Mathematical Fuzzy Logic for the Treatment of Vague Information

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Background

Preliminaries

The membership degrees of fuzzy sets can be seen as truth degrees of an infinite valued logic with $[0, 1]$ as truth degree set.

Thus the membership function μ_A becomes the truth degree function of a graded membership predicate "$\ldots \varepsilon A$": $\mu_A(x)$ is the truth degree of the formula $x \,\varepsilon\, A$.

In the same way, e.g. the intersection $A \cap B$ of fuzzy sets A, B can be seen as having the membership degree $\mu_{A\cap B}(x)$ corresponding to the truth degree of a formula $x \,\varepsilon\, A \text{ AND } x \,\varepsilon\, B$. Here AND refers to a suitably defined conjunction connective, defined according to the different possibilities one has to determine the membership degrees $\mu_{A\cap B}(x)$.

So one has a strong similarity with the theory of crisp sets presented in the language of classical first-order logic.

Hence one needs a suitable first-order logic, here an infinite valued one with truth degree set $[0, 1]$. And this logic has to have connectives as well as quantifiers.

Already ZADEH's famous 1965 paper on fuzzy sets (implicitly) offers different proposals to understand the above mentioned AND-operation:

as taking the minimum or as taking the usual, algebraic product.

These ideas have initiated an enormous development in the field of fuzzy logic in narrow sense, mainly since the mid-1990s.

As much as possible we restrict our considerations here to the propositional case.

Basic Infinite Valued Logics

If one looks for infinite valued logics of the kind which is needed as the underlying logic for a theory of fuzzy sets, one finds three main systems:

- the Łukasiewicz logic L developed during the 1920s;
- the Gödel logic G from 1932;
- the product logic Π introduced in 1996.

In their original presentations, these logics look rather different, regarding their propositional parts.

For the first order extensions, however, there is a unique strategy: one adds a universal and an existential quantifier such that quantified formulas get, respectively, as their truth degrees the infimum and the supremum of all the particular cases in the range of the quantifiers.

Here, however, we focus on a unified approach.

Gödel logic

This logic G has a conjunction $\wedge$ and a disjunction $\vee$ defined by the minimum and the maximum, respectively, of the truth degrees of the constituents:

$$u \wedge v = \min\{u, v\}\,, \qquad u \vee v = \max\{u, v\}\,.$$

The Gödel logic has also a negation $\sim$ and an implication $\to_{\mathsf{G}}$ defined by the truth degree functions

$$\sim u = \begin{cases} 1\,, & \text{if } u = 0; \\ 0\,, & \text{if } u > 0. \end{cases} \qquad u \to_{\mathsf{G}} v = \begin{cases} 1\,, & \text{if } u \leq v; \\ v\,, & \text{if } u > v. \end{cases}$$

Łukasiewicz logic

This logic L was originally designed with an implication $\to_{\mathsf{L}}$ and a negation $\neg$ as primitive connectives, given by

$$\neg u = 1 - u\ , \qquad u \to_{\mathsf{L}} v = \min\{1, 1 - u + v\}\ .$$

Further connectives are definable from these primitive ones:

$$\varphi \mathbin{\&} \psi =_{\mathrm{df}} \neg(\varphi \to_{\mathsf{L}} \neg\psi)\ , \qquad \varphi \;\underline{\vee}\; \psi =_{\mathrm{df}} \neg\varphi \to_{\mathsf{L}} \psi$$

they are a (strong) conjunction and a (strong) disjunction with truth degree functions

$$u \mathbin{\&} v = \max\{u + v - 1, 0\}\ , \qquad u \mathbin{\underline{\vee}} v = \min\{u + v, 1\}$$

The definitions

$$\varphi \wedge \psi =_{\mathrm{df}} \varphi \mathbin{\&} (\varphi \to_{\mathsf{L}} \psi) \qquad \varphi \vee \psi =_{\mathrm{df}} (\varphi \to_{\mathsf{L}} \psi) \to_{\mathsf{L}} \psi$$

give another (weak) conjunction $\wedge$ with truth degree function $\min$, and a further (weak) disjunction $\vee$ with truth degree function $\max$.

Product logic

The logic Π has a fundamental conjunction $\odot$ with the ordinary product of reals as its truth degree function, as well as an implication $\to_\Pi$ with truth degree function

$$u \to_\Pi v = \begin{cases} 1\,, & \text{if } \ u \leq v; \\ \frac{u}{v}\,, & \text{if } \ u < v. \end{cases}$$

Additionally it has a truth degree constant $\overline{0}$ to denote the truth degree zero.

A negation and a further conjunction are defined as

$$\sim \varphi =_{\mathrm{df}} \varphi \to_\Pi \overline{0}\,, \qquad \varphi \wedge \psi =_{\mathrm{df}} \varphi \odot (\varphi \to_\Pi \psi)\,.$$

Both these connectives coincide with the corresponding ones of the Gödel logic.

And also the disjunction $\vee$ of the Gödel logic becomes available, now via the definition

$$\varphi \vee \psi =_{\mathrm{df}} ((\varphi \to_\Pi \psi) \to_\Pi \psi) \wedge ((\psi \to_\Pi \varphi) \to_\Pi \varphi)\,.$$

Standard and Algebraic Semantics

These fundamental infinite valued logics have their **standard semantics** as explained: the real unit interval $[0, 1]$ as truth degree set, and the connectives (and quantifiers) as mentioned.

"As usual" one then can introduce for formulas φ their **validity in a model**, which in these logics means that φ has the truth degree 1 w.r.t. this model.

By a **model** we mean either—in the propositional case—an evaluation of the propositional variables by truth degrees, or—in the first-order case—a suitable interpretation of all the non-logical constants together with an assignment of the variables.

Logical validity of a formula φ means validity of φ in each model.

A set Σ of formulas **entails** a formula φ iff each model of Σ is also a model of φ.

In the standard terminology of many-valued logic this means that all the systems G, L, Π have the truth degree 1 as their only designated truth degree.

Besides these standard semantics, all three of these basic infinite valued logics have also **algebraic semantics** determined by suitable classes $\mathcal{K}$ of truth degree structures.

These structures have to have the same signature as the language $\mathcal{L}$ of the corresponding logic.

And they have to have—in the first-order case—suprema and infima for all those subsets which may appear as value sets of formulas. Particularly, hence, they have to be (partially) ordered, or at least preordered.

For each formula φ of the language $\mathcal{L}$ of the corresponding logic, for each such structure $\mathbf{A}$, and for each evaluation e which maps the set of propositional variables of $\mathcal{L}$ into the carrier of $\mathbf{A}$, one has to define a value $e(\varphi)$, and finally one has to define what it means that such a formula φ is **valid in** $\mathbf{A}$. Then a formula φ is **logically valid** w.r.t. the class $\mathcal{K}$ iff φ is valid in all structures from $\mathcal{K}$.

Such a class $\mathcal{K}$ of algebraic structures ideally forms a variety.

$\mathcal{K}$ is a **variety** iff **either:**

- $\mathcal{K}$ is equationally definable, **or:**
- $\mathcal{K}$ is closed under forming subalgebras, homomorphic images, and direct products.

Particular algebraic semantics:

classical logic	Boolean algebras
Gödel logic	prelinear Heyting algebras
Łukasiewicz logic	MV-algebras
Product logic	product algebras

Here:
Heyting algebra: relatively pseudo-complemented lattice

prelinearity: $(u \rightarrowtail v) \sqcup (v \rightarrowtail u) = \mathbf{1}\,.$

NB: All these structures—besides BA's—are abelian lattice ordered monoids with an additional residuation operation.

T-Norm Based Logics

T-Norm Based Connectives

There is a common uniform generalization of these three logics in which a conjunction connectives becomes a core role:

$\wedge$ in the system G, $\&$ in the system L, and $\odot$ in the system Π.

For this uniform generalization the core conjunction connective $\&$ has a truth degree function $\otimes$ which is a t-norm.

A **t-norm** T is a binary operation in $[0, 1]$ which is **associative, commutative, isotonic**, and has 1 as a **neutral** element.

Hence t-norms satisfy:

(T1) $x \otimes (y \otimes z) = (x \otimes y) \otimes z$,

(T2) $x \otimes y = y \otimes x$,

(T3) if $x \leq y$ then $x \otimes z \leq y \otimes z$,

(T4) $x \otimes 1 = x$.

Main examples:
– the minimum operation $u \wedge v$,
– the Łukasiewicz arithmetic conjunction $u \,\&\, v$,
– the ordinary product.

