evropský sociální fond v ČR

EVROPSKÁ UNIE

MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

OP Vzdělávání pro konkurenceschopnost

INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Fuzzy stable models: definition, existence, uniqueness

Nicolás Madrid
Univ of Cádiz

Manuel Ojeda-Aciego
Univ of Málaga

Olomouc, March 5, 2012

On the context of our research

FEAST, supported by Spanish Ministry of Science

Foundations and Extensions of Answer Set Technology

1. Logical foundations of ASP
2. Extensions of its language in order to accommodate uncertainty and/or temporal and modal operators.
3. Reasoning methods aimed at specifying and verifying protocols of virtual organizations.

On the context of our research

FEAST, supported by Spanish Ministry of Science

- Generalized algebraic structures (hyperalgebra) to formalize approximate reasoning with incomplete, uncertain, and/or imprecise information.
- Multi-adjoint approach to formal concept analysis
- Qualitative reasoning
- Modal $\times$ temporal logics
- Equilibrium logic
- Other applied logics aimed at temporal, causal, fuzzy extensions of ASP.

Outline of the talk

- We recall the *fuzzy answer set semantics* for normal (and extended) residuated logic programs
- The, introduce conditions which ensure existence and uniqueness of fuzzy stable models.
- It is worth to stress the intensive use of results in real analysis and vector calculus.

Preliminaries

Definition

A *residuated lattice* is a tuple $(L, \leq, *, \leftarrow)$ such that:

1. $(L, \leq)$ is a complete bounded lattice, with top and bottom elements 1 and 0.
2. $(L, *, 1)$ is a commutative monoid with unit element 1.
3. $(*, \leftarrow)$ forms an adjoint pair, i.e. $z \leq (x \leftarrow y)$ iff $y * z \leq x \quad \forall x, y, z \in L$.

Definition

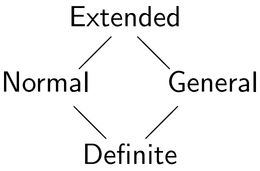
A negation operator, over $(L, \leq, *, \leftarrow)$, is any decreasing mapping $n\colon L \to L$ satisfying $n(0) = 1$ and $n(1) = 0$.

Types of programs according to negation

An extended residuated logic program $\mathbb{P}$ is said to be:

- *definite*, or *positive*, if it does not contain negation operators.
- *normal* if it does not contain **strong** negation.
- *general* if it does not contain **default** negation.

At least these two negations are needed to cope with information at a practical level.



Normal logic programs
Syntax

Definition

Given a residuated lattice with negation $(L, *, \leftarrow, \neg)$ an *normal residuated logic program* $\mathbb{P}$ is a set of weighted rules of the form

$$\langle p \leftarrow p_1 * \cdots * p_m * \neg p_{m+1} * \cdots * \neg p_n; \quad \vartheta \rangle$$

where ϑ is an element of L and $p, p_1, \ldots, p_n$ are propositional symbols.

Definition

A positive residuated logic program is a normal residuated logic program *without* default negation.

Normal logic programs
Semantics

Rules will be frequently denoted as $\langle p \leftarrow \mathcal{B}; \vartheta \rangle$. As usual, the formula $\mathcal{B}$ is called the *body* of the rule whereas p is called its *head*.

Definition

A fuzzy *L-interpretation* is a mapping $I \colon \Pi \to L$.

I satisfies a rule $\langle p \leftarrow \mathcal{B}; \quad \vartheta \rangle$ if and only if $I(\mathcal{B}) * \vartheta \leq I(p)$ or, equivalently, $\vartheta \leq I(p \leftarrow \mathcal{B})$.

I is a *model* of $\mathbb{P}$ if it satisfies all rules in $\mathbb{P}$.

Definition

Let $\mathbb{P}$ be a **positive** residuated logic program and let I be an interpretation. The immediate consequence operator of I wrt $\mathbb{P}$ is the interpretation defined as follows:

$$T_{\mathbb{P}}(I)(p) = \sup\{I(\mathcal{B}) * \vartheta : \langle p \leftarrow \mathcal{B}; \vartheta\rangle \in \mathbb{P}\}$$

Theorem

Every **positive** *program has a least model, which coincides with the least fix point of $T_{\mathbb{P}}(-)$. Hence, the least model of $\mathbb{P}$ is denoted by $lfp(T_{\mathbb{P}})$.*

The reduct of $\mathbb{P}$ w.r.t an L-interpretation.

