Aggregation Functions
Part III.

Radko Mesiar
STU Bratislava, Faculty of Civil Engineering
Dept. of Mathematics

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Olomouc

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Quantitative weights

How to introduce weights (importances) into aggregation? For an input vector $\mathbf{x} = (x_1, \ldots, x_n)$, the corresponding weights $w_1, \ldots, w_n$ can be understood as cardinalities of single inputs $x_1, \ldots, x_n$, respectively. We will deal with weighting vectors $\mathbf{w} = (w_1, \ldots, w_n)$, $w_i \in [0, \infty[\,, i \in \{1, \ldots, n\}$, and $\sum_{i=1}^{n} w_i > 0$. If $\sum_{i=1}^{n} w_i = 1$, $\mathbf{w}$ will be called a *normal weighting vector*.

Weighted aggregation function

For an extended aggregation function $A : \bigcup_{n\in\mathbb{N}} I^n \to I$, and a weighting vector $\mathbf{w} = (w_1, \ldots, w_n)$ (for some $n \in \mathbb{N}$), we will discuss an n–ary aggregation function $A_{\mathbf{w}} : I^n \to I$, which will be called a *weighted aggregation function*.

For a weighting triangle $\Delta = (w_{i,n} | n \in \mathbb{N}, i \in \{1, \ldots, n\})$ for each n, $(w_{1,n}, \ldots, w_{n,n})$ is a weighting vector,

$$A_\Delta : \bigcup_{n\in\mathbb{N}} I^n \to I$$

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We expect the next quite natural properties of weighted aggregation functions.

(W1) If $\mathbf{w} = (1, \ldots, 1) = \mathbf{1}$ then

$$A_{\mathbf{1}}(x_1, \ldots, x_n) = A(x_1, \ldots, x_n)$$

for all $(x_1, \ldots, x_n) \in I^n$.

(W2) For any $(x_1, \ldots, x_n) \in I^n$ and any $\mathbf{w} = (w_1, \ldots, w_n)$,

$$A_{\mathbf{w}}(x_1, \ldots, x_n) = A_{\mathbf{w}^*}(x_{m_1}, \ldots, x_{m_k}),$$

where $\{m_1, \ldots, m_k\} = \{i \in \{1, \ldots, n\} \mid w_i > 0\}$, $m_1 < \ldots < m_k$, $\mathbf{w}^* = (w_{m_1}, \ldots, w_{m_k})$.

(W3) If $\mathbf{w}$ is a normal weighting vector then $A_{\mathbf{w}}$ is an idempotent aggregation function.

Observe that (W1) simply embeds the aggregation function A into weighted aggregation functions. Further, due to (W2), a zero weight w_i in a weighting vector $\mathbf{w}$ means that we can omit the corresponding score x_i (and the weight $w_i = 0$) from aggregation. Finally, the property (W3) expresses the standard boundary condition for extended aggregation functions, namely, that the aggregation of a unique input x results in x, $A(x) = x$. Then $A_{\mathbf{w}}(x_1, \ldots, x_n)$ with $\sum_{i=1}^{n} w_i = 1$ can be seen as the aggregation of x with cardinality $\sum_{i=1}^{n} w_i = 1$, i.e., $A_{\mathbf{w}}(x, \ldots, x) = A(x) = x$, which is exactly the idempotency of the function $A_{\mathbf{w}}$.

Weighted sum

The standard summation on $[0, +\infty]$ can be understood as a typical aggregation on $[0, +\infty]$. For a given weighting vector $\mathbf{w} = (w_1, \dots, w_n)$, the weighted sum $\sum_{i=1}^{n} w_i x_i$ is simply the sum of inputs x_i transformed by means of weights w_i into new inputs $y_i = w_i x_i$. Note that the common multiplication of reals applied in the next transformation can be straightforwardly deduced from the original summation (and the standard order of real numbers), i.e., for $w \geq 0$, $x \in [0, +\infty]$

$$w \cdot x = \sup \left(y \in [0, +\infty] \mid \exists\, i, j \in \mathbb{N}, \frac{i}{j} < w \text{ and } u \in [0, +\infty] \text{ such that } \sum^{j} u < x \text{ and } y = \sum^{i} u \right)$$

The weighted sum $\sum_{i=1}^{n} w_i x_i$ for weights w_i such that $\sum_{i=1}^{n} w_i = 1$ is just the weighted arithmetic mean. The above discussed approach can be applied to any continuous symmetric associative aggregation function defined on $I = [0, c]$ with neutral element 0, as, for example, to any continuous t–conorm S.