A t-norm $\otimes$ makes $[0, 1]$ into an **ordered abelian monoid**, i.e. an commutative semigroup with unit element 1.

And this ordered abelian monoid is even **integral**, i.e. its unit element is also the universal upper bound of the ordering. And it has 0 as an **annihilator**:

$$0 \otimes x \leq 0 \otimes 1 = 0\,.$$

For us: The t-norms are natural candidates for truth degree functions of conjunction connectives. And from such a t-norm one is able to derive (essentially) all the other truth degree functions for further connectives.

From a t-norm $\otimes$ one finds a truth degree function $\rightarrowtail$ for an **implication** connective via the **adjointness condition**

$$x \otimes z \leq y \quad \Longleftrightarrow \quad z \leq (x \rightarrowtail y)\,.$$

This adjointness condition is **equivalent** to the condition that $\otimes$ is left continuous in both arguments.

Equivalent to this adjointness condition are the direct definition

$$x \rightarrowtail y = \sup\{z \,|\, x \otimes z \leq y\}$$

of the **residuation operation** $\rightarrowtail$, as well as the **sup-preservation property**

$$\sup_{i \to \infty}(x_i \otimes y) = (\sup_{i \to \infty} x_i) \otimes y$$

for each $y \in [0,1]$ and each non-decreasing sequence $(x_i)_{i \to \infty}$ from $[0,1]$.

These residuation based implications are called **R-implications**.

One introduces a further unary operation $-$ by

$$-x =_{\mathrm{df}} x \rightarrowtail 0\,,$$

and considers this as the truth degree function of a **negation** connective.

This means to introduce into the formal language of the logic a truth degree constant $\overline{0}$.

Finally one looks for the **weak conjunction and disjunction** connectives $\wedge, \vee$.

It is usual to add only the min-conjunction $\wedge$, because then for each left continuous t-norm $\otimes$ and its residuated implication $\rightarrowtail$ one has

$$u \vee v = ((u \rightarrowtail v) \rightarrowtail v) \wedge ((v \rightarrowtail u) \rightarrowtail u)\,.$$

All together this gives algebraic structures in $[0,1]$ which have a left continuous t-norm $\otimes$ together with its residuation operation $\rightarrowtail$, with the minimum-operation $\wedge$, and the maximum operation $\vee$ as basic operations of such an algebraic structure, and with the particular truth degrees $0, 1$ as fixed objects (i.e. as nullary operations) of the structure.

Such an algebraic structure

$$\langle [0,1], \wedge, \vee, \otimes, \rightarrowtail, 0, 1 \rangle$$

is coined to be a **t-norm algebra**.

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Each left continuous t-norm T determines a T-based logic $\mathcal{L}_T$ with just this t-algebra as its characteristic matrix.

Digression: **Residuated Implications versus S-Implications**

In classical logic one has the logical equivalence of the formulas

$$\varphi \to \psi \qquad \text{and} \qquad \neg\varphi \vee \psi \qquad \text{and} \qquad \neg(\varphi \wedge \neg\psi)$$

Thus the introduction of the implication connective directly as residuation seems to be quite sophisticated, and perhaps unnecessarily complicated.

For an **arrow-free** approach, one could start by a definition of an implication connective either via a generalized disjunction or via a generalized conjunction. **But** one had to add in any case a generalized negation.

Such an the implication is often coined **S-implication**.

NB: This approach does not really become simpler as the former one because one needs to fix either, besides the basic t-norm, an additional negation, or one has to fix a negation together with a disjunction.

The main **disadvantage** of an S-implication approach is that one looses a natural strong **soundness** property for the **rule of detachment**, the inequality:

$$u \otimes (u \rightarrowtail v) \leq v\,.$$

From the adjointness condition one has this inequaliy immediately via

$$u \otimes (u \rightarrowtail v) \leq v \quad \text{iff} \quad (u \rightarrowtail v) \otimes u \leq v \quad \text{iff} \quad u \rightarrowtail v \leq u \rightarrowtail v\,.$$

A similar property is lacking in general for the approach via S-implications. Starting from a t-norm $\otimes$ and a negation $\rightharpoondown$, the corresponding inequality becomes

$$u \otimes \rightharpoondown (u \otimes \rightharpoondown v) \leq v\,.$$

But this fails already in the case that the negation $\rightharpoondown$ is the Gödel negation $\sim$:

For any $v > 0$ one then has $\sim v = 0$, hence $u \otimes \sim v = 0$, which means $\sim (u \otimes \sim v) = 1$ and $u \otimes \sim (u \otimes \sim v) = u$. Now choose $u > v$ to see that the crucial inequality fails.

End of digression

Problem: How to axiomatize the logic $\mathcal{L}_T$ for a given t-norm T ?

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Instead:

Hájek 1998: Axiomatize the common logic of all continuous t-norms.

Esteva/Godo 1999: Axiomatize the common logic of all left continuous continuous t-norms.

New Problem: What about suitable semantics ?

Solution: Consider appropriate algebraic semantics.

However the classes of all corresponding t-algebras do **not** form varieties – neither for the continuous nor for the left continuous t-norms.

These classes are not closed under direct products.

Way out:
Find “small” varieties which extend these classes of continuous—or: left continuous—t-norms.

Next Problem:
Is there an algebraic characterization of the continuity of a t-norm ?

Definition: A t-norm algebra $\langle [0,1], \wedge, \vee, \otimes, \rightarrowtail, 0, 1 \rangle$ is **divisible** iff one has for all $a, b \in [0,1]$:

$$a \wedge b = a \otimes (a \rightarrowtail b)\,.$$

Proposition: A t-norm algebra $\langle [0,1], \wedge, \vee, \otimes, \rightarrowtail, 0, 1 \rangle$ is divisible iff the t-norm $\otimes$ is continuous.

Suitable varieties:

The classes of all structures $\langle L, \wedge, \vee, \otimes, \rightarrowtail, 0, 1\rangle$ such that

- $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded lattice with lattice ordering $\leqslant$.
- $\langle L, \otimes, 1, \leqslant\rangle$ is a lattice ordered abelian monoid.
- $\langle \otimes, \rightarrowtail\rangle$ is an adjoint pair, i.e. satisfies the adjointness condition

$$x \otimes y \leqslant z \quad \Longleftrightarrow \quad x \leqslant (y \rightarrowtail z)\,.$$

- $\vee, \rightarrowtail$ satisfy the prelinearity condition: $\quad (x \rightarrowtail y) \vee (y \rightarrowtail x) = 1\,.$

And for the continuous case additionally

- $\langle L, \wedge, \vee, \otimes, \rightarrowtail, 0, 1\rangle$ is divisible: $\quad x \wedge y = x \otimes (x \rightarrowtail y)$.

NB:

These particular residuated lattices are called **BL-algebras** (for the continuous case, i.e. including the divisibility condition) or **MTL-algebras**, respectively.

Continuous T-Norms

The continuous t-norms are the best understood.

A t-norm is continuous iff it is continuous as a real function of two variables, or equivalently, iff it is continuous in each argument.

All continuous t-norms are ordinal sums of only three of them: the Łukasiewicz arithmetic t-norm, the ordinary product t-norm, and the minimum operation.

Definition: Let $([a_i, b_i])_{i \in I}$ be a countable family of non-overlapping proper subintervals of $[0, 1]$, let $(T_i)_{i \in I}$ be a family of t-norms, and let $(\varphi_i)_{i \in I}$ be a family of order isomorphisms from $[a_i, b_i]$ onto $[0, 1]$.
Then the **(generalized) ordinal sum** of the combined family $(([a_i, b_i], \boldsymbol{t}_i, \varphi_i))_{i \in I}$ the t-norm

$$T(u, v) = \begin{cases} {\varphi_k}^{-1}(\boldsymbol{t}_k(\varphi_k(u), \varphi_k(v)), & \text{if } u, v \in [a_k, b_k] \\ \min\{u, v\} & \text{otherwise.} \end{cases}$$

Visualization of the construction of an ordinal sum:

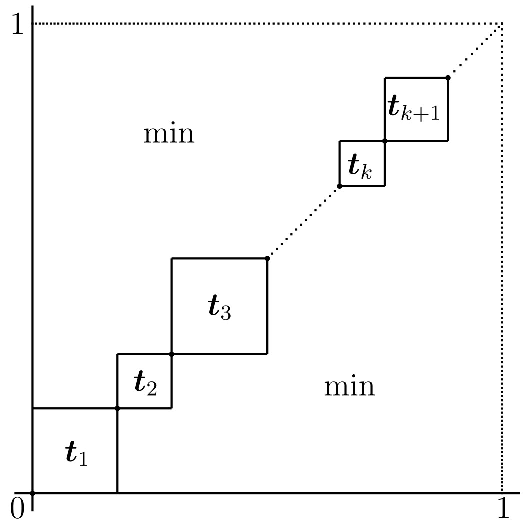

Figure 1: The basic construction of an ordinal sum

Theorem: Each continuous t-norm T is the (generalized) ordinal sum of (isomorphic) copies of the Łukasiewicz t-norm, the product t-norm, and the minimum t-norm.

