Aggregation Functions
Part II.

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Contents

Conjunctive aggregation functions

In this section we restrict our considerations to the interval $I = [0, 1]$ only. The conjunctive aggregation functions are bounded from above by *Min*, and from below by the weakest extended aggregation function $A_w : \bigcup_{n \in \mathbb{N}} [0, 1]^n \to [0, 1]$ given by

$$A_w(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } \prod_{i=1}^{n} x_i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

is also the weakest conjunctive extended aggregation function, and, obviously, *Min* is the strongest one.

For any (conjunctive) extended aggregation function A it holds

i) $A_w \leq A \leq \text{Min}$;

ii) A is idempotent if and only if $A = \text{Min}$.

Definition 1

Let $A : [0,1]^2 \to [0,1]$ be an aggregation function.

(i) A is a *boundary weak triangular norm* (*bwt-norm* for short) if it is an associative symmetric conjunctive aggregation function.

(ii) If A is an aggregation function with neutral element $e = 1$ then it is called a *conjunctor*.

(iii) A bwt-norm A which is also a conjunctor is called a *triangular norm* (*t-norm* in short).

Definition 1

(iv) A conjunctor A satisfying the Lipschitz condition with constant 1, i.e., such that for all $x, y, u, v, \in [0, 1]$ we have

$$|A(x, y) - A(u, v)| \leq |x - u| + |y - v|,$$

is called a *quasi-copula*.

(v) A conjunctor A fulfilling the *moderate growth property*, i.e., such that for all $x, y, u, v \in [0, 1]$, with $x \leq u,\ y \leq v$, we have

$$A(x, y) + A(u, v) \geq A(x, v) + A(u, y),$$

is called a *copula*.

Bwt-norms and t-norms are associative and thus their extension to extended aggregation functions is trivial, hence we will keep the same name and notation for the binary and the extended bwt-norms and t-norms. An extended conjunctor is an extended aggregation function on $[0, 1]$ with neutral element $e = 1$. Similarly, a 1-Lipschitz conjunctive extended aggregation function is called a (extended) quasi-copula.

An aggregation function $A : [0,1]^n \to \mathbb{R}$ is conjunctive if and only if, for each $n > 1$, for any $i, j \in [n]$, with $i \neq j$, and any $x \in [0,1[$ it holds $A(a_1, \ldots, a_n) \leq x$, where $a_i = x$ and $a_j = 1$ whenever $j \neq i$.

Let A be an (extended) conjunctive aggregation function. Then 0 is annihilator of A.

Proposition 1

Let A be an extended aggregation function with neutral element e. Then the following are equivalent:

(i) $e = 1$;

(ii) A is a conjunctive extended aggregation function.

Remark

Conjunctive aggregation functions on interval $[a, b]$ have necessarily a as annihilator and if they have neutral element e, then $e = b$. However, if $\mathbb{I} =]a, b[$, then a conjunctive A cannot possess neither annihilator nor neutral element.

Proposition 2

Let A be an idempotent extended aggregation function and let for $m \in \mathbb{N}, A_1, \ldots, A_m$ be conjunctive extended aggregation functions. Then the composed extended aggregation function $B : \bigcup [0, 1]^n \to [0, 1]$ given by

$$B(\mathbf{x}) = A(A_1(\mathbf{x}), \ldots, A_m(\mathbf{x}))$$

is a conjunctive extended aggregation function.

Proposition 2 remains true also if we restrict the functions $A_1, \ldots, A_m$ to be conjunctors (and then also B is a conjunctor).

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Proposition 3

Let φ be an increasing bijection from $[0,1]$ onto $[0,1]$ and let A be a conjunctive extended aggregation function. Then $A_\varphi : \bigcup [0,1]^n \to [0,1]$ given by

$$A_\varphi(\mathbf{x}) = \varphi^{-1}(A(\varphi(x_1), \ldots, \varphi(x_n))),$$

is a conjunctive extended aggregation function.

Proposition 3 remains true if we replace conjunctive extended aggregation function by bwt-norms, t-norms or conjunctors. However, it fails for quasi-copulas and copulas, in general.

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Definition 2

Let $\mathbb{I}$ be a given real interval and let a family

$$\mathfrak{F} = \{(g_n, f_{1,n}, \dots, f_{n,n}) \mid n \in \mathbb{N}\}$$

be given so that $f_{i,n} : \mathbb{I} \to [-\infty, \infty]$, $i = 1, \dots, n$, are nondecreasing (nonincreasing) functions,

$$g_n : \left\{ \sum_{i=1}^{n} u_i \mid u_i \in \operatorname{ran}(f_{i,n}) \right\} \to \mathbb{I}$$

are nondecreasing (nonincreasing) surjective functions, for all $n \in \mathbb{N}$, and $g_1 = f_{1,1}^{-1}$ (i.e., $f_{1,1}$ is strictly monotone).

Definition 2

Then $\mathfrak{F}$ is called an *extended generating system* (EGS for short) on $\mathbb{I}$, and the extended function $A : \cup_n \mathbb{I}^n \to \mathbb{I}$ given by

$$A(\mathbf{x}) = g_n\left(\sum_{i=1}^{n} f_{i,n}(x_i)\right)$$

is called a *generated extended aggregation function*. All involved one-place functions are called *additive generators* of A.

A generated extended aggregation function $A : \cup_n [0,1]^n \to [0,1]$ is conjunctive if and only if for all $n > 1, i \in \{1, \ldots, n\}$ and $x \in [0,1[$ it holds

$$g_n \left(\sum_{j \neq i}^{n} f_j(1) + f_i(x) \right) \leqslant x.$$

Definition 3

Let $[a, b]$ and $[c, d]$ be two closed subintervals of the extended interval $[-\infty, \infty]$, and let $f : [a, b] \to [c, d]$ be a monotone and nonconstant function. Then the *pseudo-inverse* $f^{(-1)} : [c, d] \to [a, b]$ is defined by

$$f^{(-1)}(y) := \sup\{x \in [a, b] \mid (f(x) - y)(f(b) - f(a)) < 0\} \quad (y \in [c, d]).$$

f continuous strictly increasing (decreasing) and $f(a) = c$ ($f(b) = c$), then

$$f^{(-1)}(y) = f^{-1}(\min\{f(b), y\}) \quad \left(f^{-1}(\min\{f(a), y\})\right)$$

on $[0,1]$

$$f(x) = 1 - x \quad [0, \infty]$$

$$g(x) = x - 1 \quad [-\infty, 0]$$

$$f^{(-1)}\left(\sum_{i=1}^{n} f(x_i)\right) = g^{(-1)}\left(\sum_{i=1}^{n} g(x_i)\right) = \max\left\{0, \sum_{i=1}^{n} x_i - (n-1)\right\}$$

Lukasiewicz *t*–norm

Definition 4:
An extended aggregation function A is a *conjunctor* if it has as a neutral element $e = 1$.

Proposition 4:
The smallest conjunctor is the *drastic product* $\mathrm{T}_\mathbf{D}$ (also notation Z is commonly used) which is associative, symmetric (and thus a triangular norm), and it is given by

$$\mathrm{T}_\mathbf{D}(\mathbf{x}) := \begin{cases} \min(x_1, \dots, x_n) & \text{if } |\{i \mid x_i < 1\}| < 2 \\ 0 & \text{else,} \end{cases}$$

and the greatest conjunctor is Min, i.e., for any conjunctor A,

$$\mathrm{T}_\mathbf{D} \leq A \leq \mathrm{Min}.$$

Moreover, a binary aggregation function A is a conjunctor if and only if $\mathrm{T}_\mathbf{D} \leq A \leq \mathrm{Min}$.