Weighted t–conorm

The weighted t–conorm $S_{\mathbf{w}} : [0,1]^n \to [0,1]$, where $n = \dim \mathbf{w}$, is simply defined as

$$S_{\mathbf{w}}(x_1, \dots, x_n) = S(w_1 \odot x_1, \dots, w_n \odot x_n)$$

where the transformed input data $w_i \odot x_i$ are obtained from the weights w_i and the original inputs x_i by means of a binary operation $\odot : [0, +\infty[\times [0,1] \to [0,1]$,

$$w \odot x = \sup \left(y \in [0,1] \mid \exists\, i, j \in \mathbb{N}, \ \frac{i}{j} < w \text{ and } u \in [0,1] \text{ such that} \right.$$

$$\left. S(\underbrace{u, \dots, u}_{j-times}) < x \text{ and } y = S(\underbrace{u, \dots, u}_{i-times}) \right).$$

Note that $0 \odot x = 0$ and $1 \odot x = x$ for all $x \in [0, 1]$. In the case when S has unit multipliers, i.e., $S(x, y) = 1$ for some $x,\ y \in [0, 1[$ we should require $\sum_{i=1}^{n} w_i \geq 1$ to keep the boundary condition $S_{\mathbf{w}}(1, \dots, 1) = 1$. Obviously, the weighted t–conorm $S_{\mathbf{w}}$ for any continuous t–conorm S fulfills axioms (W1), (W2), (W3).

- $Max_{\mathbf{w}}(x_1, \ldots, x_n) = \max(x_i \mid w_i > 0)$, (due to $w \odot x = x$ if $w > 0$);
- $S_{\mathbf{w}}$ is lower semi–continuous (left continuous);
- $S_{\mathbf{w}}$ (with some nontrivial $w_i \notin \{0, 1\}$) is continuous if and only if either $S = Max$ or S is a continuous Archimedean t–conorm;

- If S is continuous Archimedean t–conorm with an additive generator $g : [0,1] \to [0,+\infty]$, and $\mathbf{w}$ is a normal weighting vector, then $S_{\mathbf{w}}(x_1,\dots,x_n) = g^{-1}\left(\sum_{i=1}^{n} w_i g(x_i)\right)$, i.e., $S_{\mathbf{w}}$ is a weighted quasi–arithmetic mean (because $w \odot x = g^{-1}(w \cdot g(x))$ for $w \in [0,1]$). It is either cancelative (if S is a nilpotent t–conorm; e.g., the Yager t–conorm for $p = 2$, leads to the weighted quadratic mean) or it has annihilator $a = 1$ (if S is a strict t–conorm).

Weighted t–norms

Weighted t–norms can be defined by the duality, i.e.,

$$T_{\mathbf{w}}(x_1, \ldots, x_n) = 1 - S_{\mathbf{w}}(1 - x_1, \ldots, 1 - x_n),$$

where T is an arbitrary continuous t–norm and $S = T^d$ is the corresponding dual t–conorm. Note that axioms (W1), (W2) and (W3) are also fulfilled for weighted t–norms. Similarly as in the case of weighted t–conorms we have the following facts:

- $Min_{\mathbf{w}}(x_1, \ldots, x_n) = \min(x_i \mid w_i > 0)$;
- $T_{\mathbf{w}}$ is upper semi–continuous (right continuous);
- $T_{\mathbf{w}}$ (with some nontrivial $w_i \notin \{0, 1\}$) is continuous if and only if either $T = \min$ or T is a continuous Archimedean t–norm;

- If T is a continuous Archimedean t–norm with an additive generator $f : [0,1] \to [0,+\infty]$, and **w** is a normal weighting vector, then

$$T_{\mathbf{w}}(x_1, \ldots, x_n) = f^{-1}\left(\sum_{i=1}^{n} w_i f(x_i)\right),$$

i.e., $T_{\mathbf{w}}$ is a weighted quasi–arithmetic mean. It is cancelative whenever T is nilpotent and it has annihilator 0 whenever T is a strict t–norm.

