Aggregation Functions
Part I.

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Contents

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Basic notions

$$I = [a, b] \subseteq [-\infty, \infty]$$

$$\mathbf{x} = (x_1, \ldots, x_n)$$

Definition 1

(i) An ***n–ary aggregation function*** is a function $A^{(n)} : I^n \to I$ that is non–decreasing in each place and fulfills the following boundary conditions

$$\inf_{\mathbf{x}\in I^n} A^{(n)}(\mathbf{x}) = \inf I \quad \text{and} \quad \sup_{\mathbf{x}\in I^n} A^{(n)}(\mathbf{x}) = \sup I.$$

(ii) An ***extended aggregation function*** is a function $A : \bigcup_{n\in\mathbb{N}} I^n \to I$ such that for all $n > 1$, $A^{(n)} = A_{|_{I^n}}$ is an n–ary aggregation function and $A^{(1)}$ is the identity on I.

Examples

- The *sum* Σ,

$$\Sigma(x_1,\ldots,x_n)=\sum_{i=1}^{n}x_i,$$

in the case of an interval I with the left–end point $-\infty$ or 0, the right–end point 0 or ∞, and with the convention $(-\infty)+\infty=-\infty$ if necessary.

- The *product* Π,

$$\Pi(x_1,\ldots,x_n)=\prod_{i=1}^{n}x_i,$$

if I is an interval with the left–end point 0 or 1, the right–end point 1 or ∞ and with the convention $0\,.\,\infty=0$ if necessary.

Examples

- The *arithmetic mean M*,

$$M(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i,$$

on an arbitrary interval I, and if $I = [-\infty, \infty]$, the convention $(-\infty) + \infty = -\infty$ is adopted.

- The *geometric mean G*,

$$G(x_1, \dots, x_n) = \left(\prod_{i=1}^{n} x_i \right)^{1/n},$$

where $I \subseteq [0, \infty]$, and $0 \,.\, \infty = 0$ by convention.

Examples

- The *minimum Min*,

$$Min(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\} = \bigwedge_{i=1}^{n} x_i.$$

- The *maximum Max*,

$$Max(x_1, \ldots, x_n) = \max\{x_1, \ldots, x_n\} = \bigvee_{i=1}^{n} x_i.$$

Properties

Properties of n–ary aggregation functions

$$\Downarrow$$

Properties of extended aggregation functions

Strong properties concern extended aggregation functions only

Formally, weak properties concern n–ary aggregation functions only

Definition 2

For a fixed $n \in \mathbb{N} \setminus \{1\}$, let $A^{(n)} : I^n \to I$ be an n–ary aggregation function on I. Then $A^{(n)}$ is called:

(i) ***symmetric*** (anonymous) if for each permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ and each $\mathbf{x} \in I^n$

$$A^{(n)}(\mathbf{x}) = A^{(n)}(x_{\sigma(1)}, \ldots, x_{\sigma(n)})\,;$$

(ii) ***idempotent*** (unanimous) if for each $c \in I$

$$A^{(n)}(c, \ldots, c) = c\,;$$

(iii) ***strictly monotone*** if for all $x_i,\ y_i \in I$, $i \in \{1, \ldots, n\}$ such that $x_i \leq y_i$ and $(x_1, \ldots, x_n) \neq (y_1, \ldots, y_n)$ it follows that

$$A^{(n)}(x_1, \ldots, x_n) < A^{(n)}(y_1, \ldots, y_n)\,;$$

Definition 2

(iv) ***continuous*** if for each $\mathbf{x}_0 \in I^n$,

$$\lim_{\mathbf{x}\to\mathbf{x_0}} A^{(n)}(\mathbf{x}) = A^{(n)}(\mathbf{x_0}),$$

i.e., if $A^{(n)}$ is a continuous function of n variables in the usual sense;

(v) ***1–Lipschitz***, if for all $(x_1,\ldots,x_n), (y_1,\ldots,y_n) \in I^n$,

$$|A^{(n)}(x_1,\ldots,x_n) - A^{(n)}(y_1,\ldots,y_n)| \leq \sum_{i=1}^{n} |x_i - y_i|;$$