An order isomorphic copy of the minimum t-norm is again the minimum operation.

Thus the whole construction of ordinal sums of t-norms even allows to assume that the summands are formed from t-norms different from the minimum t-norm.

All the endpoints a_i, b_i of the interval family $([a_i, b_i])_{i \in I}$ give **idempotents** of the resulting ordinal sum t-norm T:

$$T(a_i, a_i) = a_i \,, \quad T(b_i, b_i) = b_i \quad \text{for all } i \in I.$$

Conversely, if one knows all the idempotents of a given continuous t-norm T, i.e. all $u \in [0, 1]$ with $\boldsymbol{t}(u, u) = u$, then one is able to give a representation of T as an ordinal sum.

The Logic BL of Continuous T-Norms (Hájek 1998)

Axiom schemata:

(Ax$_{\mathsf{BL}}1$) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))\,,$

(Ax$_{\mathsf{BL}}2$) $\varphi \mathbin{\&} \psi \to \varphi\,,$

(Ax$_{\mathsf{BL}}3$) $\varphi \mathbin{\&} \psi \to \psi \mathbin{\&} \varphi\,,$

(Ax$_{\mathsf{BL}}4$) $(\varphi \to (\psi \to \chi)) \to (\varphi \mathbin{\&} \psi \to \chi)\,,$ importation

(Ax$_{\mathsf{BL}}5$) $(\varphi \mathbin{\&} \psi \to \chi) \to (\varphi \to (\psi \to \chi))\,,$ exportation

(Ax$_{\mathsf{BL}}6$) $\varphi \mathbin{\&} (\varphi \to \psi) \to \psi \mathbin{\&} (\psi \to \varphi)\,,$

(Ax$_{\mathsf{BL}}7$) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)\,,$ prelinearity

(Ax$_{\mathsf{BL}}8$) $\overline{0} \to \varphi\,.$

Inference rules:

Only the rule of detachment (modus ponens): $\dfrac{\varphi,\ \varphi \to \psi}{\psi}$

The language $\mathcal{L}_T$ of BL is extended by definitions of additional connectives:

$$\begin{aligned} \varphi \wedge \psi &=_{\mathrm{df}} \varphi \,\&\, (\varphi \to \psi)\,, \\ \varphi \vee \psi &=_{\mathrm{df}} ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)\,, \\ \neg\varphi &=_{\mathrm{df}} \varphi \to \overline{0}\,. \end{aligned}$$

These additional connectives $\wedge, \vee$ just have the lattice operations $\cap, \cup$ as their truth degree functions.

This axiomatic system BL is sound, i.e. derives only such formulas which are valid in all BL-algebras.

Corollary: The LINDENBAUM algebra of the axiomatic system BL is a BL-algebra.

Theorem [General Completeness, Hajek 1998]: A formula φ of the language $\mathcal{L}_T$ is derivable within the axiomatic system BL iff φ is valid in all BL-algebras.

The proof method yields that each BL-algebra is (isomorphic to) a subdirect product of linearly ordered BL-algebras, i.e. of BL-chains. Thus it allows a nice modification of the previous result.

Theorem [General Completeness; Version 2]: A formula φ of $\mathcal{L}_T$ is derivable within the axiomatic system BL iff φ is valid in all BL-chains.

But even more is provable and leads back to the starting point of the whole approach: the logical calculus BL characterizes just those formulas which hold true w.r.t. all divisible t-norm algebras.

Theorem [Standard Completeness, Cignoli/Esteva/Godo/Torrens 2000]: The class of all formula which are provable in the system BL coincides with the class of all formulas which are logically valid in all t-norm algebras with a continuous t-norm.

The main steps in the proof are to show that

(i) each BL-algebra is a subdirect product of subdirectly irreducible BL-chains, i.e. of linearly ordered BL-algebras which are not subdirect products of other BL-chains,

(ii) each subdirectly irreducible BL-chain can be embedded into the ordinal sum of some BL-chains which are either trivial one-element BL-chains, or linearly ordered MV-algebras, or linearly ordered product algebras,

(iii) each such ordinal summand is locally embedable into a t-norm based residuated lattice with a continuous t-norm.

Even more can be seen from this proof: the class of BL-algebras is the **smallest variety** which contains all the divisible t-norm algebras. The algebraic reason for this is that each variety may be generated from its subdirectly irreducible elements.

Another generalization of the Completeness Theorem is important too.

Theorem [Extended General Completeness, Hájek 1998]: For each finite set $\mathcal{C}$ of axiom schemata and any formula φ of $\mathcal{L}_T$ there are equivalent:

(i) φ is derivable within BL$(\mathcal{C})$;

(ii) φ is valid in all BL$(\mathcal{C})$-algebras;

(iii) φ is valid in all BL$(\mathcal{C})$-chains.

Here a **schematic extension** of BL is an extension by (finitely many) further axiom schemata. Denote such an extension by BL$(\mathcal{C})$. A BL$(\mathcal{C})$-algebra shall be a BL-algebra which is a model of $\mathcal{C}$.

The extension of these considerations to the first-order case is also given in Hajek's 1998 monograph *Metamathematics of Fuzzy Logic*.

The algebraic machinery allows even deeper insights. The study of such subvarieties of the variety of all BL-algebras which are generated by single t-norm algebras $\langle [0,1], \wedge, \vee, \otimes, \rightarrowtail, 0, 1 \rangle$ with a continuous t-norm $\otimes$ led to (finite) axiomatizations of those t-norm based logics which have a standard semantics determined just by this continuous t-norm algebra.

Theorem: Each t-norm based fuzzy logic $\mathcal{L}_T$ determined by a continuous t-norm T is a finite axiomatic extension of the basic fuzzy logic BL.

There is even an algorithm to determine these additional axiom schemata.

Source: F. Esteva, L. Godo, F. Montagna: *Equational characterization of the subvarieties of BL generated by t-norm algebras*. Studia Logica **76** (2004), 161–200.

NB: A similar result is lacking for left continuous t-norms.

The Logic MTL of Left Continuous T-Norms

Esteva/Godo guessed 1999 that one should arrive at the logic of left continuous t-norms if one starts from the logic of continuous t-norms and simply deletes the continuity, i.e. the divisibility condition.

For the algebraic semantics that means to switch from BL-algebras to **MTL-algebras**. MTL-algebras also form a variety.

The previous axiom system needs a suitable modification:

One has to delete the definition of the connective $\wedge$, because this definition essentially codes the divisibility condition.

The definition of the connective $\vee$ remains unchanged.

Thus one now considers the logic MTL characterized semantically by the class of all MTL-algebras.

Axiom system:

(Ax$_{\mathsf{MTL}}$1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))\,,$

(Ax$_{\mathsf{MTL}}$2) $\varphi \,\&\, \psi \to \varphi\,,$

(Ax$_{\mathsf{MTL}}$3) $\varphi \,\&\, \psi \to \psi \,\&\, \varphi\,,$

(Ax$_{\mathsf{MTL}}$4) $(\varphi \to (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)\,,$ importation

(Ax$_{\mathsf{MTL}}$5) $(\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))\,,$ exportation

(Ax$_{\mathsf{MTL}}$6) $\varphi \wedge \psi \to \varphi\,,$

(Ax$_{\mathsf{MTL}}$7) $\varphi \wedge \psi \to \psi \wedge \varphi\,,$

(Ax$_{\mathsf{MTL}}$8) $\varphi \,\&\, (\varphi \to \psi) \to \varphi \wedge \psi\,,$

(Ax$_{\mathsf{MTL}}$9) $\overline{0} \to \varphi\,,$

(Ax$_{\mathsf{MTL}}$10) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)\,,$ prelinearity

Inference rules:

Only the rule of detachment (modus ponens): $\dfrac{\varphi,\, \varphi \to \psi}{\psi}$

This axiomatization for MTL is sound, i.e. derives only such formulas which are valid in all MTL-algebras.