For example, for the product t–norm Π, the relevant normal weighted function $\Pi_{\mathbf{w}}$ is just the weighted geometric mean.

Observe that if $\sum_{i=1}^{n} w_i = n$, then for a continuous Archimedean t–norm T generated by an additive generator f the corresponding weighted operator is given by $T_{\mathbf{w}}(x_1, \ldots, x_n) = f^{(-1)}\left(\sum_{i=1}^{n} w_i f(x_i)\right)$ what is just a weighted generated t–norm as proposed by Dubois and Prade in 1985.

In statistics, starting with integer weights n_i, which are simply frequencies of observations x_i, the weighted mean is

$$M_{\mathbf{n}}(x_1, \ldots, x_n) = \frac{\sum_{i=1}^{n} n_i\, x_i}{\sum_{i=1}^{n} n_i},$$

where $\mathbf{n} = (n_1, \ldots, n_n)$. Because of the strong idempotency of the standard arithmetic mean, $M_{\mathbf{n}}$ can be easily generalized into the form

$$M_{\mathbf{w}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i\, x_i, \quad w_i \geq 0, \ \sum_{i=1}^{n} w_i = 1.$$

The previous property of the standard arithmetic mean we can apply on any symmetric strongly idempotent extended aggregation function A. The strong idempotency of a symmetric extended aggregation function A allows to introduce integer and rational quantitative weights – simply looking at them as cardinalities. In fact, we repeat the standard approach applied to the arithmetic mean as mentioned above. Indeed, for inputs $x_1, \ldots, x_n \in I$ and integer weights $\mathbf{w} = (w_1, \ldots, w_n) \in (\mathbb{N} \cup \{0\})^n$, we put

$$A_{\mathbf{w}}(x_1, \ldots, x_n) = A(\underbrace{x_1, \ldots, x_1}_{w_1 - times}, \underbrace{x_2, \ldots, x_2}_{w_2 - times}, \ldots, \underbrace{x_n, \ldots, x_n}_{w_n - times}).$$

If $\mathbf{k} = (k, \ldots, k)$, $k \in \mathbb{N}$, is a constant weighting vector, the symmetry and the strong idempotency of A result in $A_{\mathbf{k}}(\mathbf{x}) = A(\mathbf{x})$. This fact allows to define consistently the weighted aggregation in the case of rational weights $w_i \in \mathbb{Q}^+$. In that case we find such an integer $k \in \mathbb{N}$ that $k\, w_i \in \mathbb{N} \cup \{0\}$ for all $i = 1, \ldots, n$, and we put

$$A_{\mathbf{w}}(\mathbf{x}) = A_{k\mathbf{w}}(\mathbf{x}).$$

$A_{\mathbf{w}} = A_{p\mathbf{w}}$ for each positive rational p and each rational weighting vector $\mathbf{w} \in (\mathbb{Q}^{+})^{n}$, $\mathbf{w} \neq (0, \ldots, 0)$. Therefore we can deal with normed (rational) weighting vectors only, that is, we may suppose that $\sum_i w_i = 1$. The last problem we need to solve, is the case when also irrational weights w_i are admitted.

Definition 1

Let $A : \bigcup_{n \in \mathbb{N}} I^n \to I$ be a symmetric strongly idempotent extended aggregation function. For any non–zero weighting vector $\mathbf{w} = (w_1, \ldots, w_n) \in [0, \infty[^n$, the corresponding n–ary weighted function $A_{\mathbf{w}} : I^n \to I$ is defined as follows:

(i) If all weights w_i are rational, we apply previous formulas.

(ii) If there is some irrational weight w_i, denote $\mathbf{w}^* = (w_1^*, \ldots, w_n^*)$ the corresponding normed weighting vector, that is,

$$\mathbf{w} = \left(\sum_i w_i \right) \mathbf{w}^*.$$

Definition 1

For any $m \in \mathbb{N}$, $i \in \{1, \ldots, n\}$, let

$$w_i^{(m)} = \min\left(\frac{j}{m!} \,|\, j \in \mathbb{N} \cup \{0\},\ \frac{j}{m!} \geq w_i^*\right),$$

and $\mathbf{w}^{(m)} = (w_1^{(m)}, \ldots, w_n^{(m)})$.
Then $w_i^{(m)} \in \mathbb{Q}^+$ and $\sum_i w_i^{(m)} \geq 1$ for all $m \in \mathbb{N}$ (and if already all weights $w_i^* \in \mathbb{Q}^+$, then also $w_i^{(m)} = w_i^*$ for all i and all sufficiently large m) and we define

$$A_{\mathbf{w}}(\mathbf{x}) = \liminf_{m \to \infty} A_{\mathbf{w}^{(m)}}(\mathbf{x}) \quad \text{for all } \mathbf{x} \in I^n.$$