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Moreover, a binary aggregation function A is a conjunctor if and only if $\mathrm{T}_{\mathbf{D}} \leq A \leq \mathrm{Min}$.

Example 1

Let an extended aggregation function A on $[0,1]$ be given by

$$A(\mathbf{x}) := \left(\prod_{i=1}^{n} x_i\right) \cdot \left(1 + \prod_{i=1}^{n}((x_i)^i \cdot (1 - x_i))\right).$$

Then A is a continuous conjunctor which is not symmetric.

Definition 5:
A triangular norm T : $\bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ is an associative symmetric aggregation function with neutral element 1.

Proposition 5:

(i) Each t-norm T is an aggregation function with annihilator 0, i.e., $T(\mathbf{x}) = 0$ for all $\mathbf{x} \in [0,1]^n$ such that $0 \in \{x_1, \ldots, x_n\}$.

(ii) The smallest t-norm is the drastic product $\mathrm{T}_\mathbf{D}$. The greatest (and the only idempotent) t-norm is the standard Min, i.e., for any t-norm T,

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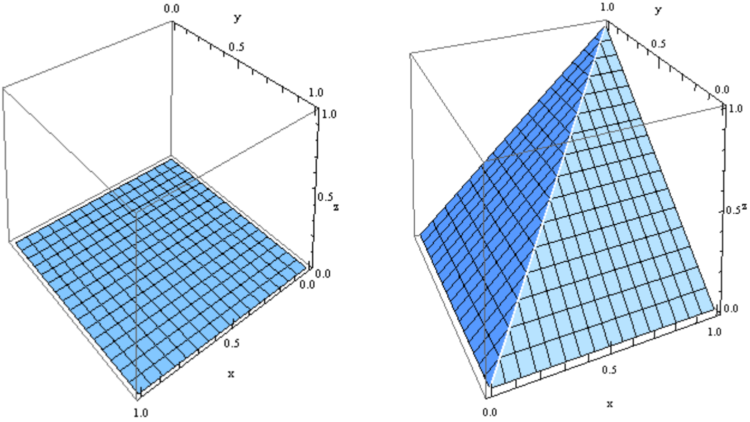

Figure: Two basic t-norms T_D (left) and Min (right)

Example 2

(i) The third basic t-norm is the product

$$\Pi(\mathbf{x}) = x_1 \cdot x_2 \cdots x_{n-1} \cdot x_n,$$

(ii) The fourth basic t-norm is the Łukasiewicz t-norm $T_{\mathbf{L}} : \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$,

$$T_{\mathbf{L}}(\mathbf{x}) := \max\left(0, \sum_{i=1}^{n} x_i - (n-1)\right)$$

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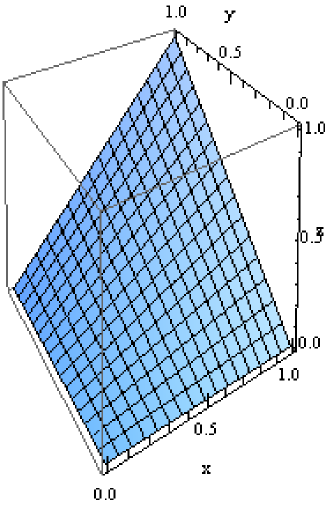

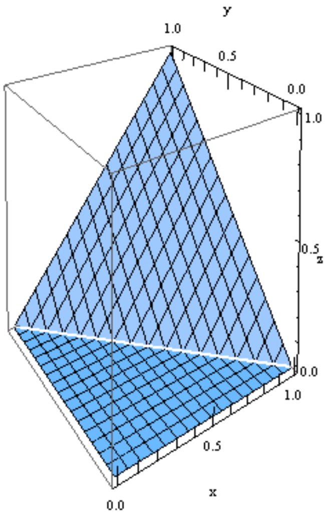

Figure: Two basic t-norms Π (left) and T_L (right)

Frank t-norms family

We list some important families of t-norms.

The family $(T^{F}_{\lambda})_{\lambda \in [0,\infty]}$ of *Frank t-norms* is given by

$$T^{F}_{\lambda}(x,y) = \begin{cases} \mathrm{Min}(x,y) & \text{if } \lambda = 0, \\ \Pi(x,y) & \text{if } \lambda = 1, \\ T_{\mathbf{L}}(x,y) & \text{if } \lambda = \infty, \\ \log_{\lambda}\left(1 + \frac{(\lambda^{x}-1)(\lambda^{y}-1)}{\lambda - 1}\right) & \text{otherwise.} \end{cases}$$

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The family $(\mathrm{T}^{\mathbf{F}}_{\lambda})_{\lambda\in[0,\infty]}$ of *Frank t-norms* is given by

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Yager t-norms family

The family $(\mathrm{T}^{\mathbf{Y}}_{\lambda})_{\lambda\in[0,\infty]}$ of *Yager t-norms* is given by

$$\mathrm{T}^{\mathbf{Y}}_{\lambda}(x,y)=\begin{cases}\mathrm{T}_{\mathbf{D}}(x,y) & \text{if } \lambda=0,\\ \mathrm{Min}(x,y) & \text{if } \lambda=\infty,\\ \max\left(0,1-((1-x)^{\lambda}+(1-y)^{\lambda})^{\frac{1}{\lambda}}\right) & \text{otherwise.}\end{cases}$$

Sugeno–Weber t-norms family

The family $(\mathrm{T}^{\mathbf{SW}}_{\lambda})_{\lambda\in[-1,\infty]}$ of *Sugeno–Weber t-norms* is given by

$$\mathrm{T}^{\mathbf{SW}}_{\lambda}(x,y)=\begin{cases}\mathrm{T}_{\mathbf{D}}(x,y) & \text{if } \lambda=-1,\\ \Pi(x,y) & \text{if } \lambda=\infty,\\ \max\left(0,\dfrac{x+y-1+\lambda xy}{1+\lambda}\right) & \text{otherwise.}\end{cases}$$

Hamacher t-norms family

The family $(\mathsf{T}^{\mathbf{H}}_{\lambda})_{\lambda\in[0,\infty]}$ of *Hamacher t-norms* is given by

$$\mathsf{T}^{\mathbf{H}}_{\lambda}(x,y)=\begin{cases}\mathsf{T}_{\mathbf{D}}(x,y) & \text{if } \lambda=\infty,\\ 0 & \text{if } \lambda=x=y=0,\\ \dfrac{xy}{\lambda+(1-\lambda)(x+y-xy)} & \text{if } \lambda\in[0,\infty[\text{ and } (\lambda,x,y)\neq(0,0,0)\end{cases}$$

Schweizer-Sklar t-norms family

The family $(\mathrm{T}^{\mathbf{SS}}_{\lambda})_{\lambda \in [-\infty,\infty]}$ of *Schweizer-Sklar t-norms* is given by

$$\mathrm{T}^{\mathbf{SS}}_{\lambda}(x,y) = \begin{cases} \mathrm{Min}(x,y) & \text{if } \lambda = -\infty, \\ \Pi(x,y) & \text{if } \lambda = 0, \\ \mathrm{T}_{\mathbf{D}}(x,y) & \text{if } \lambda = \infty, \\ \left(\max\left(0, (x^{\lambda} + y^{\lambda} - 1)\right)\right)^{\frac{1}{\lambda}} & \text{if } \lambda \in]-\infty, 0[\cup]0, \infty[. \end{cases}$$

Examples 3

The drastic product T_D is an example of a non-continuous but right-continuous (upper semi-continuous) t-norm.