Definition 2

(vi) ***bisymmetric*** if for all $n \times n$ matrices $X = (x_{ij})$, with entries $x_{ij} \in I$ for all $i, j \in \{1, \ldots, n\}$,

$$\begin{aligned} & A^{(n)}\left(A^{(n)}(x_{11}, \ldots, x_{1n}), \ldots, A^{(n)}(x_{n1}, \ldots, x_{nn})\right) \\ = \quad & A^{(n)}\left(A^{(n)}(x_{11}, \ldots, x_{n1}), \ldots, A^{(n)}(x_{1n}, \ldots, x_{nn})\right). \end{aligned}$$

We can equivalently say that, for example, an n–ary aggregation function $A^{(n)}$ is symmetric if and only if for all $\mathbf{x} \in I^n$ it holds

$$A^{(n)}(\mathbf{x}) = A^{(n)}(x_2, x_1, x_3, \ldots, x_n) = A^{(n)}(x_2, \ldots, x_n, x_1).$$

Similarly, the idempotency of $A^{(n)}$ is equivalent to the property

$$Min^{(n)} \leq A^{(n)} \leq Max^{(n)}.$$

Definition 3

For a fixed $n \in \mathbb{N} \setminus \{1\}$, let $A^{(n)} : I^n \to I$ be an n–ary aggregation function on I.

(i) An element $e \in I$ is called ***neutral element*** of $A^{(n)}$ if for each $i \in \{1, \dots, n\}$ and each $x_i \in I$ it holds that

$$A^{(n)}(e, \dots, e, x_i, e, \dots, e) = x_i \,.$$

(ii) An element $a \in I$ is called ***annihilator*** of $A^{(n)}$ if for all $(x_1, \dots, x_n) \in I^n$ it holds that
if $x_i = a$ for some $i \in \{1, \dots, n\}$ then $A^{(n)}(x_1, \dots, x_n) = a$.

Definition 4

Let $A : \bigcup_{n \in \mathbb{N}} I^n \to I$ be an extended aggregation function. Then

(i) A is ***strongly idempotent*** whenever

$$A(\underbrace{\mathbf{x}, \dots, \mathbf{x}}_{k-times}) = A(\mathbf{x})$$

for all $k \in \mathbb{N}$ and $\mathbf{x} \in \bigcup_{n \in \mathbb{N}} I^n$.

(ii) An element $e \in I$ is said to be a ***strong neutral element*** of A if for each $n \in \mathbb{N}$, each $\mathbf{x} \in \bigcup_{n \in \mathbb{N}} I^n$ and $i \in \{1, \dots, n+1\}$ it holds

$$A(\mathbf{x}) = A(x_1, \dots, x_{i-1}, e, x_i, \dots, x_n).$$

Definition 4

(iii) A is ***strongly bisymmetric*** if for any $n \times m$ matrix $X = (x_{ij})$ with all entries $x_{ij} \in I$, it holds

$$A^{(n)}(A^{(m)}(\mathbf{x}_{1.}), \ldots, A^{(m)}(\mathbf{x}_{n.})) = A^{(m)}(A^{(n)}(\mathbf{x}_{.1}), \ldots, A^{(n)}(\mathbf{x}_{.m})),$$

where for all $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$,

$$\mathbf{x}_{i.} = (x_{i1}, \ldots, x_{im}) \quad \text{and} \quad \mathbf{x}_{.j} = (x_{1j}, \ldots, x_{nj}).$$

Classical properties linking different input arities of extended aggregation functions are:

- ***associativity***, that is, for each $n, m \in \mathbb{N}$, $\mathbf{x} \in I^n$, $\mathbf{y} \in I^m$

$$A^{(n+m)}(\mathbf{x}, \mathbf{y}) = A^{(2)}(A^{(n)}(\mathbf{x}), A^{(m)}(\mathbf{y}));$$

- ***decomposability***, that is, for all integers $0 \leq k \leq n$, $n \in \mathbb{N}$, and all $\mathbf{x} \in I^n$

$$\begin{aligned} & A^{(n)}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) \\ = \; & A^{(n)}(\underbrace{A^{(k)}(x_1, \ldots, x_k)}_{k-times}, \underbrace{A^{(n-k)}(x_{k+1}, \ldots, x_n)}_{(n-k)-times}). \end{aligned}$$

The associativity of an extended aggregation function A is equivalent to the standard associativity of the corresponding binary aggregation function $A^{(2)}$,

$$A^{(2)}(x, A^{(2)}(y, z)) = A^{(2)}(A^{(2)}(x, y), z)$$

for all x, y, $z \in I$, and $A^{(n)}$ for $n > 2$, being the genuine n–ary extension of $A^{(2)}$ given by

$$A^{(n)}(x_1, \ldots, x_n) = A^{(2)}\left(A^{(n-1)}(x_1, \ldots, x_{n-1}), x_n\right)$$

defined by induction.