Proposition 1

Let $\triangle = (\mathbf{w}^{(n)})_{n=1}^{\infty}$ be a weighting triangle, i.e., for each $n \in \mathbb{N}$, let $\mathbf{w}^{(n)} = (w_{1,n}, \ldots, w_{n,n})$ be a non–zero weighting vector. Under the notations and requirements in Definition 1, define the function $A_\triangle : \bigcup\limits_{n\in\mathbb{N}} I^n \to I$, $A_\triangle(\mathbf{x}) = A_{\mathbf{w^{(n)}}}(\mathbf{x})$, whenever $\mathbf{x} \in I^n$. Then $A_\triangle$ is a well defined idempotent extended aggregation function.

Note that the approach allowing to introduce integer (rational) weights as given in previous formulas was already applied to decomposable idempotent symmetric extended aggregation functions by Fodor and Roubens in 1994. However, our results cover a wider class of symmetric strongly idempotent extended aggregation functions.

Let $g : [0,1] \to [0,1]$ be given by $g(x) = 2\,x - x^2$. Define the function $A : \bigcup_{n\in\mathbb{N}} I^n \to I$ by $A(x_1,\ldots,x_n) = \sum_{i=1}^{n} \left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right) x'_i$, where x'_i is the i–th order statistics from the sample $(x_1,\ldots,x_n)$. Then A, which is an extended OWA operator, is a symmetric strongly idempotent extended aggregation function which is not decomposable.

$A(1,2,3) = g(\frac{1}{3})\cdot 1 + (g(\frac{2}{3}) - g(\frac{1}{3}))\cdot 2 + (g(1) - g(\frac{2}{3}))\cdot 3 = \frac{5}{9}\cdot 1 + \frac{3}{9}\cdot 2 + \frac{1}{9}\cdot 3 = \frac{14}{9}$; $A(1,2) = g(\frac{1}{2})\cdot 1 + (g(1) - g(\frac{1}{2}))\cdot 2 = \frac{5}{4}$; $A\left(A(1,2), A(1,2), 3\right) = \frac{13}{9}$.

Further observe that the limit in formula in Definition 1 need not exist, in general.

Let $g : [0,1] \to [0,1]$ be given by $g(x) = 2x - x^2$. Define the function $A : \bigcup_{n\in\mathbb{N}} I^n \to I$ by $A(x_1,\dots,x_n) = \sum_{i=1}^{n} \left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right) x_i'$, where x_i' is the i–th order statistics from the sample $(x_1,\dots,x_n)$. Then A, which is an extended OWA operator, is a symmetric strongly idempotent extended aggregation function which is not decomposable.
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Qualitative weights

The idea of qualitative weights incorporation into aggregation is linked to the transformation of the inputs by means of the corresponding weights from [0, 1] (as parameters expressing the importance of the corresponding input coordinates/criteria),

$$A_{\mathbf{w}}(\mathbf{x}) = A(h(w_1, x_1), \ldots, h(w_n, x_n)),$$

where $h : [0,1] \times I \to [0,1]$ is an appropriate binary function. This idea was already applied, e.g., in expert systems, and for $I = [0,1]$ it was introduced by Yager in 2001, where h is a function called a RET operator.

Properties of function h

To ensure (W1), the following property of h is required:
(RET1) $h(1, x) = x$ for all $x \in I$.
Similarly, to ensure (W2), A is supposed to have a neutral element e and then
(RET2) $h(0, x) = e$ for all $x \in I$.
Further, to ensure the monotonicity of $A_{\mathbf{w}}$, one requires
(RET3) $h(w, \cdot)$ is non-decreasing for all $w \in [0, 1]$.
Finally, to ensure the boundary conditions of aggregation functions, one requires
(RET4) $h(\cdot, b)$ is non-decreasing for all $b \geq e$;
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Proposition 2

Let $A : \bigcup_{n \in \mathbb{N}} I^n \to I$ be an extended aggregation function with neutral element e and let $h : [0, 1] \times I \to I$ fulfil properties (RET1)–(RET5). For any weighting vector $\mathbf{w} \in [0, 1]^n$, $\max w_i = 1$, define the function $A_{\mathbf{w}}$ by previous formula. Then $A_{\mathbf{w}}$ is an n-ary aggregation function satisfying axioms (W1), (W2) and (W3).

