evropský sociální fond v ČR

EVROPSKÁ UNIE

MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

OP Vzdělávání pro konkurenceschopnost

INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Some Generalizations of Formal Concept Analysis

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Why to fuzzify?

Classical formal concept analysis

- Ganter & Wille
- an object-attribute model
  - columns – attributes – the set A
  - rows – objects – the set B
  - values – a relation $R \subseteq A \times B$
- a Galois connection $(\uparrow, \downarrow)$
  - if $X \subseteq B$ then $\uparrow(X) = \{a \in A : (\forall b \in X)\langle a, b\rangle \in R\}$
  - if $Y \subseteq A$ then $\downarrow(Y) = \{b \in B : (\forall a \in Y)\langle a, b\rangle \in R\}$
- a concept – such (X, Y) that $\uparrow(X) = Y$ and $\downarrow(Y) = X$
- $(X_1, Y_1) \leq (X_2, Y_2)$ iff $X_1 \subseteq X_2$ iff $Y_1 \supseteq Y_2$
- the set of concepts order by $\leq$ is a complete lattice called the concept lattice

Non-binary data

- what to do with these data?

	a	b	c	d	e
α	1.0	0.8	0.2	0.3	0.5
β	0.8	1.0	0.2	0.6	0.9
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- e. g. how to (re)define mappings $\uparrow$ and $\downarrow$?

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1
One-sided fuzzy approach

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- these definitions are non-symmetric!

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- $\{\gamma, \delta\}^{\uparrow} = \{\langle \mathsf{a}, 0.2\rangle, \langle \mathsf{b}, 0.3\rangle, \langle \mathsf{c}, 0.1\rangle, \langle \mathsf{d}, 0.2\rangle, \langle \mathsf{e}, 0.3\rangle\}$

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- $\{\langle \mathrm{a}, 0.3\rangle, \langle \mathrm{b}, 0.3\rangle, \langle \mathrm{c}, 0.2\rangle, \langle \mathrm{d}, 0.3\rangle, \langle \mathrm{e}, 0.5\rangle\}^{\downarrow} = \{\alpha, \beta\}$

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- or equivalently

$$f \leq \uparrow(X) \quad \text{iff} \quad X \subseteq \downarrow(f)$$

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- Bělohlávek, Sklenář, & Zacpal
    - crisply generated concepts

2
Generalized fuzzy approach

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- a natural question arise – how to unify these approaches?
- hence we try to find a common platform for them all

Types of fuzziness of subsets

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approach	*object subsets*	*attribute subsets*
classical	crisp	crisp
L-fuzzy	L-fuzzy	L-fuzzy
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generalized	D-fuzzy	C-fuzzy

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2a) if $d \in D$, $p \in P$, $X \subseteq C$ and $(\forall c \in X) c \otimes d \leq p$

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  then $\sup X \otimes d \leq p$

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  2a) if $d \in D$, $p \in P$, $X \subseteq C$ and $(\forall c \in X) c \otimes d \leq p$ then $\sup X \otimes d \leq p$
  2b) if $c \in C$, $p \in P$, $Y \subseteq D$ and $(\forall d \in Y) c \otimes d \leq p$ then $c \otimes \sup Y \leq p$

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  2b) if $c \in C$, $p \in P$, $Y \subseteq D$ and $(\forall d \in Y) c \otimes d \leq p$ then $c \otimes \sup Y \leq p$

- note that $\otimes$ need not be commutative!

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- $\uparrow : {}^{B}D \to {}^{A}C$:

$$\boxed{\uparrow(g)(a) = \sup\{c \in C : (\forall b \in B) c \otimes g(b) \leq R(a, b)\}}$$

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- $\uparrow$ and $\downarrow$ form a Galois connection:

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$$\boxed{\uparrow(g)(a) = \sup\{c \in C : (\forall b \in B) c \otimes g(b) \leq R(a,b)\}}$$

- $\downarrow : {}^{A}C \to {}^{B}D$:

$$\boxed{\downarrow(f)(b) = \sup\{d \in D : (\forall a \in A) f(a) \otimes d \leq R(a,b)\}}$$

- $\uparrow$ and $\downarrow$ form a Galois connection:
  - if $f_1, f_2 \in {}^{B}D$ and $f_1 \leq f_2$ then $\downarrow(f_1) \geq \downarrow(f_2)$
  - if $g_1, g_2 \in {}^{A}C$ and $g_1 \leq g_2$ then $\uparrow(g_1) \geq \uparrow(g_2)$
  - if $f \in {}^{B}D$ then $f \leq \uparrow(\downarrow(f))$
  - if $g \in {}^{A}C$ then $g \leq \downarrow(\uparrow(g))$