Consider a system $\mathcal{A} = (A_n^{(2)})_{n\in\mathbb{N}}$ of binary aggregation functions by induction. We define $A_{\mathcal{A}} = A$ as follows:

$$A^{(1)}(x_1) = x_1,$$
$$A^{(2)}(x_1, x_2) = A_1^{(2)}(x_1, x_2),$$
$$\vdots$$
$$A^{(n)}(x_1, \ldots, x_n) = A_{n-1}^{(2)}\left(A^{(n-1)}(x_1, \ldots, x_{n-1}), x_n\right)$$
$$\vdots$$

Extended aggregation functions $A_{\mathcal{A}} = A$ are called recursive.

Examples

The sum Σ is symmetric, associative and bisymmetric. If $0 \in I$, then 0 is the strong neutral element of Σ, if $-\infty \in I$ then this element is the annihilator of Σ, and if $+\infty \in I$ and $-\infty \notin I$ then $+\infty$ is the annihilator of Σ. The extended aggregation function Σ is 1–Lipschitz and strictly monotone if $I \subset \mathbb{R}$, continuous if $I \neq [-\infty, \infty]$.

Examples

The arithmetic mean M is recursive, symmetric, strongly idempotent and bisymmetric on any interval I. It is 1–Lipschitz and strictly monotone if $I \subset \mathbb{R}$ and continuous if $I \neq [-\infty, \infty]$. It has an annihilator a only if I is an unbounded interval, namely, $a = -\infty$ if $-\infty \in I$; $a = \infty$, if $\infty \in I$ and $-\infty \notin I$.

Let the extended aggregation function $A : \bigcup_{n \in \mathbb{N}} I^n \to I$ be given by

$$A(x_1, \ldots, x_n) = \min \left(x_1, \prod_{i=2}^{n} x_i \right)$$

whenever $n > 1$. Evidently, $e = 1$ is the neutral element of A, but it is not a strong neutral element. Indeed, if we take $(x_1, x_2) = (0.5, 0.5)$ then, for $i = 1$ we have $A(1, x_1, x_2) = 0.25$, for $i = 2$ and $i = 3$ we have $A(x_1, 1, x_2) = A(x_1, x_2, 1) = 0.5$. Observe that A is 1-Lipschitz.

Basic classification

The basic classification of aggregation functions takes into account the main fields of applications. Following Dubois and Prade, we will distinguish four classes of (n–ary/extended) aggregation functions:

- *conjunctive aggregation functions*: aggregation functions $A \leq Min$;
- *averaging aggregation functions*: aggregation functions A, $Min \leq A \leq Max$, or, equivalently, idempotent aggregation functions;
- *disjunctive aggregation functions*: aggregation functions $A \geq Max$;
- *mixed aggregation functions*: aggregation functions which do not belong to any of other three classes.

There are refined approaches to the classification of aggregation functions (on POSETS) due to Komorníková & Mesiar, see also Marichal for CHAINS;
$A^{(n)} : I^n \to I$:

- *k–disjunctive (k–tolerant)*: if card $\{i \,|\, A(\mathbf{x}) \geq x_i\} \geq k$;
- *k–conjunctive (k–intolerant)*: if card $\{i \,|\, A(\mathbf{x}) \leq x_i\} \geq k$;

and equality is attained for some $\mathbf{x}$.

conjunctive $\equiv$ n–intolerant
disjunctive $\equiv$ n–tolerant

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and equality is attained for some **x**.

conjunctive $\equiv$ n–intolerant
disjunctive $\equiv$ n–tolerant

Observe that the interval I may be crucial for the classification of a discussed aggregation function. For example, the product Π is a conjunctive aggregation function on $[0, 1]$, disjunctive on $[1, \infty]$ and mixed on $[0, \infty]$.

