Models of generalized measures representation based on aggregation functions

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Monotone (generalized) measures

Let X be a finite set. A set function $\mu : 2^X \to [0,1]$ is called a monotone measure if

1. $\mu(\emptyset) = 0$, $\mu(X) = 1$ (norming);
2. $A \subseteq B$ implies $\mu(A) \leqslant \mu(B)$ (monotonicity).

Notation:

- $M_{mon}(X)$ is the set of all monotone measures on 2^X;
- $\mu_1 \leqslant \mu_2$ for $\mu_1, \mu_2 \in M_{mon}(X)$ if $\mu_1(A) \leqslant \mu_2(A)$ for all $A \in 2^X$.

Basic concepts of imprecise probabilities

- Classical probability theory works with single probability measures.
- The theory of imprecise probabilities works with sets of probability measures.

In this lecture we consider probability measures defined on the powerset 2^X of a finite set $X = \{x_1, ..., x_n\}$.

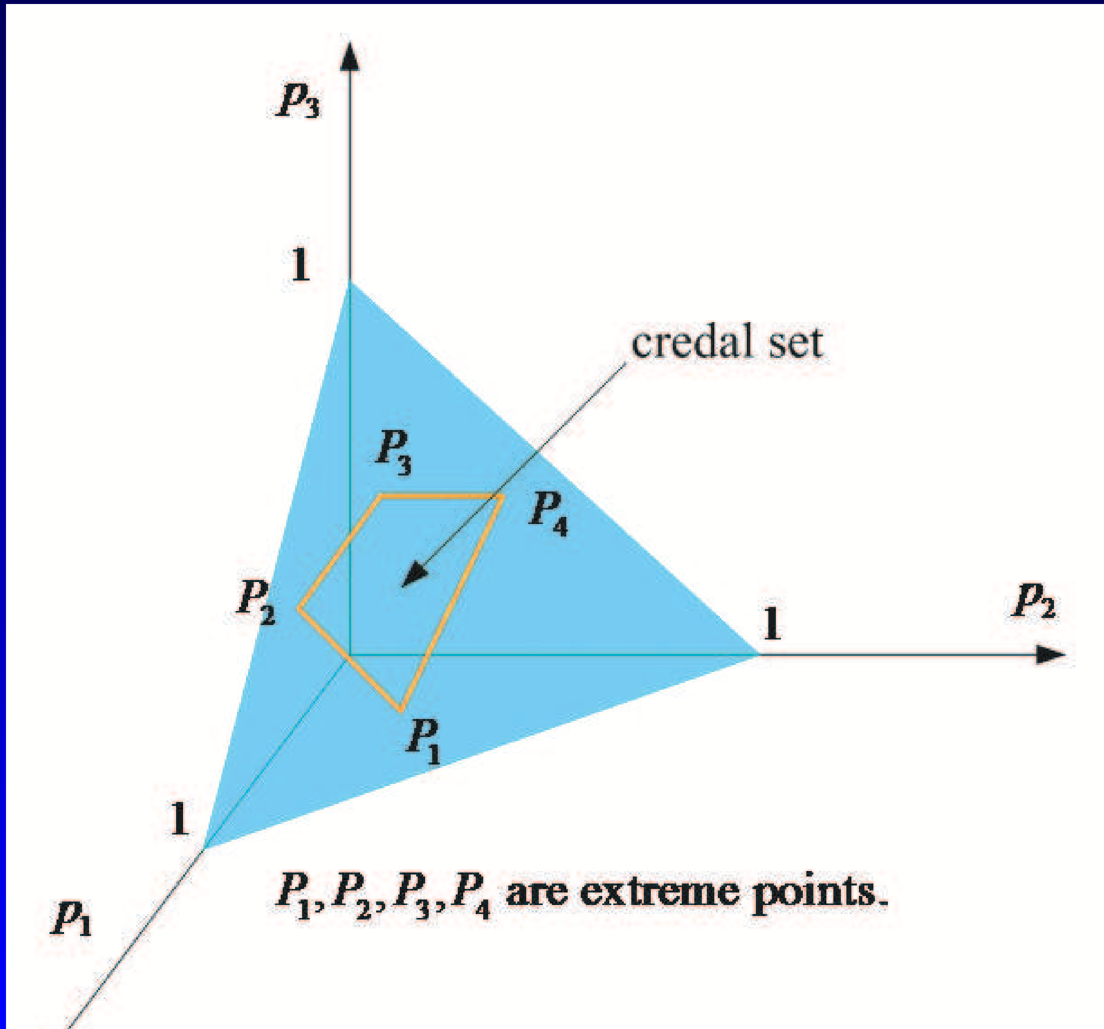
$M_{pr}(X)$ is the set of all probability measures on 2^X.