Let $\mathbb{P}$ and I be a **normal** residuated logic program and a L-interpretation respectively, we will construct a new **positive** program $\mathbb{P}_I$ by substituting each rule in $\mathbb{P}$ of the form:

$$\langle p \leftarrow p_1 * \cdots * p_m * \neg p_{m+1} * \cdots * \neg p_n; \quad \vartheta\rangle$$

by the rule

$$\langle p \leftarrow p_1 * \cdots * p_m; \quad \neg I(p_{m+1}) * \cdots * \neg I(p_n) * \vartheta\rangle$$

Definition

The program $\mathbb{P}_I$ is called the reduct of $\mathbb{P}$ wrt the interpretation I.

Fuzzy Answer Sets

Definition

Let $\mathbb{P}$ be a general residuated logic program, an L-interpretation I is said to be a **fuzzy answer set** (or **fuzzy stable model**) of $\mathbb{P}$ iff I is the least model of $\mathbb{P}_I$.

Proposition

Let $\mathbb{P}$ be a positive residuated logic program. Then the unique fuzzy answer set of $\mathbb{P}$ is the least model of $\mathbb{P}$.

Theorem

Any fuzzy answer set of $\mathbb{P}$ is a **minimal** *model of $\mathbb{P}$.*

Inconsistent Programs Exist

Remark

Not every normal program has fuzzy answer sets.

For example, if we consider $L = [0,1]$ and the negation operator

$$n_{0.4}(x) = \begin{cases} 0 & \text{if} \quad x \geq 0.4 \\ 1 & \text{if} \quad x < 0.4 \end{cases}$$

the program $\mathbb{P} = \langle p \leftarrow \neg p \ ; \ 0.7 \rangle$ has not fuzzy answer sets.

Definition

A general residuated logic program $\mathbb{P}$ is **consistent** if there is a fuzzy answer set of $\mathbb{P}$. Otherwise, $\mathbb{P}$ is inconsistent.

Existence of fuzzy stable models

Case $L = [0, 1]$

The existence of stable models can be guaranteed by simply imposing conditions on the underlying residuated lattice:

Theorem

Let $\mathcal{L} \equiv ([0,1], \leq, *, \leftarrow, \neg)$ be a residuated lattice with negation. If $*$ and $\neg$ are continuous, then every finite normal program $\mathbb{P}$ defined over $\mathcal{L}$ has at least a fuzzy stable model.

Proof (sketch).

The idea is to apply Brouwer's fix-point theorem to the operator $\mathcal{R}(I) = \mathit{lfp}(T_{\mathbb{P}_I})$, which is proved to be continuous.
Operator $\mathcal{R}$ can be seen as a composition of two operators $\mathcal{F}_1(I) = \mathbb{P}_I$ and $\mathcal{F}_2(\mathbb{P}) = \mathit{lfp}(T_{\mathbb{P}})$.
Both $\mathcal{F}_1$ and $\mathcal{F}_2$ are proved to be continuous. $\square$

Existence of fuzzy stable models

Case $L = [0, 1]$

Key points of the proof

1. $\mathcal{F}_1$ can be interpreted as an operator from the set of $[0, 1]$-interpretations to the Euclidean space $[0, 1]^k$ where k is the number of rules in $\mathbb{P}$. Continuity follows from hypotheses.
2. $\mathcal{F}_2$ can be seen as a function from $[0, 1]^k$ to the set of interpretations, which can be seen as a finite composition of continuous operators. Thus, $\mathcal{F}_2$ is also continuous.
3. Hence we can apply Brouwer's fix-point theorem to $\mathcal{R}(I)$ and ensure that it has at least a fix-point.
4. To conclude, we only have to note that every fix-point of $\mathcal{R}(I)$ is actually a fuzzy answer set of $\mathbb{P}$.

Uniqueness of fuzzy stable models

Case $L = [0,1]$ and product t-norm

For finite programs, the operator $T_{\mathbb{P}}$ can be seen as a real function from $[0,1]^n$ to $[0,1]^n$ where $n = \#\Pi_{\mathbb{P}}$.

Lemma

Let $\mathbb{P}$ be a normal residuated logic program such that at most one rule which head is p appears in $\mathbb{P}$. Let I and J be two $[0,1]$-interpretations such that $J \leq I$, then:

$$\sum_{j=1}^{n} \left| \frac{\partial (T_{\mathbb{P}})_i}{\partial p_j}(J(p_1), \dots, J(p_n)) \right| \leq$$

$$\sum_{j=1}^{h} I(q_1) \cdot \ldots \cdot I(q_{j-1}) \cdot I(q_{j+1}) \cdot \ldots \cdot I(q_h) \cdot \vartheta \, +$$

$$+ \, (k-h)\big(I(q_1) \cdot \ldots \cdot I(q_h) \cdot \vartheta\big)$$

where $\langle p_i \leftarrow q_1 * \cdots * q_h * \neg q_{h+1} * \cdots * \neg q_k \; ; \vartheta \rangle$ is the rule in $\mathbb{P}$ whose head is p_i.

