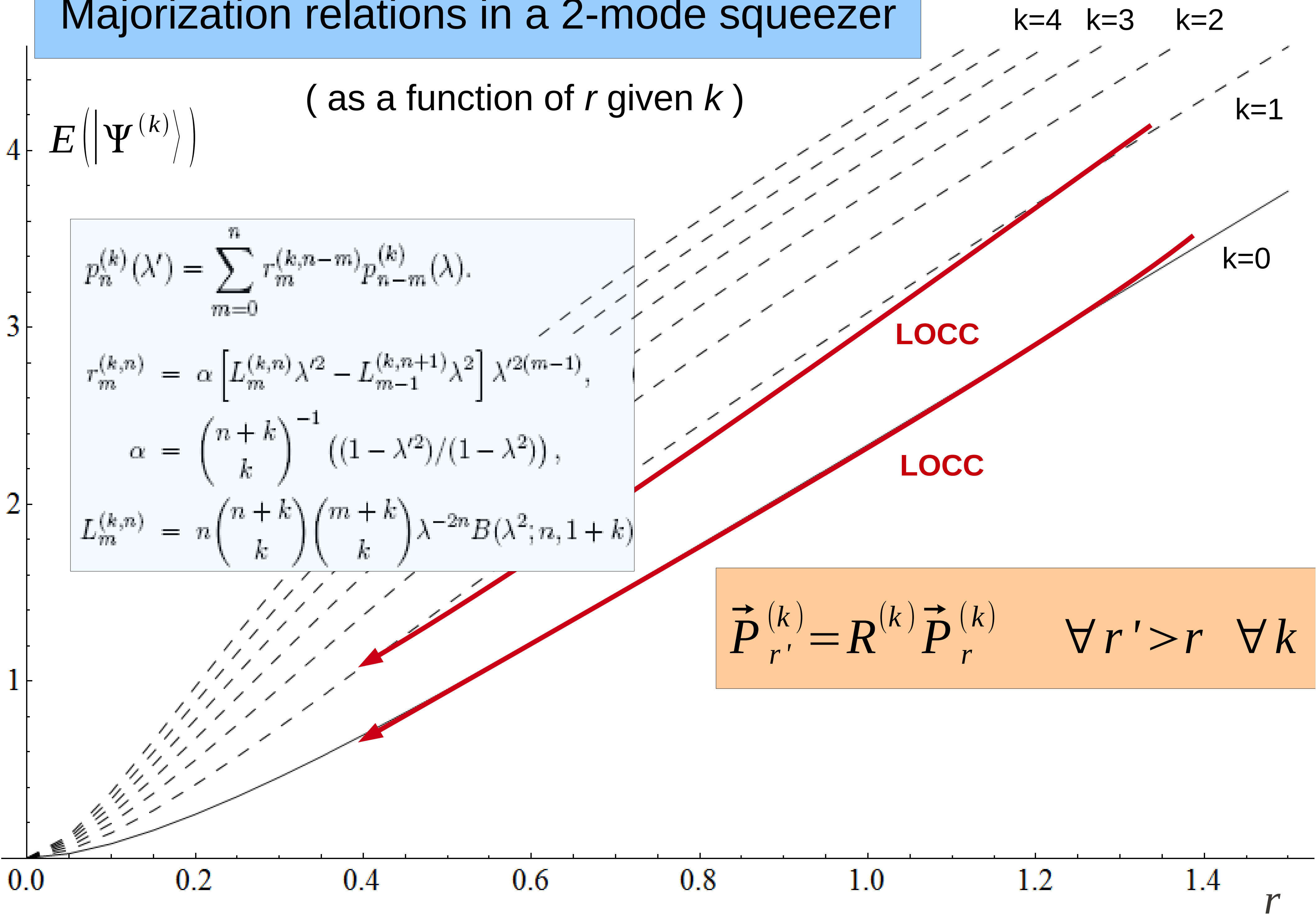
$$(A_{NO}\times 1)\,|\Psi_{k+1}\rangle = \sqrt{\lambda^2}\ \sum_{n=0}^{\infty}\sqrt{p_n(k+1)}\ |n+k+1\rangle|n+1\rangle \rightarrow \sqrt{\lambda^2}\ |\Psi_{k+1}\rangle \quad \text{NO}$$

$$\uparrow$$

If "NO" she communicates it to Bob who applies $U = \sum_{m=0}^{\infty} |m\rangle\langle m+1|$ and then they start a new round again

Majorization relations in a 2-mode squeezer

(as a function of *r* given *k*)