An important example of a left-continuous (lower semi-continuous) non-continuous t-norm is the *nilpotent minimum* $T^{nM} : \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$,

$$T^{nM}(\mathbf{x}) := \begin{cases} 0 & \text{if } x_{(1)} + x_{(2)} \leqslant 1, \\ x_{(1)} & \text{else,} \end{cases}$$

where $(x_{(1)}, \ldots, x_{(n)})$ is a non-decreasing permutation of $(x_1, \ldots, x_n)$.

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where $(x_{(1)}, \ldots, x_{(n)})$ is a non-decreasing permutation of $(x_1, \ldots, x_n)$.

Proposition 6

Let

$$x_{\mathbf{T}}^{(n)} := \mathbf{T}^{(n)}(x, \dots, x).$$

For a t-norm **T** the following are equivalent:

(i) **T** is Archimedean, if for each $x, y \in]0,1[$ there is an $n \in \mathbb{N}$ such that

$$x_{\mathbf{T}}^{(n)} < y.$$

(ii) For every $x \in]0,1[$

$$\lim_{n \to \infty} x_{\mathbf{T}}^{(n)} = 0.$$

Proposition 7

If a t-norm $\mathbf{T}$ is continuous, then it is Archimedean if and only if we have

$$\mathbf{T}(x, x) < x \qquad \text{for each } x \in]0, 1[\ .$$

Theorem 1

Representation theorem of Ling:

An aggregation function $\mathrm{T}: \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ is a continuous Archimedean t-norm if and only if there is a continuous strictly decreasing mapping $t: [0,1] \to [0,\infty]$, $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that

$$\mathrm{T}(x_1,\ldots,x_n) = t^{-1}\left(\min\left(t(0), \sum_{i=1}^{n} t(x_i)\right)\right).$$

Definition 6

A continuous Archimedean t-norm with unbounded additive generator is called a *strict t-norm*. Non-strict continuous Archimedean t-norms are called *nilpotent*.

Theorem 2

The following holds.

(i) A strict t-norm is strictly monotone on $\bigcup_{n\in\mathbb{N}}]0,1]^n$. All strict t-norms are mutually isomorphic, i.e., if $\mathbf{T}_1, \mathbf{T}_2$ are strict t-norms, then there exists a bijection $\varphi : [0,1] \to [0,1]$ such that

$$\varphi^{-1}(\mathbf{T}_1(\varphi(x), \varphi(y))) = \mathbf{T}_2.$$

A function $\mathbf{T} : [0,1]^2 \to [0,1]$ is a strict t-norm if and only if it is isomorphic to the product Π.

Theorem 2

(ii) All nilpotent t-norms are mutually isomorphic, i.e., if $\mathbf{T}_1, \mathbf{T}_2$ are nilpotent t-norms, then there exists a bijection $\varphi : [0,1] \to [0,1]$ such that

$$\varphi^{-1}(\mathbf{T}_1(\varphi(x), \varphi(y))) = \mathbf{T}_2.$$

A function $\mathbf{T} : [0,1]^2 \to [0,1]$ is a nilpotent t-norm if and only if it is isomorphic to the Łukasiewicz t-norm $\mathrm{T}_{\mathbf{L}}$.

Example 4

(i) A family of additive generators $(t_\lambda^{\mathbf{F}} : [0,1] \to [0,\infty])_{\lambda \in [0,\infty]}$ for the family $(T_\lambda^{\mathbf{F}})_{\lambda \in [0,\infty]}$ of Frank t-norms is given by

$$t_\lambda^{\mathbf{F}}(x) = \begin{cases} -\log x & \text{if } \lambda = 1, \\ 1 - x & \text{if } \lambda = \infty, \\ -\log \dfrac{\lambda^x - 1}{\lambda - 1} & \text{otherwise.} \end{cases}$$

Example 4

(ii) A family of additive generators $(t_\lambda^{\mathbf{Y}} : [0,1] \to [0,1])_{\lambda \in]0,\infty[}$ for the family $(T_\lambda^{\mathbf{Y}})_{\lambda \in]0,\infty[}$ of continuous Yager t-norms is given by

$$t_\lambda^{\mathbf{Y}}(x) = (1-x)^\lambda.$$

For $\lambda = 0$, an additive generator $t_0^{\mathbf{Y}} : [0,1] \to [0,2]$ for Archimedean (noncontinuous) t-norm $T_0^{\mathbf{Y}} = \mathbf{T_D}$ is given by

$$t_{\mathbf{D}}(x) = \begin{cases} 2-x & \text{if } x \in [0,1[\,, \\ 0 & \text{if } x = 1. \end{cases}$$

Example 4

(iii) For the family of continuous Sugeno–Weber's t-norms $(T_\lambda^{\mathbf{SW}})_{\lambda \in]-1,\infty]}$, the corresponding additive generators are given by

$$t_\lambda^{\mathbf{SW}}(x) = \begin{cases} 1 - x & \text{if } \lambda = 0, \\ -\log x & \text{if } \lambda = \infty \\ 1 - \dfrac{\log(1+\lambda x)}{\log(1+\lambda)} & \text{if } \lambda \in]-1,\infty[\setminus \{0\}. \end{cases}$$

Example 4

(iv) For the family of continuous Hamacher's t-norms $(T^{\mathbf{H}}_{\lambda})_{\lambda \in [0,\infty[}$, the corresponding additive generators are given by

$$t^{\mathbf{H}}_{\lambda}(x) = \begin{cases} \dfrac{1-x}{x} & \text{if } \lambda = 0, \\ \log\left(\dfrac{\lambda + (1-\lambda)x}{x}\right) & \text{if } \lambda \in]0,\infty[\,. \end{cases}$$

Example 4

(v) For the family of Schweizer-Sklar's t-norms $(T_\lambda^{\mathbf{SS}})_{\lambda\in]-\infty,\infty[}$, the corresponding additive generators $(t_\lambda^{\mathbf{SS}} : [0,1] \to [0,\infty])_{\lambda\in]-\infty,\infty]}$ are given by

$$t_\lambda^{\mathbf{SS}}(x) = \begin{cases} -\log x & \text{if } \lambda = 0, \\ \dfrac{1 - x^\lambda}{\lambda} & \text{if } \lambda \in]-\infty,\infty[\setminus \{0\}. \end{cases}$$

Proposition 8

Let $t : [0,1] \to [0,\infty]$ be an additive generator of some continuous Archimedean t–norm T. Then for all $\lambda \in]0,\infty[$, also t^λ generates a continuous Archimedean t-norm $\mathrm{T}_{(\lambda)}$. The family $\left(\mathrm{T}_{(\lambda)}\right)_{\lambda \in]0,\infty[}$ is increasing and

$$\lim_{\lambda \to \infty} \mathrm{T}_{(\lambda)} = \mathrm{T}_{(\infty)} = \mathrm{Min}$$

uniformly,

$$\lim_{\lambda \to 0^+} \mathrm{T}_{(\lambda)} = \mathrm{T}_{(0)} = \mathrm{T}_{\mathbf{D}}$$

pointwisely.