Corollary: The LINDENBAUM algebra of this logical calculus MTL is an MTL-algebra.

Theorem [General Completeness, Esteva/Godo 2001]: A formula φ is derivable within this axiomatic system MTL iff φ is valid in all MTL-algebras.

Again the proof method yields that each MTL-algebra is (isomorphic to) a subdirect product of MTL-chains.

Theorem [General Completeness; Version 2]: A formula φ is derivable within the axiomatic system MTL iff φ is valid in all MTL-chains.

Even more is provable.

Theorem [Standard Completeness, Jenei/Montagna 2002]: A formula φ is provable in the axiomatic system MTL iff φ is logically valid in all t-norm algebras with a **left continuous** t-norm.

This result again means, as the similar one for the logic of continuous t-norms, that the variety of all MTL-algebras is the **smallest** variety which contains all t-norm algebras with a left continuous t-norm.

Because the BL-algebras are the divisible MTL-algebras, one gets another adequate axiomatization of the basic t-norm logic BL if one extends the axiom system MTL with the additional axiom schema

$$\varphi \wedge \psi \to \varphi \,\&\, (\varphi \to \psi)\,.$$

The simplest way to prove that this implication is sufficient is to show that the inequality $x * (x \rightarrowtail y) \leq x \cap y$, which corresponds to the converse implication, holds true in each MTL-algebra.

Theorem [Extended General Completeness]: For each finite set $\mathcal{C}$ of axiom schemata and any formula φ of $\mathcal{L}_T$ the following are equivalent:

(i) φ is derivable within the axiom system MTL $+\,\mathcal{C}$;

(ii) φ is valid in all MTL($\mathcal{C}$)-algebras;

(iii) φ is valid in all MTL($\mathcal{C}$)-chains.

The extension to the first-order case is similar to the BL-case.

Some Extensions and Generalizations

The standard approach toward t-norm based logics has been modified in various ways. The main background ideas are the extension or the modification of the expressive power of these logical systems.

A first, quite fundamental addition to the standard vocabulary of the languages of t-norm based systems was proposed by M. Baaz from Vienna:

a unary propositional operator $\triangle$ which has for t-norm algebras the semantics

$$\triangle(x) = 1 \quad \text{for } x = 1\,, \qquad \triangle(x) = 0 \quad \text{for } x \neq 1\,.$$

This unary connective can be added to BL and to MTL via the additional axioms

$$\begin{array}{ll}
(\Delta 1) & \triangle\varphi \vee \neg\triangle\varphi\,, \\
(\Delta 2) & \triangle(\varphi \vee \varphi) \to (\triangle\varphi \vee \triangle\psi)\,, \\
(\Delta 3) & \triangle\varphi \to \varphi\,, \\
(\Delta 4) & \triangle\varphi \to \triangle\triangle\varphi\,, \\
(\Delta 5) & \triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)\,.
\end{array}$$

This addition leaves all the essential theoretical results, like correctness and completeness theorems, valid: of course w.r.t. suitably expanded algebraic structures.

A second extension discusses the addition of an **idempotent negation**, i.e. a negation which satisfies the double negation law, for those cases where the standard negation of the t-norm based system is not idempotent.

This is e.g. the case for the product logic as well as for all those t-norm logics based upon a t-norm which does not have zero-divisors.

Zero-divisors of a t-norm T are such reals $0 < u, v < 1$ for which $T(u, v) = 0$ holds.

A resulting system is e.g. coined IMTL.

A general approach is given in

Esteva/Godo/Hájek/Navara: Residuated fuzzy logic with an involutive negation, Archive for Mathematical Logic **39** (2000), 103–124.

Another development is devoted to a unified treatment of **different** t-norms and their related connectives within one logical system.

Particularly interesting is the join of the Łukasiewicz t-norm and the product t-norm.

The great advantage of this unification is that the Łukasiewicz t-norm T_{L} essentially allows to treat the addition, and that the product t-norm T_{P} adds the treatment of the usual product:
and this means that the elementary arithmetic (in the unit interval) can be "used" in this combined system.

This combination comes in two strongly related forms: ŁΠ and Ł$\Pi\frac{1}{2}$.

The distinction between both systems is that

- ŁΠ has both t-norms T_{L} and T_{P} together with their related R-implications and negations among their basic connectives
- and that Ł$\Pi\frac{1}{2}$ adds also a truth degree constant for the truth degree $\frac{1}{2}$.

Basic contributions came from P. Cintula.

A fourth type of genralization weakens the systems BL and MTL in such a way that one deletes the explicit reference to the truth degree constant $\overline{0}$ and considers the **falsity free** fragments of the previous systems.

From the algebraic point of view their characteristic structures become the **hoops**:

algebraic structures $H = \langle H, *, \Rightarrow, \mathbf{1} \rangle$ such that $\langle H, *, \mathbf{1} \rangle$ is an abelian monoid and that the further binary operation $\Rightarrow$ satisfies the equations

$$\begin{aligned} x \Rightarrow x &= \mathbf{1}\,, \\ x * (x \Rightarrow y) &= y * (y \Rightarrow x)\,, \\ (x * y) \Rightarrow z &= x \Rightarrow (y \Rightarrow z)\,. \end{aligned}$$

The definition

$$x \sqsubseteq y \ =_{\mathrm{def}} \ x \Rightarrow x = \mathbf{1}$$

provides an ordering $\sqsubseteq$ with universal upper bound $\mathbf{1}$ making $\langle H, *, \mathbf{1} \rangle$ an ordered monoid with the additional property that $*, \Rightarrow$ become an adjoint pair w.r.t. $\sqsubseteq$.

In particular, hoops with the additional property

$$x \Rightarrow (y \Rightarrow z) \ \sqsubseteq \ (y \Rightarrow (x \Rightarrow z)) \Rightarrow z$$

can in a natural way be generated from t-norm algebras with continuous t-norms:

Aglianò/Ferreirim/Montagna: Basic hoops: an algebraic study of continuous t-norms, Studia Logica **87** (2007), 73–98.

For this kind of algebraic semantics one can find adequate axiomatizations for corresponding **hoop logics** quite similar to the approaches for BL and MTL.

A basic paper is

Esteva/Godo/Hájek/Montagna: Hoops and fuzzy logic, Journ. Logic and Comput. **13** (2003), 531–555.

And a fifth stream discusses the generalization of the algebraic semantics from the case of abelian lattice ordered monoids with residuation to the case of non-commutative lattice ordered semigroups.

In this context one defines non-commutative BL-algebras or non-commutative MTL-algebras, and similarly defines non-commutative t-norms, also called pseudo-t-norms. And these considerations become combined with the design of an adequate axiomatization, with similar results as for BL and MTL.