Uniqueness of fuzzy stable models

Case $L = [0,1]$ and product t-norm

Theorem

Let $\mathbb{P}$ be a finite normal residuated logic program, and p a propositional symbol occurring in $\mathbb{P}$. If, for every rule $\langle p \leftarrow q_1 * \cdots * q_h * \neg q_{h+1} * \cdots * \neg q_k \,; \vartheta \rangle \in \mathbb{P}$, the inequality

$$\left(\sum_j^h \vartheta_{q_1} \cdot \ldots \cdot \vartheta_{q_{j-1}} \cdot \vartheta_{q_{j+1}} \cdot \ldots \cdot \vartheta_{q_h} \cdot \vartheta \right) + (k-h)(\vartheta_{q_1} \cdot \ldots \cdot \vartheta_{q_h} \cdot \vartheta) < 1$$

holds, then there is only one stable model of $\mathbb{P}$, where we write $\vartheta_p = \max\{\vartheta_j \colon \langle p \leftarrow \mathcal{B} \,; \vartheta_j \rangle \in \mathbb{P}\}$

Uniqueness of fuzzy stable models

Case $L = [0,1]$ and product t-norm

Proof (Sketch).

- $T_{\mathbb{P}}$ is a contractive map with respect to the norm $||.||_\infty$ in a specific subset $A \subseteq [0,1]^n$, for this part we use that each element in A can be seen as one $[0,1]$-interpretation. Then, by applying Banach's fix-point theorem, $T_{\mathbb{P}}$ has only one fix-point in A.
- We show as well that, if $I \in [0,1]^n$ is a fix-point of $T_{\mathbb{P}}$, then I necessarily belongs to A. Therefore $T_{\mathbb{P}}$ has only one fix-point in $[0,1]^n$.
- Finally, taking into account that:
  - Every stable model of $\mathbb{P}$ is actually a fix-point of $T_{\mathbb{P}}$
  - We know that there exists at least one stable model of $\mathbb{P}$

then the unique fix-point of $T_{\mathbb{P}}$ is its unique stable model.

$\square$

Uniqueness of fuzzy stable models

Case $L = [0, 1]$ and product t-norm

Corollary

Consider a rule $\langle p \leftarrow q_1 * \cdots * q_h * \neg q_1 * \cdots * \neg q_k \; ; \vartheta \rangle$ in a finite normal residuated logic program $\mathbb{P}$. If the inequality

$$h \cdot (\max\{\vartheta_{q_1}, \ldots, \vartheta_{q_h}, \vartheta\})^h + k \cdot (\max\{\vartheta_{q_1}, \ldots, \vartheta_{q_h}, \vartheta\})^{h+1} < 1$$

holds then the rule satisfies the inequality required in the statement of the previous theorem.

Corollary

Let $\mathbb{P}$ be a finite normal residuated logic program. If every rule has a weight strictly less than 1 and at most one propositional symbol appears in the body of each rule, then $\mathbb{P}$ has only one stable model.

Computing the unique stable model of $\mathbb{P}$

Case $L = [0,1]$ and product t-norm

Theorem

Under the hypothesis of the theorem, the sequence $I_0 = I_\perp,\ I_{i+1} = T_\mathbb{P}(I_i)$ converges to the unique stable model of $\mathbb{P}$, although it may require ω many steps.

Remark

Banach's fix-point theorem provides a method to compute the unique stable model by computing the limit of the sequence:

$$I_{i+1} = T_\mathbb{P}(I_i)$$

where the initial I_0 can be any $[0,1]$-interpretation.

Conclusions and future work

- A lot of well-know results in real vector analysis can be used in the framework of fuzzy logic programming.
- For future work, we are planning a fuzzy approach to Here-and-There and Equilibrium logics.

References

- Madrid & Ojeda-Aciego. Measuring inconsistency in fuzzy answer set semantics. *IEEE Transactions on Fuzzy Systems*, 19(4):605-622, 2011.
- Madrid & Ojeda-Aciego. On the existence and unicity of stable models in normal residuated logic programs. *Intl J of Computer Mathematics*, 89(3):310-324, 2012.

Advertisements

Forthcoming related workshops in Málaga

CLA'12	**Concept Lattices and Applications**
	Oct 11-14, 2012, Fuengirola (Málaga)
Website	`http://www.matap.uma.es/cla2012/`
Deadline	June 29 (abstracts required on June 22)

Fuzzy stable models: definition, existence, uniqueness

Nicolás Madrid

Univ of Cádiz

Manuel Ojeda-Aciego

Univ of Málaga

Olomouc, March 5, 2012