Definition 7

Let $(\mathrm{T}_k)_{k\in K}$ be a family of t-norms and $(]a_k, b_k[)_{k\in K}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. The extended function $\mathrm{T} : \cup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ defined by

$$\mathrm{T}(\mathbf{x}) = \begin{cases} a_k + (b_k - a_k)\mathrm{T}_k\left(\dfrac{\min(x_1, b_k) - a_k}{b_k - a_k}, \ldots, \dfrac{\min(x_n, b_k) - a_k}{b_k - a_k}\right), \\ \qquad \text{if } \min\limits_{1\leqslant i\leqslant n} x_i \in\,]a_k, b_k[\,, \\ \\ \mathrm{Min}(x_1, \ldots, x_n), \\ \qquad \text{else} \end{cases}$$

is called the *(t-norm) ordinal sum* of summands $< a_k, b_k, \mathrm{T}_k >, k \in K$, and we denote it by

$$\mathrm{T} = (< a_k, b_k, \mathrm{T}_k >)_{k\in K}.$$

Theorem 3

A function $\mathrm{T} : \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$ is a continuous t-norm if and only if it is an ordinal sum of continuous Archimedean t-norms, i.e., there exists a family $(\mathrm{T}_k)_{k \in K}$ of continuous Archimedean t-norms such that

$$\mathbf{T} = (< a_k, b_k, \mathrm{T}_k >)_{k \in K}.$$

General copulas

Let for $\mathbf{x}, \mathbf{y} \in [0,1]^n$ such that $x_i \leqslant y_i, i = 1, \ldots, n$, $[\mathbf{x}, \mathbf{y}]$, be an n-box, and let $\mathbf{z} = (z_1, \ldots, z_n)$ be a vertex of $[\mathbf{x}, \mathbf{y}]$. Then we define $\mathrm{sign}_{[\mathbf{x},\mathbf{y}]}(\mathbf{z})$ in the following way

$$\mathrm{sign}_{[\mathbf{x},\mathbf{y}]}(\mathbf{z}) = \begin{cases} 1 & \text{if } z_m = y_m \text{ for an even number of } m's, \\ -1 & \text{if } z_m = y_m \text{ for an odd number of } m's. \end{cases}$$

If the vertices of n-box $[\mathbf{x}, \mathbf{y}]$ are not all distinct then $\mathrm{sign}_{[\mathbf{x},\mathbf{y}]}(\mathbf{z}) = 0$.

Definition 8

For a fixed $n \geq 2$, let $C : [0,1]^n \to [0,1]$ be an n-ary aggregation function with a neutral element $e = 1$ such that for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$, $\mathbf{x} \leqslant \mathbf{y}$, the following inequality (n-increasingness, moderate growth) is fulfilled:

$$\sum \mathrm{sign}_{[\mathbf{x},\mathbf{y}]}(\mathbf{z}) C(\mathbf{z}) \geqslant 0,$$

where the sum is taken over all vertices $\mathbf{z}$ of $[\mathbf{x}, \mathbf{y}]$. Then C is called an *n-copula*.

An extended aggregation function $C : \cup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$ such that for each $n \geqslant 2$ the corresponding n-ary aggregation function $C^{(n)}$ is an n-copula is called a *general copula.*

Definition 8

For a fixed $n \geq 2$, let $C : [0,1]^n \to [0,1]$ be an n-ary aggregation function with a neutral element $e = 1$ such that for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$, $\mathbf{x} \leqslant \mathbf{y}$, the following inequality (n-increasingness, moderate growth) is fulfilled:

$$\sum \mathrm{sign}_{[\mathbf{x},\mathbf{y}]}(\mathbf{z}) C(\mathbf{z}) \geqslant 0,$$

where the sum is taken over all vertices $\mathbf{z}$ of $[\mathbf{x}, \mathbf{y}]$. Then C is called an *n-copula*.
An extended aggregation function C : $\cup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$ such that for each $n \geqslant 2$ the corresponding n-ary aggregation function $C^{(n)}$ is an n-copula is called a *general copula.*

Remark 1

(i) The condition of n-increasingness can be also written in the following form

$$\sum_{\mathbf{d}} \prod_{i=1}^{n} d_i \, C(u_1^{(d_1)}, \ldots, u_n^{(d_n)}) \geqslant 0,$$

where the sum is taken over all n-tuples $\mathbf{d} = (d_1, \ldots, d_n) \in \{-1, 1\}^n$ and where $u_i^{(-1)} = x_i$, $u_i^{(1)} = y_i$.

(ii) Recall that in the case of 2-copulas $C : [0,1]^2 \to [0,1]$, the moderate growth, 2-increasingness, whenever $0 \leqslant x_1 \leqslant y_1 \leqslant 1$ and $0 \leqslant x_2 \leqslant y_2 \leqslant 1$ (i.e., for 2-box $[(x_1, x_2), (y_1, y_2)]$) can be rewritten into

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Example 5

(i) The product Π^n is a general copula. The product Π produces the joint distribution in the case of independent marginal random variables.

(ii) Min as an extended aggregation function is a general copula
positive total dependence

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Theorem 4 - Sklar theorem

(i) If $H : [-\infty, \infty]^n \to [0, 1]$ is an n-dimensional distribution function with one dimensional marginal distribution functions $F_1, \ldots, F_n : [-\infty, \infty] \to [0, 1]$ then there is an n-copula C such that

$$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}$. If $F_1, \ldots, F_n$, are continuous then C is unique; otherwise C is uniquely determined on $\mathrm{ran}(F_1) \times \cdots \times \mathrm{ran}(F_n)$.

(ii) For any one-dimensional distribution functions $F_1, \ldots, F_n$, and any n-copula C, the function H is an n-dimensional distribution function with one dimensional margins $F_1, \ldots, F_n$.

A nice probabilistic characterization of the three basic continuous t-norms is the next one: for events $E_1, \ldots, E_n$, let $P(E_1), \ldots, P(E_n)$ be their respective probabilities.

What can we say about the probability x of the intersection $\bigcap_{i=1}^{n} E_i$?

Then the probability $P(E_1 \cap \cdots \cap E_n)$ can be computed by means of a (in most cases unknown) copula C,

$$P(E_1 \cap \cdots \cap E_n) = C(P(E_1), \ldots, P(E_n)).$$

Due to the fact that

$$\mathrm{T}_{\mathbf{L}} \leqslant C \leqslant \mathrm{Min}$$

for any copula C we have the (best) estimation

$$\mathrm{T}_{\mathbf{L}}(P(E_1), \ldots, P(E_n)) \leqslant P(E_1 \cap \cdots \cap E_n) \leqslant \mathrm{Min}(P(E_1), \ldots, P(E_n)).$$

If the events $E_1, \ldots, E_n$ are jointly independent, then $C = \Pi$ and $P(E_1 \cap \cdots \cap E_n) = \Pi(P(E_1), \ldots, P(E_n))$.

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For any $\mathbf{x}, \mathbf{y} \in [0,1]$ we have

$$|C(x_1, \ldots, x_n) - C(y_1, \ldots, y_n)| \leqslant \sum_{i=1}^{n} |x_i - y_i|,$$

Proposition 9

A mapping $C : [0,1]^2 \to [0,1]$ is an Archimedean copula if and only if there is a convex strictly decreasing continuous function $t : [0,1] \to [0,\infty]$ with $t(1) = 0$ such that

$$C(x,y) = t^{-1}(\min(t(x) + t(y), t(0)).$$

C is a strict copula if and only if $t(0) = \infty$.

Proposition 10

Let **T** be a continuous Archimedean t-norm with an additive generator t such that its pseudo-inverse $t^{(-1)} : [0, \infty] \to [0, 1]$ given by $t^{(-1)}(u) = t^{-1}(\min(u, t(0))$. $t^{(-1)}$ is a completely monotone function on $]0, \infty[$, i.e., it has all derivatives on $]0, \infty[$ and these derivatives alter their signs, if and only if **T** is a general copula, i.e., for each $n > 1$ it is an n-copula.

Example 6

(i) A typical example of a general Archimedean copula is the product Π with an additive generator $t : [0, 1] \to [0, \infty]$ given by $t(x) = -\log x$.