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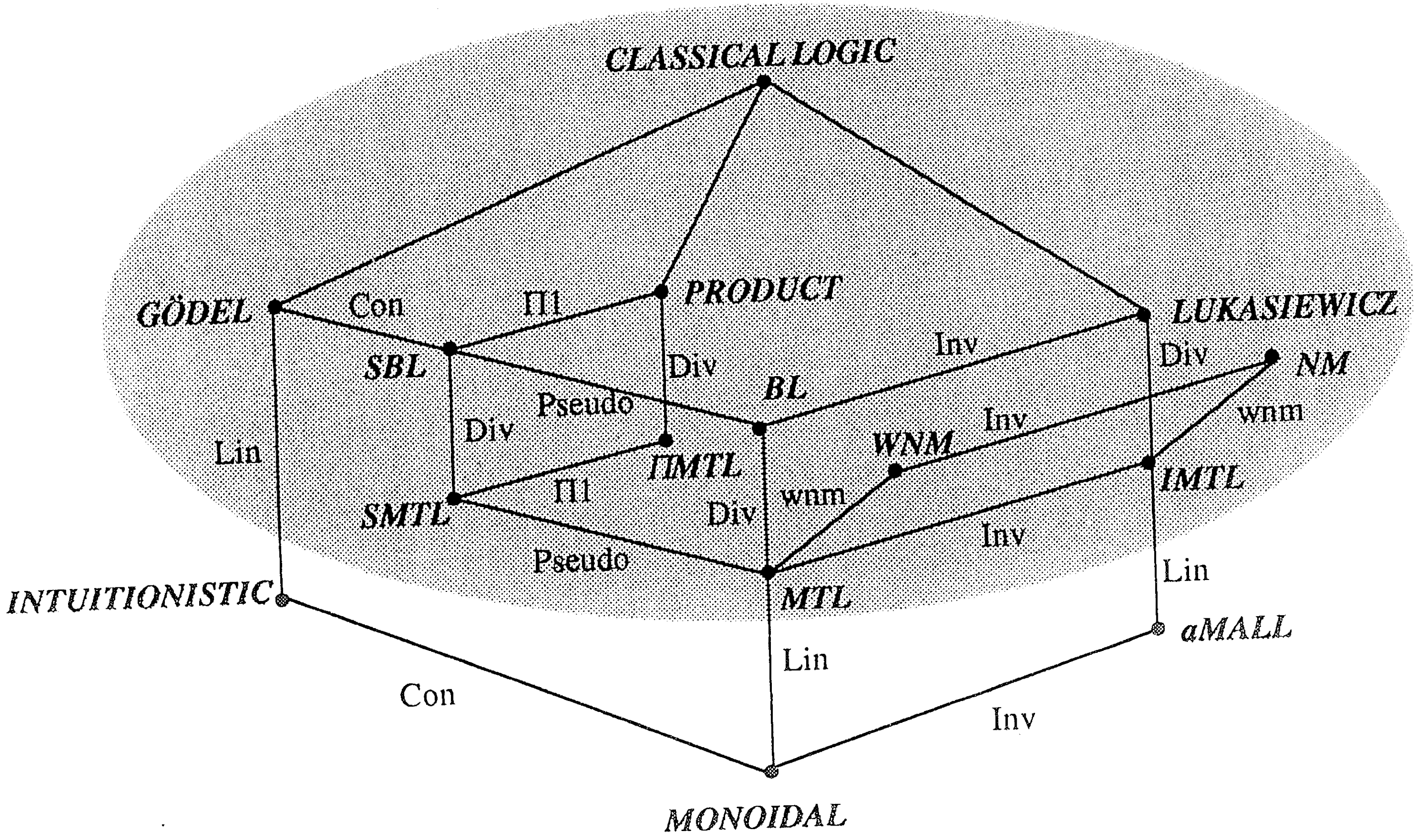
And finally it should be mentioned that Hájek even gives a common generalization of all of these generalized fuzzy logics, thus giving up divisibility, the falsity constant, and commutativity. The corresponding algebras are called **fleas**, or **flea algebras**, and the logic is the **flea logic** FlL.

There are examples of fleas on $(0, 1]$ **not satisfying** divisibility, nor commutativity, and having no least element.



Inv	$\neg\neg\varphi \to \varphi$	inv~~ersion~~ olution
Con	$\varphi \to \varphi \,\&\, \varphi$	contraction
Lin	$(\varphi \to \psi) \vee (\psi \to \varphi)$	prelinearity
wnm	$(\varphi \,\&\, \psi \to \underline{0}) \vee (\varphi \wedge \psi \to \varphi \,\&\, \psi)$	weak nilpotent minimum
Π1	$\varphi \wedge \neg\varphi \to \underline{0}$	
Div	$\varphi \wedge \psi \longleftrightarrow \varphi \,\&(\varphi \to \psi)$	divisibility
Pseudo	$\neg\neg\chi \to ((\varphi \,\&\, \chi \to \psi \,\&\, \chi) \to (\varphi \to \psi))$	

Pavelka Style Extensions

Fuzzy logics, also in their form as formalized logical systems, should be a (mathematical) tool for approximative reasoning.

Hence they should be able to deal with **graded inferences**.

But, all the previous systems of t-norm based logics have crisp notions of consequence, i.e. of entailment and of provability.

Problem: How to generalize these approaches to the case that one starts from **fuzzy sets of formulas**, and that one gets from them as consequence hulls again fuzzy sets of formulas

This problem was first treated by J. Pavelka 1979.
The basic monograph elaborating this approach is:

Novák/Perfilieva/Močkoř: Mathematical Principles of Fuzzy Logic, Kluwer Acad. Publ., Boston/Dordrecht/London 1999.

Remark: There is also a more algebraic approach toward consequence operations for the classical case, originating from Tarski:

consequence operations are **closure operations**.

NB: Closure operations are idempotent, increasing, and isotonic operations within a power set.

This type of approach has been generalized to closure operations in classes of fuzzy sets of formulas by G. Gerla and shall be discussed later on.

*

The Pavelka-style approach has to deal with **fuzzy sets** $\Sigma^\sim$ of formulas, i.e. besides formulas φ also their membership degrees $\Sigma^\sim(\varphi)$ in $\Sigma^\sim$.

These membership degrees are just truth degrees and shall form a residuated lattice $\mathbf{L} = \langle L, \cap, \cup, *, \rightarrowtail, 0, 1 \rangle$.

Semantic Version:

The Pavelka-style approach is quite natural for the entailment relationship.

An evaluation e is a **model** of a fuzzy set $\Sigma^{\sim}$ of formulas iff

$$\Sigma^{\sim}(\varphi) \leqslant e(\varphi)$$

holds for each formula φ.

Now the **entailment relation**, i.e. the semantic consequence hull of $\Sigma^{\sim}$ should be characterized by the membership degrees

$$\mathcal{C}^{\mathsf{sem}}(\Sigma^{\sim})(\psi) = \bigwedge \{ e(\psi) \mid e \text{ model of } \Sigma^{\sim} \} \, .$$

Be careful:

- consequences come with degrees,
- but the notion of model is crisp.

Syntactic Version:

For derivations one needs some logical calculus $\mathbb{K}$ which treats formulas together with truth degrees.

So the language of $\mathbb{K}$ should extend the language of the basic logical system by having also symbols for the truth degrees.

Further on we indicate these symbols by overlined letters like $\overline{a}, \overline{c}$. And we realize the common treatment of formulas and truth degrees by considering **evaluated formulas**, i.e. ordered pairs $(\overline{a}, \varphi)$ consisting of a truth degree symbol and a formula.

This trick transforms each fuzzy set $\Sigma^{\sim}$ of formulas into a (crisp) set of evaluated formulas, again denoted $\Sigma^{\sim}$.

So $\mathbb{K}$ has to derive evaluated formulas out of sets of evaluated formulas, of course using suitable axioms and rules of inference.

These axioms are usually only formulas φ which, however, are used in the derivations as the corresponding evaluated formulas $(\overline{1}, \varphi)$.

Derivations in $\mathbb{K}$ out of some set $\Sigma^{\sim}$ of evaluated formulas are finite sequences of evaluated formulas which either are axioms, or elements of (the support of) $\Sigma^{\sim}$, or result from former evaluated formulas by application of one of the inference rules.

Each $\mathbb{K}$-derivation of an evaluated formula $(\overline{a}, \varphi)$ counts as a derivation of φ **to the degree** $a \in L$. The **provability degree** of φ from $\Sigma^{\sim}$ in $\mathbb{K}$ is the **supremum** over all these degrees.

Thus the syntactic consequence hull of $\Sigma^{\sim}$ is the fuzzy set $\mathcal{C}^{\mathsf{syn}}_{\mathbb{K}}$ with membership function:

$$\mathcal{C}^{\mathsf{syn}}_{\mathbb{K}}(\Sigma^{\sim})(\psi) = \bigvee \{a \in L \mid \mathbb{K} \text{ derives } (\overline{a}, \psi) \text{ out of } \Sigma^{\sim}\}\,. \tag{1}$$

Despite the fact that $\mathbb{K}$ is a standard calculus, this is an **infinitary** notion of provability.

For the infinite-valued Łukasiewicz logic L this machinery works particularly well because it needs essentially the continuity of the residuation operation. In this case we can form a calculus $\mathbb{K}_{\mathsf{L}}$ which gives an **adequate** axiomatization for the graded notion of entailment in the sense that one has suitable soundness and completeness results.

The calculus $\mathbb{K}_{\mathsf{L}}$ has as axioms any axiom system of the infinite-valued Łukasiewicz logic L which provides an adequate axiomatization of L—together with the rule of detachment.

But $\mathbb{K}_{\mathsf{L}}$ replaces this standard **rule of detachment** by a generalized form for evaluated formulas:

$$\frac{(\overline{a}, \varphi) \qquad (\overline{c}, \varphi \to \psi)}{(\overline{a * c}, \psi)}$$

The calculus $\mathbb{K}_{\text{L}}$ is **sound** in the sense that the $\mathbb{K}_{\text{L}}$-provability of an evaluated formula $(\overline{a}, \varphi)$ says that $a \leq e(\varphi)$ holds for every valuation e, i.e. that the formula $\overline{a} \to \varphi$ is valid—however as a formula of an **extended** propositional language which has all the truth degree constants among its vocabulary.