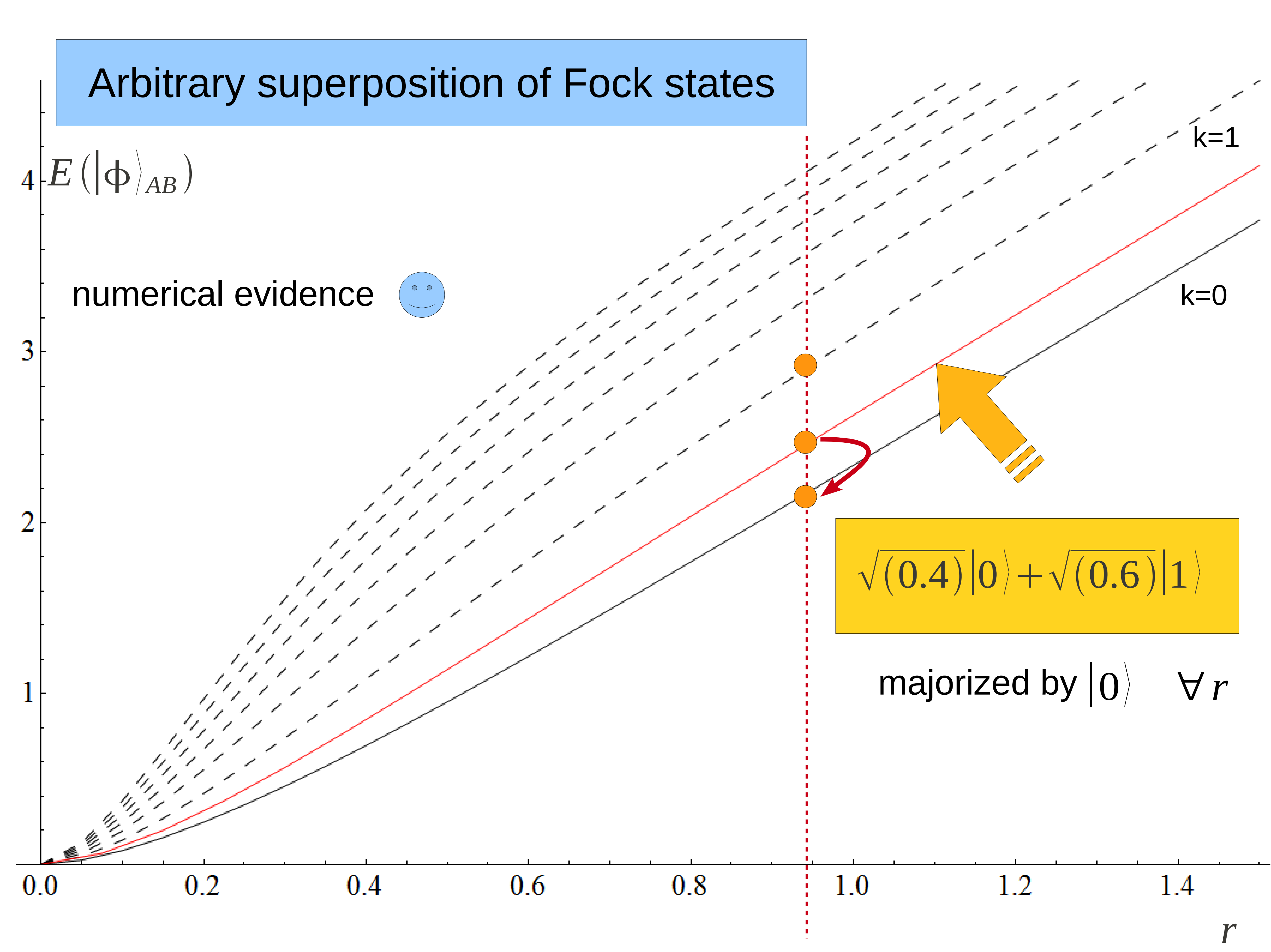
$$p_n^{(k)}(\lambda') = \sum_{m=0}^{n} r_m^{(k,n-m)} p_{n-m}^{(k)}(\lambda).$$

$$r_m^{(k,n)} = \alpha \left[L_m^{(k,n)} \lambda'^2 - L_{m-1}^{(k,n+1)} \lambda^2\right] \lambda'^{2(m-1)},$$