(ii) *Hamacher product* (or *Ali-Mikhail-Haq copula* with parameter 0,) C_H is a general Archimedean copula generated by an additive generator $t : [0, 1] \to [0, \infty]$, given by $t(x) = \frac{1-x}{x}$.

Example 7

(i) The Farlie-Gumbel-Morgenstern family of copulas (symmetric but non-associative for $\lambda \neq 0$) is given by

$$C_\lambda(x, y) = xy + \lambda xy(1-x)(1-y) \text{ for } \lambda \in [-1, 1] .$$

(ii) The family of cubic 2-copulas (non-associative and asymmetric)is given by

$$C_{\alpha,\beta,\gamma,\delta}(x, y) =$$

$$= xy + xy(1-x)(1-y)(\alpha xy + \beta x(1-y) + \gamma y(1-x) + \delta(1-x)(1-y)).$$

Example 7

(iii) A linear combination $\mathrm{C} = p \cdot \Pi + (1-p) \cdot \mathrm{Min}$ is an example of a general non-associative symmetric copula (whenever $p \in]0,1[$). In its binary form C is given by

$$\mathrm{C}(x,y) = pxy + (1-p)\min(x,y) = x'(1 - p(1-y')).$$

Proposition 11

Let $t : [0,1] \to [0,\infty]$ be a convex strictly decreasing function such that $t(1) = 0$ (i.e., t is an additive generator of some 2-copula C_t) and let $D : [0,1] \to [0,1]$ be a convex function bounded from below by $\max(x, 1-x)$ (D is so-called dependence function). Then the mapping $C_{t,D} : [0,1]^2 \to [0,1]$ given by

$$C_{t,D}(x,y) = t^{-1}\left(\min\left(t(0), (t(x)+t(y)) \cdot D\left(\frac{t(x)}{t(x)+t(y)}\right)\right)\right)$$

is a 2-copula. This copula is called an *Archimax copula.*

Quasi-copulas

In order to generalize the notion of n-copulas, quasi-copulas of dimension n were introduced as special n-ary functions Q defined on $[0,1]^n$ such that for any continuous random variables $X_1, \ldots, X_n$ with support on $[0,1]$ there is a copula C such that

$$Q(F_{X_1}(t), \ldots, F_{X_n}(t)) = C(F_{X_1}(t), \ldots, F_{X_n}(t))$$

for all $t \in [0,1]$. In particular, if the random variables $X_1, \ldots, X_n$ have the same distribution function, we obtain that for each quasi-copula Q there is a copula C (with the same diagonal section), i.e., $Q(u, \ldots, u) = C(u, \ldots, u)$ for all $u \in [0,1]$.
Some of properties of copulas (1 is neutral element and they are non-decreasing 1-Lipschitz functions) are herited by quasi-copulas, but for example not the n-increasingness.

Theorem 5

A function $Q : [0,1]^n \to [0,1]$, $n \geq 2$, is a quasi-copula if and only if it is a 1-Lipschitz conjunctive aggregation function.

Disjunctive aggregation functions

Dual functions to conjunctive aggregation functions are called disjunctive aggregation functions. Disjunctive aggregation functions are those which are stronger than Max (i.e., than the strongest idempotent aggregation function). Though also disjunctive aggregation functions can be discussed on an arbitrary real interval $\mathbb{I}$, we will restrict our considerations to the case $\mathbb{I} = [0, 1]$, similarly as in the case of conjunctive aggregation functions. There is a genuine connection between these two classes of aggregation functions.

Lemma 1

Let $\varphi : [0, 1] \to [0, 1]$ be a decreasing bijection. Then the (extended) aggregation function A is disjunctive if and only if its transform A_φ is conjunctive.

Remark 2

(i) In fuzzy logic, an involutive decreasing bijection $\mathbf{neg} : [0,1] \to [0,1]$ (i.e., $\mathbf{neg}(\mathbf{neg}(x)) = x$ for all $x \in [0,1]$) plays the role of negation. For any aggregation function A it holds $(A_{\mathbf{neg}})_{\mathbf{neg}} = A$, i.e., **neg**-transformation brings a kind of duality into the class of aggregation functions. Due to Lemma 1, this duality also connects the class of disjunctive aggregation functions and the class of conjunctive aggregation functions. Moreover, if a t-norm $\mathbf{T}$ models the conjunction in fuzzy logic and **neg** models the negation, then the triplet $(\mathbf{T}, \mathbf{T}_{\mathbf{neg}}, \mathbf{neg})$ is called a de Morgan triplet.

Remark 2

(ii) For $N : [0,1] \to [0,1]$ given by $N(x) = 1 - x$, the dual aggregation function A^d can be introduced as $A^d = A_N$. Thus disjunctive aggregation functions are just dual functions to conjunctive aggregation functions, and therefore we can derive all their properties from the corresponding properties of conjunctive aggregation functions. So, for example, the smallest and the only idempotent disjunctive aggregation function is Max. In fuzzy logic N is called the standard negation.

(i) An extended aggregation function A is disjunctive if and only if

$$A_s \geqslant A \geqslant \text{Max}.$$

(ii) Each disjunctive aggregation function has 1 as its annihilator and if a disjunctive aggregation function has a neutral element e, then necessarily $e = 0$.

(iii) Disjunctors are aggregation functions on $[0, 1]$ with neutral element 0. Additively generated disjunctors are defined by means of strictly increasing functions $g : [0, 1] \to [0, \infty]$, $g(0) = 0$ as follows:

$$A_g(x_1, \ldots, x_n) = g^{(-1)}\left(\sum_{i=1}^{n} g(x_i)\right),$$

Definition 9

The dual aggregation function to a t-norm $\mathrm{T} : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$, i. e., an associative symmetric aggregation function $\mathrm{S} : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ with neutral element 0 is called a *triangular conorm*.

By duality, t-conorms have annihilator $a = 1$. For each t-conorm S, we have

$$\mathrm{Max} \leqslant \mathrm{S} \leqslant \mathrm{S}_{\mathbf{D}},$$

where

$$\mathrm{S}_{\mathbf{D}}(x_1, \ldots, x_n) = \begin{cases} x_i & \text{if for all } j \neq i,\ x_j = 0, \\ 1 & \text{else.} \end{cases}$$

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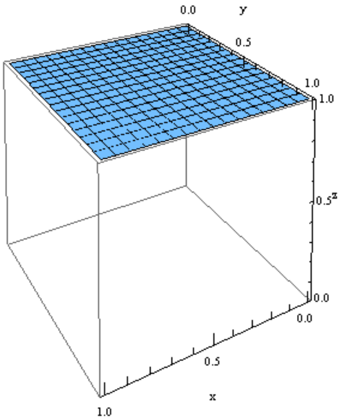

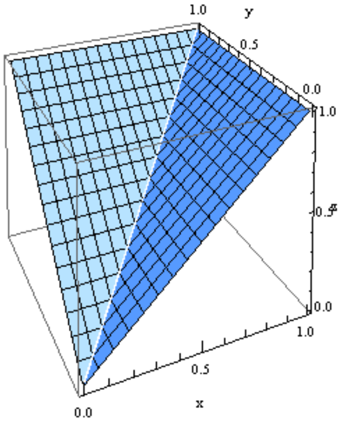

Figure: Figures of two basic t-conorms S_D (left) and Max (right)