Of course, the evaluations e have also to satisfy $e(\overline{a}) = a$.

The soundness and completeness results for $\mathbb{K}_{\text{L}}$ say that a **strong completeness theorem** holds true giving

$$\mathcal{C}^{\text{sem}}(\Sigma^{\sim})(\psi) \;=\; \mathcal{C}^{\text{syn}}_{\mathbb{K}_{\text{L}}}(\Sigma^{\sim})(\psi)$$

for each formula ψ and each fuzzy set $\Sigma^{\sim}$ of formulas.

In a more standard way one can extend the language of propositional L by truth degree constants for all degrees $a \in [0,1]$, and can read each evaluated formula $(\overline{a}, \varphi)$ as the formula $\overline{a} \to \varphi$.

Then a slight modification $\mathbb{K}_{\mathsf{L}}^{+}$ of the former calculus $\mathbb{K}_{\mathsf{L}}$ provides an adequate axiomatization for our graded notion of entailment.

One has, however, to add the **bookkeeping axioms**

$$\begin{aligned} (\overline{a} \,\&\, \overline{c}) \;&\equiv\; \overline{a * c}\,, \\ (\overline{a} \to \overline{c}) \;&\equiv\; \overline{a \rightarrowtail_{\mathsf{L}} c}\,. \end{aligned}$$

together with the **degree introduction rule**

$$\frac{(\overline{a}, \varphi)}{\overline{a} \to \varphi}\;.$$

A similar result is available which refers only to a notion of derivability over a countable language, and again does not use evaluated formulas.

The completeness result, for $\mathbb{K}_{\mathsf{L}}^{+}$ instead of $\mathbb{K}_{\mathsf{L}}$, becomes already provable if one adds truth degree constants **only** for all the **rationals** in $[0,1]$. (Hájek 1998)

The resulting logic is called **Rational Pavelka Logic**.

The Rational Pavelka Logic is a **conservative** extension of the (standard) infinite-valued Łukasiewicz logic L, i.e. $\mathbb{K}_{\mathsf{L}}^{+}$ proves only such **constant-free** formulas of the language with rational constants which are already L-provable.

Graded Consequences – Algebraically

Already in classical logic the syntactic as well as the semantic consequence relations, i.e. the provability as well as the entailment relations, are closure operators within the set of formulas.

Basic reference:
G. Gerla: Fuzzy Logic. Mathematical Tools for Approximate Reasoning. Kluwer Acad. Publ.: Dordrecht 2001.

The same holds true for the Pavelka style extensions:
the operators $\mathcal{C}^{\mathsf{sem}}$ and $\mathcal{C}^{\mathsf{syn}}$ are generalized closure operators.

Gerla's context is that of L-fuzzy sets, with $\mathbf{L} = \langle L, \leqslant \rangle$ an arbitrary complete lattice.

A **closure operator in $\mathbf{L}$** is a mapping $J : L \to L$ satisfying

$$\begin{aligned} & x \leqslant J(x)\,, && \text{(increasingness)} \\ & x \leqslant y \Rightarrow J(x) \leqslant J(y)\,, && \text{(isotonicity)} \\ & J(J(x) = J(x)\,. && \text{(idempotency)} \end{aligned}$$

And a **closure system** in $\mathbf{L}$ is a subclass $C \subseteq L$ which is closed under arbitrary lattice meets.

For mathematical **fuzzy logics** such closure operators and closure systems are considered in the lattice $\mathcal{F}_L(\mathbb{F})$ of all fuzzy subsets of the set $\mathbb{F}$ of formulas of some suitable formalized language.

Syntactic Version:

An **abstract fuzzy deduction system** is an ordered pair $\mathcal{D} = (\mathcal{F}_L(\mathbb{F}), D)$ determined by a closure operator D in the lattice $\mathcal{F}_L(\mathbb{F})$.

And the **fuzzy theories** T of such an abstract fuzzy deduction system, also called $\mathcal{D}$-theories, are the **fixed points** of D:

$$T = D(T)\,,$$

i.e. the **deductively closed** fuzzy sets of formulas.

Semantic Version:

An **abstract fuzzy semantics** $\mathcal{M}$ is nothing but a class of elements of the lattice $\mathcal{F}_L(\mathbb{F})$, i.e. a class of fuzzy sets of formulas.

These fuzzy sets of formulas are called **models**.

The only restriction is that the universal set over $\mathbb{F}$, i.e. the fuzzy subset of $\mathbb{F}$ which has always membership degree one, is not allowed as a model.

The background idea here is that, for each standard interpretation $\mathfrak{A}$ (in the sense of many-valued logic – including an evaluation of the individual variables) for the formulas of $\mathbb{F}$, a model M is determined as the fuzzy set which has for each formula $\varphi \in \mathbb{F}$ the truth degree of φ in $\mathfrak{A}$ as membership degree.

Accordingly the satisfaction relation $\models_{\mathcal{M}}$ coincides with inclusion: for models $M \in \mathcal{M}$ and fuzzy sets Σ of formulas one has:

$$M \models_{\mathcal{M}} \Sigma \quad\Leftrightarrow\quad \Sigma \subseteq M \,.$$

In this setting, one has a semantic and a syntactic consequence operator, both being closure operators, i.e. one has for each fuzzy set Σ of formulas from $\mathbb{F}$ a semantic as well as a syntactic consequence hull, given by

$$\mathcal{C}^{\mathsf{sem}}(\Sigma) = \bigcap \{ M \in \mathcal{M} \mid M \models_{\mathcal{M}} \Sigma \} \,, \qquad \mathcal{C}^{\mathsf{syn}}(\Sigma) = D(\Sigma) \,.$$

Similar to the classical case one has $\mathcal{C}^{\mathsf{sem}}(M) = M$ for each model $M \in \mathcal{M}$, i.e. each such model provides a $\mathcal{C}^{\mathsf{sem}}$-theory.

However, a general completeness theorem is not available.

One needs instead, in search for a completeness result, specifications which restrict the full generality of this approach, and lead mainly back to situations which have been discussed previously.

Applications of Mathematical Fuzzy Logics

Fuzzy Set Theory

The application to fuzzy set theory has been one of the motivations behind the development of mathematical fuzzy logics.

Indeed, a fuzzy set theory in the form of an axiomatic theory has been designed and developed by Hájek and Hanikova. They use the (first-order) logic BL of continuous t-norms, extended with the $\triangle$-operator, to develop a **ZF-like axiomatization** for fuzzy set theory together with a kind of **standard model** constructed in the style of Boolean valued models for (standard) set theory.

Axioms: Suitable versions of the axioms of extensionality, pairing, union, powerset, $\in$-induction (i.e. foundation), separation, collection (i.e. comprehension), and infinity,

together with an axiom stating the existence of the **support** of each fuzzy set.

The standard model for this theory is formed w.r.t. some complete BL-chain $\mathbf{L} = \langle L, \wedge, \vee, *, \to, 0, 1 \rangle$ and given by the transfinite hierarchy

$$V_0^{\mathbf{L}} = \emptyset\,, \qquad V_{\alpha+1}^{\mathbf{L}} = \left\{ f \in {}^{\mathsf{dom}(u)}L \mid \mathsf{dom}(u) \subseteq V_\alpha^{\mathbf{L}} \right\},$$

with unions at limit stages.

The primitive predicates $\in, \subseteqq, =$ are interpreted using as definitions for their truth degrees $[\![\ldots]\!]$:

$$\begin{aligned}
[\![x \in y]\!] \ &= \ \bigcup_{u \in \mathsf{dom}(y)} \left([\![u = x]\!] * y(u) \right), \\
[\![x \subseteqq y]\!] \ &= \ \bigcap_{u \in \mathsf{dom}(x)} \left(x(u) \Rightarrow [\![u \in y]\!] \right), \\
[\![x = y]\!] \ &= \ \triangle [\![x \subseteqq y]\!] * \triangle [\![y \subseteqq x]\!] \,.
\end{aligned}$$

The last condition forces the equality to be crisp.

Besides this "global" approach toward a generalization of the cumulative set universe for fuzzy sets, there is also a more "local" one which aims to give a unified treatment of a theory of fuzzy subsets of a given universe of discourse,

i.e. which–in a suitable sense–restricts the considerations to the first level(s) of the transfinite hierarchy of fuzzy sets of higher levels.