$$\alpha = \binom{n+k}{k}^{-1} \left((1-\lambda'^2)/(1-\lambda^2)\right),$$

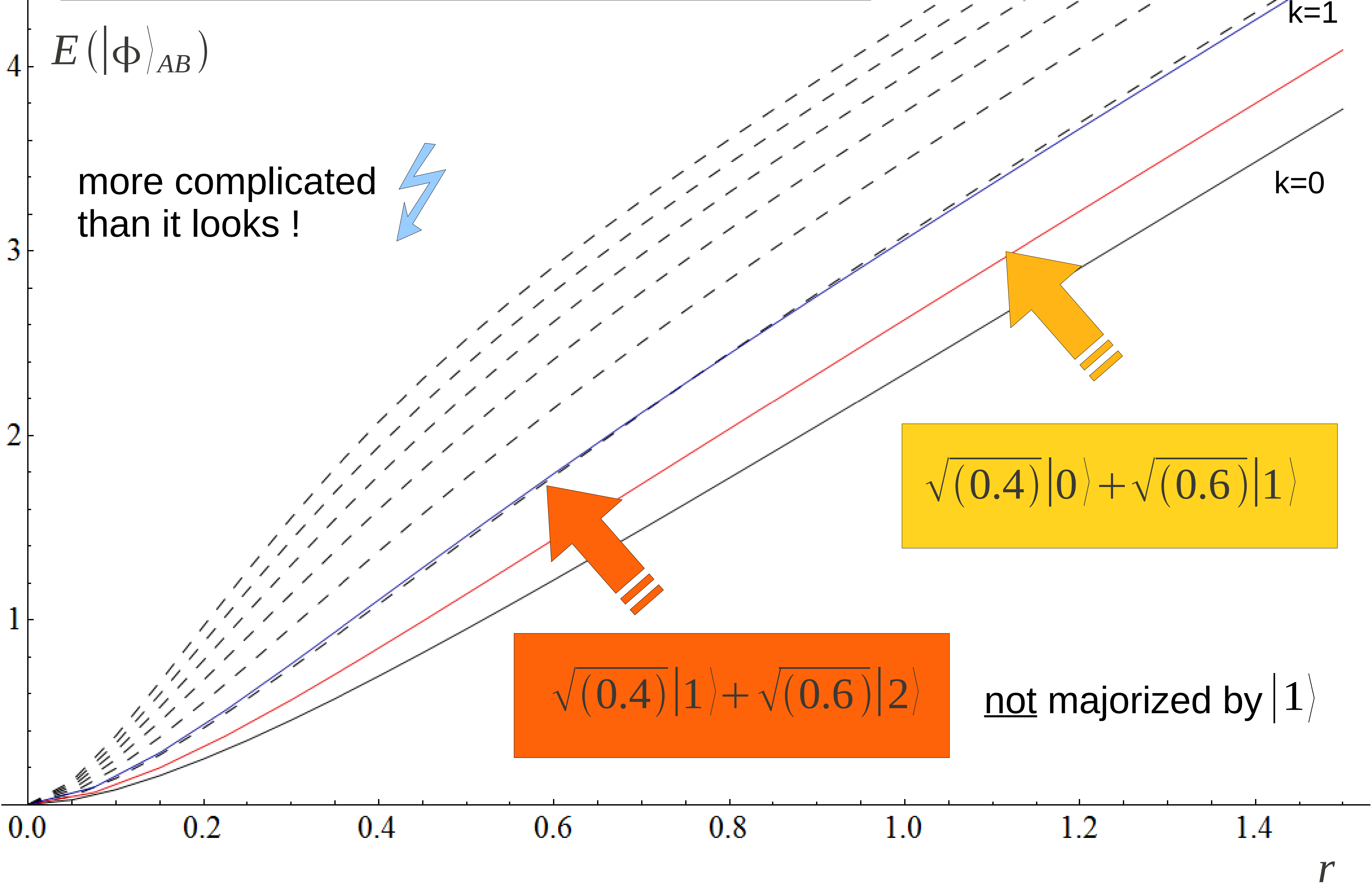
$$L_m^{(k,n)} = n\binom{n+k}{k}\binom{m+k}{k}\lambda^{-2n} B(\lambda^2; n, 1+k)$$

$$\vec{P}_{r'}^{(k)} = R^{(k)} \vec{P}_r^{(k)} \qquad \forall\, r' > r \quad \forall\, k$$



Arbitrary superposition of Fock states
$E(|\phi\rangle_{AB})$
numerical evidence
k=1
k=0
$\sqrt{(0.4)}|0\rangle + \sqrt{(0.6)}|1\rangle$
majorized by $|0\rangle$ $\forall r$
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1
2
3
4
r

Arbitrary superposition of Fock states



This is a fundamental problem

(even if you don't care about quantum channels !)

Bogoliubov transformation $\hat{a}_i' = \sum_j \left(u_{ij} \hat{a}_j + v_{ij} \hat{a}_j^+ \right)$ is everywhere

(e.g., quantum optics, supraconductivity, Hawking radiation, Unruh effect,...)

… this conjecture may have deeper implications !

ALTERNATE TITLE: can we prove that ...

"Nothing is less than vacuum" ?

(less random than the vacuum state)

Take-home message

"Nothing is less than the vacuum"

very plausible but not (yet) proven !

New approach to solve the "Minimum Output Entropy Conjecture"
for Gaussian bosonic channels

- Reduction to ideal amplifier channel
- Output entanglement of a two-mode squeezer
- Link with majorization theory (currently, Fock states only)

Numerical analysis for random input states
strongly suggests that (majorization) conjecture is true...

See: R. Garcia-Patron, C. Navarrete-Benlloch, S. Lloyd,
J. H. Shapiro & N. J. Cerf, PRL 108, 110505 (2012).