Example of RET operator

Typical example of a RET operator given by

$$h: \; h(w, x) = (x - e)w + e.$$

If $e = 0$ and $I = [0, 1]$, any binary semicopula fulfills (RET1)–(RET5), while for $e = 1$, any fuzzy implication satisfying the neutrality principle, which corresponds to (RET1), can be applied.

In some special cases, h can also be defined for weights exceeding 1, that is, h maps $[0,\infty[\times I$ into I. For example, recall the introduction of weights for continuous t–norms and t–conorms. Take, e.g., a strict t-norm T with an additive generator $f:[0,1]\to[0,\infty]$. Then $h(w,x)=f^{-1}\left(wf(x)\right)$, and for an arbitrary weighting vector $\mathbf{w}$ (the only constraint is $\sum w_i>0$) we can put $T_{\mathbf{w}}(\mathbf{x})=f^{-1}\left(\sum w_i f(x_i)\right)$.

Recall that special classes of anonymous (i.e., symmetric) aggregation functions with neutral elements appropriate for qualitative weights incorporation are triangular norms, triangular conorms, uninorms.

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For a fixed score $(x_1, x_2, \ldots, x_n)$, we will look for an appropriate "projection" to the subspace of all unanymous scores $(r, r, \ldots, r)$, $r \in I$, applying some defuzzification method. Thus, in fact, we will define a function with inputs and outputs from some real interval I. In the special case of the *MOM* defuzzification method we will rediscover a generalization of the penalty method introduced by Yager and Rybalov in 1997.

Dissimilarity function

For a fixed real interval I and $n \in \mathbb{N}$ we introduce a dissimilarity function $D : I^n \times I^n \to [0, \infty[$ by

$$D(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} D_i(x_i, y_i),$$

where all $D_i : I^2 \to [0, \infty[$ are particular one-dimensional dissimilarity functions, $D_i(x, y) = K_i(f_i(x) - f_i(y))$, with $K_i :]-\infty, \infty[\to]-\infty, \infty[$ a convex function with the unique minimum $K_i(0) = 0$, and $f_i : I \to]-\infty, \infty[$, a strictly monotone continuous real function. If K_i are even functions then D is a metric on I^n.

Definition 2

For a given dissimilarity D, the function $U : I^n \to [0,1]^I$ which assigns to a score $\mathbf{x}$ the fuzzy subset $U_{\mathbf{x}}$ of I with the membership function

$$U_{\mathbf{x}}(r) = \frac{1}{1 + D(\mathbf{x}, \mathbf{r})},$$

where $\mathbf{r} = (r, \ldots, r)$, will be called a D–fuzzy utility function.

Proposition 3

Each D–fuzzy utility function U assigns to each score $\mathbf{x} \in I^n$ a continuous quasi-convex fuzzy quantity $U_{\mathbf{x}}$, i.e., for all $r, s \in I$, $\lambda \in [0, 1]$,

$$U_{\mathbf{x}}(\lambda \cdot r + (1 - \lambda)s) \geq \min(U_{\mathbf{x}}(r), U_{\mathbf{x}}(s)),$$

and thus for any $\alpha \in]0, 1]$ the α–cut $U_{\mathbf{x}}^{\alpha} = \{r \in I \mid U_{\mathbf{x}}(r) \geq \alpha\}$ is a closed subinterval of I in the standard topology.

For each defuzzification method *DEF* acting on quasi–convex (continuous) fuzzy quantities, we can assign to each score $\mathbf{x}$ a characteristic $DEF(U_{\mathbf{x}})$. Supposing that for any fuzzy quantity Q, $DEF(Q) \in supp(Q)$, $DEF(U)$ is an $I^n \to I$ function. In general, this function must be neither idempotent nor non–decreasing. *MOM* defuzzification method (Mean of Maxima) yields both of these properties and thus we will illustrate our approach on the *MOM* defuzzification. Note that $MOM(U)(\mathbf{x}) = \frac{1}{2}\left(\inf\, U_{\mathbf{x}}^{\alpha^*} + \sup\, U_{\mathbf{x}}^{\alpha^*}\right)$, where $\alpha^* = \sup\{\alpha \in]0,1] \mid U_{\mathbf{x}}^{\alpha} \neq \emptyset\}$.