The dual operator to the product Π is called the *probabilistic sum* and it is denoted by $S_{\mathbf{P}}$,

$$S_{\mathbf{P}}(x_1, \ldots, x_n) = 1 - \prod_{i=1}^{n}(1 - x_i).$$

The Łukasiewicz t-conorm $S_{\mathbf{L}}$ is often called the *bounded sum* because of

$$S_{\mathbf{L}}(x_1, \ldots, x_n) = \min\left(1, \sum_{i=1}^{n} x_i\right).$$

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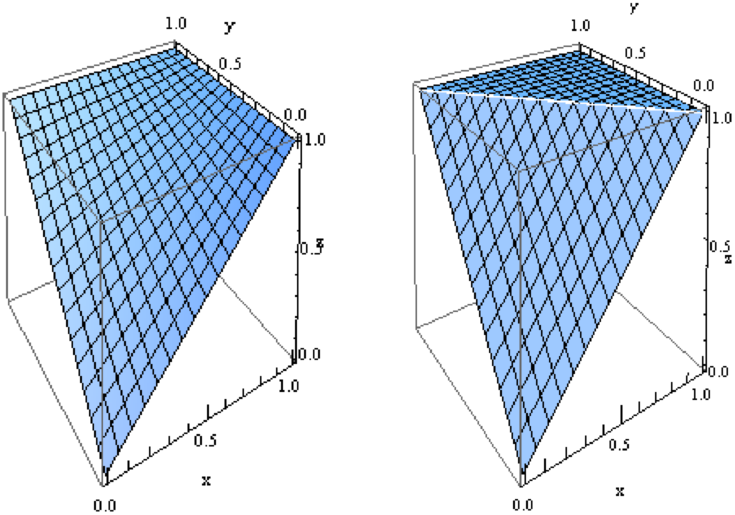

Figure: Figures of two basic t-conorms S_P (left) and S_L (right)

The family $(S_\lambda^{\mathbf{Y}})_{\lambda\in[0,\infty]}$ of *Yager t-conorms* is given by

$$S_\lambda^{\mathbf{Y}}(x,y)=\begin{cases} S_{\mathbf{D}}(x,y) & \text{if } \lambda=0,\\ \mathrm{Max}(x,y) & \text{if } \lambda=\infty,\\ \min\left(1,(x^\lambda+y^\lambda)^{\frac{1}{\lambda}}\right) & \text{otherwise.}\end{cases}$$

The family $(S_\lambda^{\mathbf{SW}})_{\lambda\in[-1,\infty]}$ of *Sugeno–Weber t-conorms* is given by

$$S_\lambda^{\mathbf{SW}}(x,y)=\begin{cases} S_{\mathbf{P}}(x,y) & \text{if } \lambda=-1,\\ S_{\mathbf{D}} & \text{if } \lambda=\infty,\\ \min(1,x+y+\lambda xy) & \text{otherwise.}\end{cases}$$

The family $(S_\lambda^{\mathbf{Y}})_{\lambda\in[0,\infty]}$ of *Yager t-conorms* is given by

$$S_\lambda^{\mathbf{Y}}(x,y) = \begin{cases} S_{\mathbf{D}}(x,y) & \text{if } \lambda = 0, \\ \mathrm{Max}(x,y) & \text{if } \lambda = \infty, \\ \min\left(1, (x^\lambda + y^\lambda)^{\frac{1}{\lambda}}\right) & \text{otherwise.} \end{cases}$$

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A continuous Archimedean t-conorm S is characterized by the diagonal inequality $\mathrm{S}(x, x) > x$ for all $x \in]0, 1[$, and it is always related to some continuous strictly increasing additive generator $s : [0, 1] \to [0, \infty]$, $s(0) = 0$

$$\mathrm{S}(x_1, \ldots, x_n) = s^{-1}\left(\min\left(s(1), \sum_{i=1}^{n} s(x_i)\right)\right).$$

Representation of continuous t-norms is reflected by the dual representation of continuous t-conorms,

$$S(x_1,\ldots,x_n)=\begin{cases} s_k^{-1}\left(\min\left(s_k(b_k),\sum\limits_{i=1}^{n} s_k(\max(x_i,a_k))\right)\right) \\ \quad \text{if } \max\ x_i\in\,]a_k,b_k[\,, \\ \max(x_1,\ldots,x_n) & \text{else,}\end{cases}$$

where $(]a_k,b_k[)_{k\in K}$ is a system of pairwise disjoint subintervals of $[0,1]$, and $s_k:[a_k,b_k]\to[0,\infty]$, $\;s_k(a_k)=0$

Uninorms

One of the prominent aggregation functions on $[0, \infty]$ is the product Π, which is symmetric, associative, and its neutral element $e = 1$ is an inner point of the domain $[0, \infty]$. This operator is not continuous, independently of the choice of the convention $0 \cdot \infty$ (0 or ∞). Further, restriction of the product to $[0, 1]$ is a triangular norm, while its restriction to $[1, \infty]$ acts as a t-conorm (i.e., neutral element is the lowest domain element). On $[0, 1]$, operators of the above mentioned nature have been introduced under the name uninorms by Yager and Rybalov.

Definition 10:
An aggregation function $\mathrm{U} : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ which is symmetric, associative and possesses a neutral element $e \in \,]0,1[$ is called a *uninorm*.

Proposition 12:
Let $U : \cup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ be a uninorm with neutral element$e \in \,]0,1[$. Denote $a_{\mathrm{U}} := U(0,1)$. Then the following hold:

(i) $a_{\mathrm{U}} \in \{0,1\}$.
(ii) a_{U} ia an annihilator of U.
(iii) U is not continuous.

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Proposition 13

Let $U : [0,1]^2 \to [0,1]$ be a (binary) uninorm with neutral element $e \in]0,1[$. Then there are three binary aggregation functions $\mathbf{T}, \mathbf{S}, \mathbf{H} : [0,1]^2 \to [0,1]$ such that $\mathbf{T}$ is a t-norm, $\mathbf{S}$ a t-conorm and $\mathbf{H}$ is a symmetric mean aggregation function, and for any $\mathbf{x} \in [0,1]^2$ it holds

$$U(\mathbf{x}) = \begin{cases} \mathbf{T}(\mathbf{x}) & \text{if } \mathbf{x} \in [0,e]^2, \\ \mathbf{S}(\mathbf{x}) & \text{if } \mathbf{x} \in [e,1]^2, \\ \mathbf{H}(\mathbf{x}) & \text{else.} \end{cases}$$

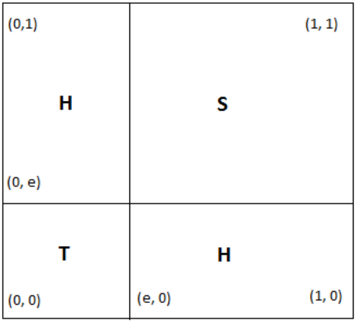

Figure: The representation of uninorm from Proposition 13

Proposition 14:
Let $e \in]0,1[$ be a given constant and let $\mathbf{T} = (< 0, e, \mathbf{T}_U >)$ and $\mathbf{S} = (< e, 1, \mathbf{S}_U >)$ be an ordinal sum t-norm and an ordinal sum t-conorm, respectively. Then the following holds.

(i) For any uninorm U characterized by $e, \mathbf{T}_U$ and $\mathbf{S}_U$, we have $\mathrm{T} < \mathrm{U}_{e,\mathrm{T},\mathrm{S}} \leqslant \mathrm{U} \leqslant \mathrm{U}_{\mathrm{T},\mathrm{S},e} < \mathrm{S}$, where

$$\mathrm{U}_{e,\mathrm{T},\mathrm{S}}(x,y) = \begin{cases} e \cdot \mathrm{T}\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot \mathrm{S}\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ \min(x,y) & \text{else,} \end{cases}$$

$$\mathrm{U}_{\mathrm{T},\mathrm{S},e}(x,y) = \begin{cases} e \cdot \mathrm{T}\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot \mathrm{S}\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ \max(x,y) & \text{else.} \end{cases}$$

(ii) $U_{e,\mathrm{T},\mathrm{S}}$ and $U_{\mathrm{T},\mathrm{S},e}$ are uninorms.