One version of this approach (Behounek/Cintula 2005) uses the (first-order) fuzzy logic $Ł\Pi_\triangle$ as basic logical system. More specifically: a two-sorted version with one sort of variables for objects of the universe of discourse and the other sort for fuzzy sets, and with the primitive predicates $\in, =$.

The advantage of this choice is that this logic:
(i) is well understood,
(ii) has sufficiently high expressive power such that former approaches, which used a mixture of object and metalanguage considerations, can be given in a uniform way.

So one can e.g. express the **comprehension axiom** by the schema

$$\exists X \triangle \forall x (x \in X \leftrightarrow \varphi(x))$$

which has $\varphi(x)$ as an arbitrary formula of the language (not containing the set variable X free).

And one can express the **axiom of extensionality** by

$$\forall x \triangle (x \in X \leftrightarrow x \in Y) \to X = Y \, .$$

This allows to denote fuzzy sets by class terms with the guiding principle

$$a \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(a) \, .$$

To guarantee the existence of fuzzy sets which are not crisp ones, one may either start with the logic Ł$\Pi\frac{1}{2}$ or add a specific **axiom of fuzziness** reading

$$\exists X \exists x (x \in X \leftrightarrow \neg_{\mathsf{L}} (x \in X)) \, .$$

Some examples shall illustrate the expressive power of this language:

$$\begin{array}{lll}
\{x \mid \neg_{\Pi}\neg_{\mathsf{L}}(x \in X)\} & \text{defines} & \text{the kernel of } X\,, \\
\{x \mid \neg_{\Pi}\neg_{\Pi}(x \in X)\} & \text{defines} & \text{the support of } X\,, \\
\{x \mid \triangle(\overline{\alpha} \to x \in X)\} & \text{defines} & \text{the (closed) } \alpha\text{-cut of } X\,,
\end{array}$$

of course with $\overline{\alpha}$ as truth degree constant to denote the truth degree α.

A further expansion of the language with additional sorts of variables allows to develop a machinery to discuss also fuzzy sets of higher level, and finally also a kind of fuzzy type theory and higher order fuzzy logics. Actually this is work in progress.

As large parts of standard mathematics can be developed in the realm of set theory, such an approach offers the possibility to do **fuzzy mathematics** in the style of “elementary‘” mathematical theories within the framework of a mathematical fuzzy logic, instead of the framework of classical logic.

Behounek is actually the main proponent of such approaches toward fuzzy mathematics.

By the way: another kind of fuzzy type theory has been developed by Novák and is applied e.g. in approaches toward natural language modeling.

Non-monotonic Fuzzy Reasoning

One of the core areas for the application of logic in computer science is artificial intelligence. And inside AI, non-monotonic reasoning has a prominent position.

Problem: Can the basic ideas of non-monotonic inference be generalized from the crisp to the fuzzy case?

This means e.g. the cases in which either the knowledge comes with degrees of vagueness, or of confidence, or in which the defaults are accepted only to some degrees.

A first idea was offered 1996 by Lehmke/Thiele. They generalized the circumscription approach in a straightforward way from classical logic to the infinite-valued Łukasiewicz logic, and give some basic properties of the non-monotonic inference operator defined via minimal models.

Another idea, based upon the abstract approach toward fuzzy logic via generalized closure operators, has more recently been offered by Richter and by Booth 2002/05.

They consider for an abstract fuzzy semantics $\mathcal{M}$ the **model class of** a fuzzy set Σ of formulas as

$$\mathbf{mod}_{\mathcal{M}}(\Sigma) = \{M \in \mathcal{M} \mid M \models_{\mathcal{M}} \Sigma\}\,,$$

and define the **theory of** a class $\mathbb{K} \subseteq \mathcal{M}$ of models as

$$\mathbf{th}(\mathbb{K}) = \bigcup\{u \in \mathcal{F}_L(\mathbb{F}) \mid M \models u \text{ for all } M \in \mathbb{K}\}$$

Hence one has

$$\mathbf{th}(\mathbb{K}) = \bigcup\Big\{u \in \mathcal{F}_L(\mathbb{F}) \mid u \subseteq \bigcap \mathbb{K}\Big\} = \bigcap \mathbb{K}\,.$$

For each fuzzy set $\Sigma \in \mathcal{F}_L(\mathbb{F})$ of formulas this means

$$\mathcal{C}^{\mathsf{sem}}(\Sigma) = \mathbf{th}(\mathbf{mod}_{\mathcal{M}}(\Sigma))\,,$$

thus $\mathbf{th}(\mathbf{mod}_{\mathcal{M}}(\Sigma))$ is a $\mathcal{C}^{\mathsf{sem}}$-theory.

This allows to adapt the model theoretic method of non-monotonic inference to connect with sets Σ of formulas as their **non-monotonic inference hull** $\mathrm{C}_\sim(\Sigma)$ the theory of a subclass $\Phi(\mathbf{mod}(\Sigma))$ of the class $\mathbf{mod}(\Sigma)$ of all models of Σ:

$$\mathbf{C}_\sim(\Sigma) = \mathbf{th}(\Phi(\mathbf{mod}(\Sigma)))\,.$$

Usually these subclasses are the classes of all normal or of all minimal models.

In this generalized setting one can prove quite similar theoretical results as in the crisp case.

Also another tool from non-monotonic reasoning has a natural generalization to a fuzzy setting: **Poole systems**.

A crisp **Poole system** P is determined by a pair (D, C) of sets of sentences understood as the relevant **defaults** and **constraints**.

For each set Σ of formulas and a suitably chosen closure operator $\mathbf{C}$ it defines a class $\mathbf{E}_P$ of extensions by

$$\mathbf{E}_P(\Sigma) = \{\mathbf{C}(\Sigma \cup D_m) \mid D_m \subseteq D \text{ maximal w.r.t. consistency of } \Sigma \cup C \cup D_m\}\,,$$

and an inference operator $\mathcal{C}_P$ by

$$\mathcal{C}_P(\Sigma) = \bigcap \mathbf{E}_P(\Sigma)\,.$$

All these definitions allow, in the abstract setting of generalized closure operators, a natural extension to the case of fuzzy sets of defaults and constraints.

Even a more practical application becomes available: **fuzzy belief revision**.

A **fuzzy belief base** B is just a fuzzy set of formulas $B \in \mathcal{F}_L(\mathbb{F})$.

The **revision information** (φ/a), understood as the fuzzy singleton of φ with membership degree a, tells that a “new” formula φ should be integrated with degree a.

As in the **AGM framework** of Alchourron/Gärdenfors/Makinson (1985) for the crisp case this may happen in the following steps:

1. Form the family $B \perp (\varphi/a)$ of all maximal $X \in \mathcal{F}_L(B)$ consistent with (φ/a).
2. Select a subset $\gamma(B \perp (\varphi/a)) \subseteq B \perp (\varphi/a)$ and form its meet.
3. Add the revision information to get the revised belief base
$$B \star (\varphi/a) = \bigcap \gamma(B \perp (\varphi/a)) \cup (\varphi/a)\,.$$

A similar **revision of fuzzy theories** is not as straightforward as in the crisp case, but can also be handled sufficiently well with some extra care regarding the moment for taking (deductive) closures.

Some Remarks
on
Fuzzy Control

A control function, in general, is a mapping F from an input space $\mathbb{X}$ into an output space $\mathbb{Y}$.

In the case of **fuzzy control** such a “fuzzy” control function should be a mapping from the **class of fuzzy subsets** of the input space $\mathbb{X}$ into the class of fuzzy subsets of the output space $\mathbb{Y}$.

Zadeh’s idea of a fuzzy control approach via a list of **linguistic control rules**

$$\text{IF } \alpha \text{ is } A_i, \text{ THEN } \beta \text{ is } B_i, \quad i = 1, \ldots, n\,,$$

here for simplicity with one input variable α and one output variable β only, was to **extract** from such a list a fuzzy relation R, and to apply to this fuzzy relation and any fuzzy input A the **compositional rule of inference** to get a corresponding fuzzy output as

$$R''A = A \circ R = \{y \mid \exists x(A(x) \,\&\, R(x,y))\}\,,$$

which means for the membership degrees

$$A \circ R(y) = \sup_{x \in \mathbb{X}} T(A(x), R(x,y))\,.$$

One of the mathematical ways to understand this approach is its transformation the list of linguistic control rules into a system of fuzzy relation equations

$$B_i = A_i \circ R = \{y \mid \exists x(A_i(x) \mathbin{\&} R(x,y))\}\,, \quad i = 1, \ldots, n\,,$$

with an unknown fuzzy relation R, and to ask for solutions of this system.