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Example 1

(i) For $D(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} (f(x_i) - f(y_i))^2$, we have $A_D(\mathbf{x}) = f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right)$, i.e., A_D is a quasi–arithmetic mean.

(ii) For $D(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} \mid x_i - y_i \mid$, we have $A_D(\mathbf{x}) = \mathrm{med}(x_1,\ldots,x_n)$, i.e., the median operator.

(iii) For $n = 2$, $D(\mathbf{x},\mathbf{y}) = \mid x_1 - y_1 \mid + (x_2 - y_2)^2$, we have $A_D(\mathbf{x}) = \mathrm{med}(x_1, x_2 - 1/2, x_2 + 1/2)$.

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Example 1

(iv) For $D(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} D_c(x_i, y_i)$, where
$D_c(x,y) = \begin{cases} c(y-x), & \text{if } x \leq y \\ x-y, & \text{else} \end{cases}$, A_D is the α–quantil (order statistics) with $\alpha = \frac{1}{(1+c)}$.

(v) For $D(\mathbf{x},\mathbf{y}) = \max_{i=1}^{n} \mid x_i - y_i \mid$ we have $A_D(\mathbf{x}) = \frac{\min_i x_i + \max_i x_i}{2}$, i.e., A_D is a special OWA operator.

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Weighted dissimilarity $D_{\mathbf{w}}$

Dissimilarity based approach to aggregation functions allows a straightforward incorporation of weights. For a weighting vector $\mathbf{w} = (w_1, \ldots, w_n)$, the weighted dissimilarity $D_{\mathbf{w}}$ will be given by $D_{\mathbf{w}}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} w_i \, D_i(x_i, y_i)$ and then we will apply Definition 3 to obtain the corresponding weighted aggregation function. In the case of standard aggregation functions we have obtained in Example 1 (i) and (ii), the standard weighted quasi–arithmetic mean and the weighted median are obtained, respectively. The weighted aggregation function corresponding to Example 1 (iii) is given by $A_{D_{\mathbf{w}}}(\mathbf{x}) = \mathrm{med}(x_1, x_2 - \frac{w_1}{2w_2}, x_2 + \frac{w_1}{2w_2})$.

Definition 4

Let $A_{\mathbf{w}} : I^n \to I$ be a weighted aggregation function. Then the operator $A'_{\mathbf{w}} : I^n \to I$ given by $A'_{\mathbf{w}}(\mathbf{x}) = A_{\mathbf{w}}(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, where $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation for which $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)}$, will be called an OWAF.

Evidently, starting from a weighted arithmetic mean $M_{\mathbf{w}}$, Definition 4 yields the OWA operator $M'_{\mathbf{w}}$. Note that the ordered weighted *t*–norm $T'_{(0,1,1)}(x,y,z) = \beta \cdot \gamma$ and its dual ordered weighted *t*–conorm $S'_{(1,1,0)}(x,y,z) = \alpha + \beta - \alpha\beta$, $\alpha = \min(x,y,z)$, $\beta = \text{med}(x,y,z)$, $\gamma = \max(x,y,z)$, were found to be important in the study of fuzzy preference structures.

Concluding remarks

We have discussed some aspects of the theory of aggregation functions, including the review of some properties and classes of aggregation functions, and some construction methods. Especially, we have splitted the properties of extended aggregation functions into local properties, i.e., the properties of relevant n–ary aggregation functions for each fixed n, and into global properties which are often called “strong".

Concluding remarks

Global properties properties constraint different arities functions involved in each extended aggregation function and thus, in the next development of the theory of aggregation functions they should be investigated in more detail. We expect interesting generalizations based on modifications of these standard approaches in the near future.

Concluding remarks

For example, copulas are due to their probabilistic nature strongly connected with the standard operations, especially with the sum. Switching to the possibilistic background which is related to the maximum, we end up with semicopulas. However, there are many appropriate pseudo–additions (t–conorms) varying between the sum and maximum.

Thanks for your attention!