The smallest uninorm U_e:

$$U_e(x,y) := \begin{cases} 0 & \text{if } (x,y) \in [0,e[^2, \\ \max(x,y) & \text{if } (x,y) \in [e,1]^2, \\ \min(x,y) & \text{else }. \end{cases}$$

The strongest uninorm U^e:

$$U^e(x,y) := \begin{cases} \min(x,y) & \text{if } (x,y) \in [0,e[^2, \\ 1 & \text{if } (x,y) \in [e,1]^2, \\ \max(x,y) & \text{else }. \end{cases}$$

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$$U_e(x,y) := \begin{cases} 0 & \text{if } (x,y) \in [0,e[^2\,, \\ \max(x,y) & \text{if } (x,y) \in [e,1]^2\,, \\ \min(x,y) & \text{else}\,. \end{cases}$$

The strongest uninorm U^e:

$$U^e(x,y) := \begin{cases} \min(x,y) & \text{if } (x,y) \in [0,e[^2\,, \\ 1 & \text{if } (x,y) \in [e,1]^2\,, \\ \max(x,y) & \text{else}\,. \end{cases}$$

Two typical idempotent uninorms related to a given neutral element $e \in]0,1[$ are given by

$$\mathrm{U}_{e,\mathrm{Min},\mathrm{Max}}(x_1,\ldots,x_n) = \begin{cases} \mathrm{Max}(x_1,\ldots,x_n) & \text{if } \min\ x_i \geqslant e, \\ \mathrm{Min}(x_1,\ldots,x_n) & \text{else,} \end{cases}$$

and

$$\mathrm{U}_{\mathrm{Min},\mathrm{Max},e}(x_1,\ldots,x_n) = \begin{cases} \min(x_1,\ldots,x_n) & \text{if } \max\ x_i \leqslant e, \\ \max(x_1,\ldots,x_n) & \text{else.} \end{cases}$$

Example 8

Let $\varphi : [0,1] \to [0,1]$ be an increasing bijection and define $U^{(\varphi)} : [0,1]^2 \to [0,1]$ by

$$U^{(\varphi)}(x,y) := \begin{cases} \min(x,y) & \text{if } \varphi(x) + \varphi(y) \leqslant 1, \\ \max(x,y) & \text{else.} \end{cases}$$

Then $U^{(\varphi)}$ is a (left-continuous, conjunctive) uninorm with neutral element $e = \varphi^{-1}(0.5)$.

Theorem 6

A function $\mathrm{U} : \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$ is a uninorm continuous and cancellative on $\bigcup_{n \in \mathbb{N}}]0,1[^n$ if and only if there exists a monotone bijection $h : [0,1] \to [-\infty, \infty]$ such that

$$\mathrm{U}(x_1, \ldots, x_n) = h^{-1}\left(\sum_{i=1}^{n} h(x_i)\right),$$

with convention $+\infty + (-\infty) = -\infty$. The uninorm U is then called a generated uninorm with additive generator h.

Example 9

A typical example of a conjunctive generated uninorm is the 3-Π-operator E given by

$$\mathrm{E}(x_1,\ldots,x_n)=\frac{\prod_{i=1}^{n}x_i}{\prod_{i=1}^{n}x_i+\prod_{i=1}^{n}(1-x_i)},\ \text{with convention } \frac{0}{0}=0.$$

Its additive generator $h:[0,1]\to[-\infty,\infty]$ (necessarily unique up to a positive multiplicative constant) is given by

$$h(x)=\log\frac{x}{1-x}.$$

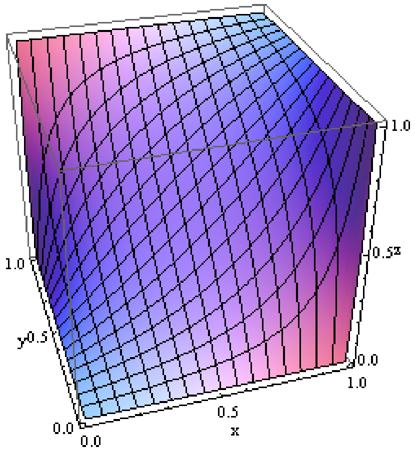

Figure: The uninorm 3-Π

Generated uninorms are always related to strict t–norms and strict t-conorms. For corresponding additive generators h, t, s of $\mathrm{U}, \mathrm{T}, \mathrm{S}$ we have the next relationships

$$h(x) = \begin{cases} -t\left(\frac{x}{e}\right) & \text{if } x \in [0, e], \\ s\left(\frac{x-e}{1-e}\right) & \text{if } x \in \,]e, 1]; \end{cases}$$

The freedom in the choice of an additive generator of a given strict t-norm T and a given strict t-conorm S allows to construct a parameterized class of (conjunctive) generated uninorms related to T and S.

Let $t : [0, 1] \to [0, \infty]$ be an (unique) additive generator of a given strict t-norm T such that $t(0.5) = 1$, and similarly, let $s : [0, 1] \to [0, \infty], \quad s(0.5) = 1$, be an additive generator of a given strict t-conorm S. For a given parameter $p \in]0, \infty[$, define an additive generator $h_p : [0, 1] \to [-\infty, \infty]$ related to a generated uninorm U_p,

$$h_p(x) = \begin{cases} -t\left(\frac{x}{e}\right) & \text{if } x \in [0, e]; \\ p \cdot s\left(\frac{x-e}{1-e}\right) & \text{if } x \in]e, 1]. \end{cases}$$

For each $p \in]0, \infty[$, U_p is related to T and S. Further, the family $(\mathrm{U}_p)_{p \in]0,\infty[}$ is non-decreasing and its limit member is

$$\mathrm{U}_0 = \lim_{p \to 0^+} \mathrm{U}_p = \mathrm{U}_{e,\mathrm{T},\mathrm{S}}.$$

The other limit member

$$\mathrm{U}_\infty = \lim_{p \to \infty} \mathrm{U}_p$$

coincides with $\mathrm{U}_{\mathrm{T},\mathrm{S},e}$ on $\bigcup_{n \in \mathbb{N}}]0, 1]^n$.

The associativity of uninorms, t–norms and t-conorms allow to derive for each uninorm U the n–ary operator $U^{(n)}$ from the binary operator $U^{(2)}$. Let $\mathbf{T}$ and $\mathbf{S}$ be a t–norm and a t-conorm, respectively, which are related to U. Then

$$U^{(n)}(x_1,\ldots,x_n) =$$
$$U^{(2)}\left(\mathbf{T}\left(\min(x_1,e),\ldots,\min(x_n,e)\right),\mathbf{S}\left(\max(x_1,e),\ldots,\max(x_n,e)\right)\right).$$

Definition 11 (Nullnorm)

A symmetric associative aggregation function $\mathrm{V} : [0,1]^2 \to [0,1]$ is called a *nullnorm* if there is an element $a \in]0,1[$ such that

$$\mathrm{V}(x,0) = x \quad \text{for all} \quad x \leqslant a, \quad \mathrm{V}(x,1) = x \quad \text{for all} \quad x \geqslant a.$$

$$V(x_1,\ldots,x_n)=\begin{cases} a\cdot S_V\left(\frac{x_1}{a},\ldots,\frac{x_n}{a}\right) & \text{if } \max\ x_i\leqslant a,\\ a+(1-a)\cdot T_V\left(\frac{x_1-a}{1-a},\ldots,\frac{x_n-a}{1-a}\right) & \text{if } \min\ x_i\geqslant a,\\ a & \text{else.}\end{cases}$$