Often this problem is treated with set theoretic means from fuzzy set theory. And often one is also satisfied with “approximate” solutions, or better: with **pseudo-solutions** of the problem, and discusses particularly two of them:

$$\begin{aligned} R_S &= \bigcap_{i=1}^{n} \{(x,y) \mid x \in A_i \rightarrow y \in B_i\}\,, \\ R_{MA} &= \bigcup_{i=1}^{n} \{(x,y) \mid x \in A_i \mathbin{\&}\ y \in B_i\}\,, \end{aligned}$$

resulting out of approaches by Sanchez and by Mamdani/Assilian.

Discussing Relation Equations via BL-provability

Here we will discuss an embedding of these considerations into provability questions inside the mathematical fuzzy logic BL.

What we have to consider now are two groups of formulas $\varphi_1(x), \ldots, \varphi_m(x)$, and $\psi_1(y), \ldots, \psi_m(y)$, with one free variable each, together with some fixed (consistent) **BL-theory** T.

The problem is to find a formula $\varrho(x, y)$ with two free variables such that the provability relationships

$$T \vdash \exists x \bigl(\varphi_i(x) \,\&\, \varrho(x, y)\bigr) \leftrightarrow \psi_i(y) \qquad \text{for all} \quad i = 1, \ldots, n$$

hold.

If such a formula $\varrho(x, y)$ exists then we say that our system of fuzzy relation equations is **solvable in** T and $\varrho(x, y)$ is its solution.

To discuss the particular relations R_S and R_{MA}, we introduce the formulas

$$\varrho_S(x,y) \;:=\; \bigwedge_{j=1}^{m} \big(\varphi_j(x) \to \psi_j(y)\big)\,, \qquad \varrho_{MA}(x,y) = \bigvee_{i=1}^{m} \big(\varphi_i(x) \,\&\, \psi_i(y)\big)\,,$$

which characterize just these fuzzy relations.

Lemma: The following is provable in every theory T of the logic BL:

$$T \;\vdash\; \exists x \big(\varphi_i(x) \,\&\, \varrho_S(x,y)\big) \to \psi_i(y)\,.$$

NB: This follows easily from the BL-provability of the formulas $\varphi \,\&\, (\varphi \to \psi) \to \psi$ and $\varphi \,\&\, (\psi \,\&\, \chi) \to \varphi \,\&\, \psi$.

In the standard terminology, this result just says that one **always** has

$$A_i \circ R_S \subseteq B_i\,,$$

i.e. the Sanchez pseudo-solution has the **subset property** and thus is a lower bound for solutions.

The following result presents in a purely syntactical way the well known fundamental result on the solvability of systems of fuzzy relation equations:

- that our system of relation equations is solvable iff R_S is a solution;
- and in this case R_S is the maximal solution.

Theorem: The provability relationships

$$T \vdash \exists x \big(\varphi_i(x) \,\&\, \varrho(x,y) \big) \leftrightarrow \psi_i(y) \qquad \text{for all} \quad i = 1, \dots, n$$

hold true in T iff the following provability relationships hold true:

$$T \ \vdash \ \exists x \big(\varphi_i(x) \,\&\, \varrho_S(x,y)) \big) \leftrightarrow \psi_i(y) \qquad \text{for all} \quad i = 1, \dots, n \,.$$

And under these assumptions one also has

$$T \vdash \varrho(x,y) \to \varrho_S(x,y) \,.$$

Another well known result is that the Mamdani/Assilian pseudosolution has the **superset** property, i.e. is a lower bound for solutions of our system of fuzzy relation equations, which means

$$A_i \circ R_S \supseteq B_i \, , \qquad \text{for all } i = 1, \ldots, n.$$

In our logical rewriting the result gets the form:

Lemma: Assume $T \vdash \exists x \varphi_i(x)$ for all $i = 1, \ldots, m$. Then

$$T \vdash \psi_i(y) \to \exists x \big(\varphi_i(x) \mathbin{\&} \varrho_{MA}(x, y) \big) \, .$$

Finally we take a look at a necessary and sufficient condition that the Mamdani-Assilian fuzzy relation R_{MA} is a solution of our system of fuzzy relation equations, given by Klawonn 2000.

This is the inequality

$$\bigvee_{x\in\mathbb{X}} T(A_i(x), A_j(x)) \leq \bigwedge_{y\in\mathbb{Y}} Eq_T(B_i(y), B_j(y))$$

for the R-biimplication function

$$Eq_T(u,v) = (u \rightarrowtail v) \wedge (v \rightarrowtail u)\,.$$

This can again be proved inside BL purely syntactically.

Theorem: Suppose $T \vdash \exists x \varphi_i(x)$ for all $i = 1, \dots, m$. Then the provability relationships

$$T \vdash \exists x\big(\varphi_i(x) \,\&\, \varrho_{MA}(x,y)\big) \leftrightarrow \psi_i(y) \qquad \text{for all} \quad i = 1, \dots, n$$

are satisfied iff

$$T \vdash \exists x(\varphi_i(x) \,\&\, \varphi_j(x)) \to \forall y(\psi_i(y) \leftrightarrow \psi_j(y)), \quad \text{for all} \quad i,j = 1, \dots, m.$$

Fuzzy Control as Interpolation

Independent of any consideration on t-norm based logics we should take a **fresh look** at the fuzzy control approach.

The starting point of fuzzy control is an **incomplete and fuzzy** description of a **control function** Φ from an input space $\mathbb{X}$ to an output space $\mathbb{Y}$ via a list

$$\mathcal{D} = (\langle A_i, B_i \rangle)_{1 \le i \le n} \,,$$

usually presented as a list of **linguistic control** rules:

$$\text{IF} \;\; x \text{ is } A_i \;\; \text{THEN} \;\; y \text{ is } B_i, \qquad i = 1, \dots, n \,.$$

The main **mathematical problem** of fuzzy control then is the **interpolation problem** to find a function $\Phi^* : \mathcal{F}(\mathbb{X}) \to \mathcal{F}(\mathbb{Y})$ which interpolates these data:

$$\Phi^*(A_i) = B_i \qquad i = 1, \dots, n \,,$$

giving thus a **fuzzy representation** for the control function Φ.

The standard approach is to look for an interpolating function which should be **uniformly** (and globally) defined over $\mathcal{F}(\mathbf{X})$ using the data list $\mathcal{D}$.

Additionally, the search for a solution has to happen in a **restricted** class $\mathcal{IF}$ of **interpolating functions**.

This is the reason why within such a class $\mathcal{IF}$ of interpolating functions the interpolation problem might become unsolvable.

And the class of interpolating functions proposed by Zadeh has been the class of all **relation representable** functions

$$\Phi_R : A \mapsto A \circ R\,.$$

As is well known, this class is to small to solve every one of the interpolation problems presented by data lists $\mathcal{D}$.

As a way out one often chooses the methodology to consider some **approximate solution** of that interpolation problem.

In two different ways this has been exemplified by the approaches of Mamdani/Assilian characterized by

$$\Psi^*_{MA}(A)(y) = \bigvee_{x \in \mathbf{X}} \big(A(x) * R_{MA}(x,y) \big) \, ,$$

and by Holmblad/Ostergaard characterized by

$$\Psi^*_{HO}(A)(y) \; = \; \bigcup_{i=1}^{n} (\mathsf{hgt}\,(A \cap A_i) \cdot B_i(y)) \, .$$

The Mamdani/Assilian approach offers an interpolating function from the class of relation representable functions.

The Holmblad/Ostergaard approach interpolates with another class of functions.

Problem: What are reasonable classes of interpolation functions here?

Ways toward a solution:

- Look at the Mamdani/Assilian and Holmblad/Ostergaard approaches as particular cases of FATI and FITA strategies, respectively—and vary the involved aggregation operations. (Gottwald)
- Consider interpolation functions which are superpositions of simple functions. (Perfilieva)
- Consider interpolation functions which have simple definitions, e.g. via normal forms of logic formulas. (Perfilieva)
- Make a more heuristic interpolation of the linguistic rules which are “neighbors”. (Koszy)

Subproblem: Are there suitable **localizations** of the interpolation process?

In general however:

The interpolation problem is unsolved.