For any decreasing one–to–one mapping $\varphi : I \to I$, $A : \bigcup_{n\in\mathbb{N}} I^n \to I$ is a conjunctive (disjunctive) extended aggregation function if and only if the function $A_\varphi : \bigcup_{n\in\mathbb{N}} I^n \to I$ given by

$$A_\varphi(x_1, \dots, x_n) = \varphi^{-1}\left(A(\varphi(x_1), \dots, \varphi(x_n))\right)$$

is a disjunctive (conjunctive) extended aggregation function. This duality allows to investigate, construct and discuss conjunctive aggregation functions only, and to transfer all the results by this duality to the disjunctive aggregation functions.

Averaging aggregation functions

- The *arithmetic mean* M,

$$M(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i .$$

- *Quasi–arithmetic means* M_f, where $f : I \to [-\infty, \infty]$ is a continuous strictly monotone function and

$$M_f(x_1, \ldots, x_n) = f^{-1}(M(f(x_1), \ldots, f(x_n)),$$

as, for example, the geometric, harmonic and quadratic means.

Averaging aggregation functions

- *Weighted arithmetic means* $M_{\mathbf{w}}$, where $\mathbf{w} = (w_1, \ldots, w_n)$, $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$ and

$$M_{\mathbf{w}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i\, x_i,$$

- *Weighted quasi–arithmetic means* $M_{f,\mathbf{w}}$,

$$M_{f,\mathbf{w}}(x_1, \ldots, x_n) = f^{-1}\left(\sum_{i=1}^{n} w_i\, f(x_i)\right).$$

Averaging aggregation functions

- *OWA* (*ordered weighted average*) *operator* $M'_{\mathbf{w}}$,

$$M'_{\mathbf{w}}(x_1, \ldots, x_n) = M_{\mathbf{w}}(x'_1, \ldots, x'_n) = \sum_{i=1}^{n} w_i\, x'_i ,$$

  where x'_i is the i–th order statistics from the sample $(x_1, \ldots, x_n)$.
- *OWQA* (*ordered weighted quasi–arithmetic*) *operator* $M'_{f,\mathbf{w}}$,

$$M'_{f,\mathbf{w}}(x_1, \ldots, x_n) = M_{f,\mathbf{w}}(x'_1, \ldots, x'_n) = f^{-1}\left(\sum_{i=1}^{n} w_i\, f(x'_i)\right) .$$

Averaging aggregation functions

- *Idempotent uninorms*,
- *Idempotent nullnorms*, i.e., a–medians, given for a fixed $a \in I$ by

$$Med_a(x_1, \ldots, x_n) = \text{med}(x_1, a, x_2, a, x_3, a, \ldots, a, x_n).$$

Fuzzy integrals

Recall that for any 2–copula $C : [0,1]^2 \to [0,1]$ (for the definition of a copula see the next section) and for any fuzzy measure $m : \mathcal{P}(\{1,\dots,n\}) \to [0,1]$, i.e., a non–decreasing set function such that $m(\emptyset) = 0$ and $m(\{1,\dots,n\}) = 1$, we can define a fuzzy integral $F_{C,m} : [0,1]^n \to [0,1]$ by

$$F_{C,m}(x_1,\dots,x_n) = \sum_{i=1}^{n} \left(C\left(x'_i, m(\{j \mid x_j \geq x'_i\})\right) - C\left(x'_{i-1}, m(\{j \mid x_j \geq x'_i\})\right) \right),$$

with the convention $x'_0 = 0$, where x'_i is the i–th order statistics from the sample $(x_1,\dots,x_n)$.

Then $F_{\Pi,m}$ is the Choquet integral and $F_{Min,m}$ is the Sugeno integral. Also observe that if m is additive then $F_{\Pi,m} = M_{\mathbf{w}}$ is the weighted arithmetic mean with the weights given by $w_i = m(\{i\})$. Similarly, if m is symmetric, i.e., $m(A) = h\left(\frac{cardA}{n}\right)$ for some increasing function $h : [0,1] \to [0,1]$, then $F_{\Pi,m}$ is the OWA operator $M'_{\mathbf{w}}$ with the weights $w_i = h\left(\frac{i}{n}\right) - h\left(\frac{i-1}{n}\right)$.

Averaging aggregation functions are closed under composition, i.e., for any averaging (extended) aggregation functions A, $A_1, \ldots, A_n$ on I, also the function $D = A(A_1, \ldots, A_n) : \bigcup_{n \in \mathbb{N}} I^n \to I$, given by $D(\mathbf{x}) = A(A_1(\mathbf{x}), \ldots, A_n(\mathbf{x}))$, is an averaging extended aggregation function.