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- of course, in the classical and one-sided cases
  we have to use the canonical equivalency
  of subsets and their characteristic functions

3
Hedge approach

R. Bělohlávek, V. Vychodil (et al.)

Hedge

Hedge

- a (complete) residuated lattice $\langle L, \vee, \wedge, \otimes, \rightarrow, 0, 1\rangle$:
  - $x \otimes y \leq z$ iff $x \leq y \rightarrow z$
  - $\otimes$ – isotone in both their arguments
  - $\rightarrow$ – antitone in the first argument, isotone in the second one
  - $\otimes$ – commutative
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  - $x \otimes 1 = 1 \otimes x = x$
- a hedge [Hájek] – a function $*$ on L s. t.:
  - $1_L^* = 1_L$
  - $a^* \leq a$
  - $(a \rightarrow b)^* \leq a^* \rightarrow b^*$
  - $a^{**} = a^*$ (or equivalently $* \circ * = *$)

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- operations:
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$$\boxed{\uparrow(g)(a) = \sup\{c \in L : (\forall b \in B) c \otimes (g(b))^{*_B} \leq R(a,b)\}}$$

  - $\downarrow : {}^A L \to {}^B L$:

$$\boxed{\downarrow(f)(b) = \sup\{d \in L : (\forall a \in A)(f(a))^{*_A} \otimes d \leq R(a,b)\}}$$

A concept lattice with hedges (2/3)

- for arbitrary $h : U \to L$ define

$$\lfloor h \rfloor = \{\langle u, a\rangle \in U \times L : a \leq h(u)\}$$

- for arbitrary $H \subseteq U \times L$ define

$$\lceil H \rceil(u) = \bigvee \{a \in L : \langle u, a\rangle \in H\}$$

- for arbitrary $h : U \to L$ and $* : L \to L$ define

$$h^*(u) = (h(u))^*$$

- for arbitrary $H \subseteq U \times L$ and $* : L \to L$ define

$$H^* = \{\langle x, a^*\rangle : \langle x, a\rangle \in H\}$$

A concept lattice with hedges (3/3)

- $Y^{\curlyvee} = \lfloor \lceil Y \rceil^{\uparrow} \rfloor^{*_A}$
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- $R_{\langle \curlywedge, \curlyvee \rangle}$ is a classical set
- CLH(...) is isomorphic to the ordinary concept lattice $\mathrm{CL}(A \times *_A[L], B \times *_B[L], \curlywedge, \curlyvee, R_{\langle \curlywedge, \curlyvee \rangle})$

Relationship between these generalizations

- the lattices

$$\mathrm{GCL}(A, B, *_A[L], *_B[L], L, R, \otimes)$$

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$$\mathrm{CL}(A \times *_A[L], B \times *_B[L], \curlywedge, \curlyvee, R_{\langle \curlywedge, \curlyvee \rangle})$$

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  - if $g : B \to *_B[L]$, $f : A \to *_A[L]$ then

$$\phi(\langle g, f \rangle) = \langle \lfloor g \rfloor, \lfloor f \rfloor \rangle$$

  - if $S \subseteq B \times *_B[L]$, $T \subseteq A \times *_A[L]$ then

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- GCL(...) and CLH(...) are (canonically) isomorphic

3
Heterogeneous approach

joined work with my colleague Ondrej Krídlo
and my students L'. Antoni, B. Macek, and L. Pisková

Motivation

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- they bring this original idea into formal concept analysis and take one $\otimes$ for each object
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- moreover we diversify all what can be diversified

Heterogeneous formal context

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  which is isotone and left-continuous in both arguments
- R is a function from $A \times B$ s. t.
  for each $a \in A$ and $b \in B$,
  $R(a, b) \in P_{a,b}$

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- define $\langle g_1, f_1 \rangle \leq \langle g_2, f_2 \rangle$ iff $g_1 \leq g_2$ iff $f_1 \geq f_2$
- a heterogeneous concept lattice $\mathrm{HCL}(A, B, \mathcal{P}, R, \mathcal{C}, \mathcal{D}, \downarrow, \uparrow, \leq)$ – the poset of all such concepts ordered by $\leq$

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- all these follow that ξ is a wanted isomorphism

4
Galois-connection approach

J. Pócs (MÚ SAV, Košice)

Galois-connection formal context

- A and B are non-empty sets
- for each $a \in A$,
  C_a is a complete lattice
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- for each $b \in B$,
  $\phi_{a,b}$, $\psi_{a,b}$ are mappings s. t.
  $\phi_{a,b}$ and $\psi_{a,b}$ form a Galois connection between C_a and D_b

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  - $R(a, b) = 0$ (!)

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5
Future work

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Thank you for your attention

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