Example 10

We define nullnorm V : $[0,1]^2 \to [0,1]$ in the following way

$$V(x,y) = \begin{cases} \mathrm{Max}(x,y) & \text{if } (x,y) \in [0,1/3]^2, \\ \frac{3xy-x-y+1}{2} & \text{if } (x,y) \in [1/3,1]^2, \\ 1/3 & \text{otherwise.} \end{cases}$$

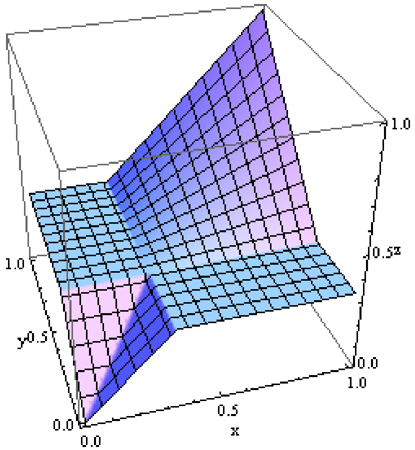

Figure: The nullnorm from Example 10

For a given annihilator $a \in]0,1[$, there is a unique idempotent nullnorm (related to $S = \text{Max}$ and $T = \text{Min}$), namely Med_a (a-median)

$$\text{Med}_a(x_1, \cdots, x_n) = \mathrm{med}(x_1, a, x_2, a, \cdots, x_{n-1}, a, x_n).$$

These important operators were introduced by Fung and Fu and further studied by J. Fodor.

Proposition 15

An aggregation function $\mathrm{V} : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ is a nullnorm if and only if there is a t–norm T, a t-conorm S and an element $a \in \,]0,1[$ such that V is a composed aggregation function,

$$\mathrm{V} = \mathrm{Med}_a(\mathrm{T}, \mathrm{S}),$$

that is,

$$\mathrm{V}(x_1, \ldots, x_n) = \operatorname{med}\left(\mathrm{T}(x_1, \ldots, x_n), \mathrm{S}(x_1, \ldots, x_n), a\right).$$

Example 11

Applying Proposition 15 to the Łukasiewicz t–norm $T_{\mathbf{L}}$ and t-conorm $S_{\mathbf{L}}$, we can find an interesting nullnorm $V_{\mathbf{L},a} : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ given by

$$V_{\mathbf{L},a}(x_1,\ldots,x_n) := \mathrm{med}\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i - (n-1), a\right).$$

This nullnorm is Archimedean as all nullnorms based on an Archimedean t-norm T and an Archimedean t-conorm S. Moreover, it is also continuous as all nullnorms based on a continuous t-norm T and a continuous t–conorm S.

Proposition 16

Let $A : \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$ be a continuous associative aggregation function and let $A(0,1) = A(1,0) = a$. Then:

(i) if $a = 0$, A is a t-norm;

(ii) if $a = 1$, A is a t-conorm;

(iii) if $a \in \,]0,1[$, A is a nullnorm with annihilator a.

Gamma operators

The gamma operators

$$\Gamma_\gamma : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$$

were introduced by Zimmermann and Zysno and applied in the car control.

For a parameter $\gamma \in [0,1]$, the gamma operator Γ_γ is given by $\Gamma_\gamma = \Pi^{1-\gamma} S_P^\gamma$, that is,

$$\Gamma_\gamma(x_1,\ldots,x_n) = \left(\prod_{i=1}^n x_i\right)^{1-\gamma} \left(1 - \prod_{i=1}^n (1-x_i)\right)^\gamma .$$

Exponential convex T-S-operators

Gamma operators are a special subclass of so called exponential convex T-S-operators, that is, of weighted geometric means of a t-norm T, and a t-conorm S (not necessarily a dual pair),
$\mathrm{E}_{\mathrm{T},\mathrm{S},\gamma} : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$,

$$\mathrm{E}_{\mathrm{T},\mathrm{S},\gamma}(x_1,\ldots,x_n) = (\mathrm{T}(x_1,\ldots,x_n))^{1-\gamma}\,(\mathrm{S}(x_1,\ldots,x_n))^{\gamma}\,.$$

Linear convex T-S-operators

Another composed aggregation approach based on t-norms and t-conorms is related to the weighted arithmetic mean (as the outer operator). A linear convex T-S-operator $\mathrm{L}_{\mathrm{T},\mathrm{S},\gamma} : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ is given by

$$\mathrm{L}_{\mathrm{T},\mathrm{S},\gamma}(x_1,\ldots,x_n) = (1-\gamma)\cdot \mathrm{T}(x_1,\ldots,x_n) + \gamma\cdot \mathrm{S}(x_1,\ldots,x_n).$$

We consider following linear convex operator

$$L_{T_{\mathbf{L}},\mathrm{Max},0.3}(x,y) = 0.7 \cdot T_{\mathbf{L}}(x,y) + 0.3 \cdot \mathrm{Max}(x,y)$$

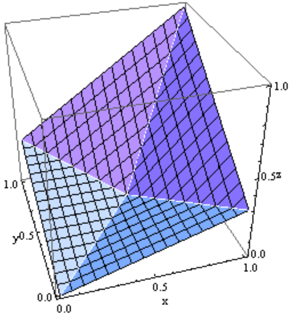

Figure: Linear convex $L_{T_{\mathbf{L}},\mathrm{Max},0.3}$-operator

Symmetric sums related to t-norms and t-conorms

Yager and Filev

The symmetric sum $\mathrm{T}^{\sharp}$

$$\mathrm{T}^{\sharp}(x_1,\ldots,x_n)=\frac{\mathrm{T}(x_1,\ldots,x_n)}{\mathrm{T}(x_1,\ldots,x_n)+\mathrm{T}(1-x_1,\ldots,1-x_n)},$$

where convention $\frac{0}{0}=\frac{1}{2}$ (for symmetric sums) can be replaced by some other convention, e. g., $\frac{0}{0}=0$. Especially, if $\mathrm{T}=\Pi$, we obtain the $3-\Pi$–operator $\mathrm{E}=\Pi^{\sharp}$

t-conorm-based symmetric sums can be introduced by

$$S^\sharp(x_1,\dots,x_n) = \frac{S(x_1,\dots,x_n)}{S(x_1,\dots,x_n)+S(1-x_1,\dots,1-x_n)}.$$

$$\mathrm{Min}^\sharp(x_1,\dots,x_n) = \frac{x'_1}{x'_1+1-x'_n},$$

and

$$\mathrm{Max}^\sharp(x_1,\dots,x_n) = \frac{x'_n}{x'_n+1-x'_1},$$

where $(x'_1,\dots,x'_n)$ is a non-decreasing permutation of $(x_1,\dots,x_n)$.

t-conorm-based symmetric sums can be introduced by

$$S^\sharp(x_1,\ldots,x_n) = \frac{S(x_1,\ldots,x_n)}{S(x_1,\ldots,x_n) + S(1-x_1,\ldots,1-x_n)}.$$

$$\mathrm{Min}^\sharp(x_1,\ldots,x_n) = \frac{x_1'}{x_1' + 1 - x_n'},$$

and

$$\mathrm{Max}^\sharp(x_1,\ldots,x_n) = \frac{x_n'}{x_n' + 1 - x_1'},$$

where $(x_1',\ldots,x_n')$ is a non-decreasing permutation of $(x_1,\ldots,x_n)$.

Thanks for your attention!