An interesting class of averaging aggregation functions are the *internal* aggregation functions characterized by $A(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$. Continuous internal aggregation functions are exactly lattice polynomials, whose prescription formula contains inputs $x_1, \ldots, x_n$, symbols for join $\vee$ and meet $\wedge$, i.e., *Max* and *Min* in infix form, and parentheses. Independently of the interval I, they have the same formula, and on any open interval I they are the only aggregation functions invariant under any increasing $I \to I$ one–to–one transformation φ. On $[0, 1]$, they are in a one–to–one correspondence with $\{0, 1\}$– valued fuzzy measures (and then we can apply any fuzzy integral based on a copula C, e.g., the Choquet or Sugeno integrals). As an example we give all 18 ternary aggregation functions which are internal and continuous on any interval I:

$$m(E) = A^{(3)}(1_E), \quad A^{(3)}(x_1, x_2, x_3) =$$

$$\begin{array}{lll}
x_1; & x_2; & x_3; \\
x_1 \wedge x_2; & x_1 \wedge x_3; & x_2 \wedge x_3; \\
x_1 \vee x_2; & x_1 \vee x_3; & x_2 \vee x_3; \\
x_1 \wedge (x_2 \vee x_3); & x_2 \wedge (x_1 \vee x_3); & x_x \wedge (x_1 \vee x_2); \\
x_1 \vee (x_2 \wedge x_3); & x_2 \vee (x_1 \wedge x_3); & x_3 \vee (x_1 \wedge x_2); \\
x_1 \wedge x_2 \wedge x_3 = x_1'; & (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3) = x_2'; & x_1 \vee x_2 \vee x_3 = x_3'
\end{array}$$

Another interesting and still not completely described family of averaging extended aggregation functions are the mixture operators $M^g : \bigcup_{n \in \mathbb{N}} I^n \to I$ given by

$$M^g(x_1, \ldots, x_n) = \frac{\sum_{i=1}^{n} g(x_i)\, x_i}{\sum_{i=1}^{n} g(x_i)},$$

where $g : I \to]0, \infty[$ is a given weighting function

Evidently, mixture operators are idempotent and they generalize the arithmetic mean M, since $M = M^g$ for any constant weighting function g. Mixture operators are extended aggregation functions if and only if they are monotone, which is not a general case. For example, let $I = [0, b]$ and let $g : I \to]0, \infty[$ be given by $g(x) = x + 1$. Then M^g is an averaging extended aggregation function only if $b \in]0, 1]$. Till now, only some sufficient conditions ensuring the monotonicity of mixture operators M^g are known, as, for example, for a non–decreasing differentiable function g the next two conditions:

(i) $g(x) \geq g'(x)\, l(I)$ for all $x \in I$, where $l(I)$ is the length of the interval I;

(ii) $g(x) \geq g'(x)\,(x - \inf I)$ for all $x \in I$.

Example:

$$g(x) = x,$$

$$I = [0,1],\ M^g(x,y) = \frac{x^2+y^2}{x+y},$$

$$M^g(0,1) = 1,\ M^g(\frac{1}{2},1) = \frac{5}{6} \quad x \geq 1 \text{ is not true}$$

BUT:

$$I = [1,2] \quad x \geq 1 \text{ is true!}$$

M^g is aggregation functions!

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M^g is aggregation functions!

Also other generalizations of mixture operators are interesting, as, for example, the *quasi–mixture operators* M_f^g, defined by

$$M_f^g(x_1,\ldots,x_n) = f^{-1}\left(\frac{\sum_{i=1}^{n} g(x_i)\, f(x_i)}{\sum_{i=1}^{n} g(x_i)}\right),$$

generalized mixture operators $M^{\mathbf{g}}$, where $\mathbf{g} = (g_1, \ldots, g_n)$ is a vector of weighting functions, defined by

$$M^{\mathbf{g}}(x_1, \ldots, x_n) = \frac{\sum_{i=1}^{n} g_i(x_i)\, x_i}{\sum_{i=1}^{n} g_i(x_i)},$$

and *ordered generalized mixture operators* $M'^{\mathbf{g}}$,

$$M'^{\mathbf{g}}(x_1, \ldots, x_n) = M^{\mathbf{g}}(x'_1, \ldots, x'_n).$$

Thanks for your attention!