Credal sets

In this lecture a credal set is understood as a closed convex set of probability measures with a finite number of extreme points. If $\mathbf{P}$ is a credal set and $P_k \in M_{pr}(X)$, $k = 1, ..., m$, are its extreme points then

$$\mathbf{P} = \left\{ \sum_{k=1}^{m} a_i P_i | a_i \geqslant 0, \sum_{k=1}^{m} a_i = 1 \right\}.$$

Let $X = \{x_1, x_2, x_3\}$, then any credal set is convex subset of triangle consisting of points (p_1, p_2, p_3): $p_i \geqslant 0$, $p_1 + p_2 + p_3 = 1$.



Lower probabilities

A monotone measure μ is called a *lower probability* if there is a $P \in M_{pr}$ such that $\mu \leqslant P$.

Any lower probability μ defines a credal set

$$\mathbf{P}(\mu) = \{P \in M_{pr}(X) | P \geqslant \mu\}.$$

Let μ be a lower probability on 2^X, where $X = \{x_1, x_2, x_3\}$, then extreme points of $\mathbf{P}(\mu)$ can be found by solving the following inequalities:

$$\begin{cases} p_1 \geqslant \mu\left(\{x_1\}\right), \\ p_2 \geqslant \mu\left(\{x_2\}\right), \\ p_3 \geqslant \mu\left(\{x_3\}\right), \\ p_1 + p_2 \geqslant \mu\left(\{x_1, x_2\}\right), \\ p_1 + p_3 \geqslant \mu\left(\{x_1, x_3\}\right), \\ p_2 + p_3 \geqslant \mu\left(\{x_2, x_3\}\right), \\ p_1 + p_2 + p_3 = 1. \end{cases}$$

Clearly lower probabilities are less general than credal sets.

Upper probabilities

A monotone measure μ is called an *upper probability* if there is a $P \in M_{pr}$ such that $\mu \geqslant P$.

Any upper probability generate a credal set $\{P \in M_{pr}(X) | P \leqslant \mu\}$.

It is possible to consider only lower probabilities. Let μ be an upper probability. Introduce into consideration a measure $\mu^d(A) = 1 - \mu(A^c)$. The measure μ^d is called dual of μ. Clearly μ^d and μ generate the same credal set

Coherent lower probabilities

A lower probability μ is called a *coherent lower probability* if for any $A \in 2^X$ there is a $P \in M_{pr}$ such that $\mu \leqslant P$ and $\mu(A) = P(A)$.

Any coherent lower probability can be generated as follows: if $\mathbf{P}$ is a credal set then

$$\mu(A) = \min_{P \in \mathbf{P}} P(A),\ A \in 2^X,$$

is a coherent lower probability.

Coherent upper probabilities

An upper probability μ is called a *coherent upper probability* if for any $A \in 2^X$ there is a $P \in M_{pr}$ such that $\mu \geqslant P$ and $\mu(A) = P(A)$.

Any coherent upper probability can be generated as follows: if $\mathbf{P}$ is a credal set then

$$\mu(A) = \max_{P \in \mathbf{P}} P(A),\ A \in 2^X,$$

is a coherent upper probability.

Generalized coherent lower probabilities

A monotone measure μ is a *generalized coherent lower probability* if for any B ($\mu(B) > 0$) a monotone measure μ_B defined by $\mu_B(A) = \mu(A \cap B)/\mu(B)$ is a lower probability.

Proposition. μ is a generalized coherent lower probability iff for any $B \in 2^X$ there is an additive measure P ($P(X) \neq 1$ in general) such that $\mu \leqslant P$ and $\mu(B) = P(B)$.

2-monotone measures

A monotone measure is called 2-*monotone* if the following inequality holds:

$$\mu(A) + \mu(B) \leqslant \mu(A \cap B) + \mu(A \cup B).$$

for the dual measure the following inequality holds:

$$\mu^d(A) + \mu^d(B) \geqslant \mu^d(A \cap B) + \mu^d(A \cup B).$$

This measure is called 2-*alternative*. It is known that any 2-monotone measure is a coherent lower probability, and any 2-alternative measure is a coherent upper probability.

Example. Let μ is a lower envelope of probability measures P_1 and P_2 with values

$$P_1(\{x_1\}) = 1/4,\ P_1(\{x_2\}) = 0,\ P_1(\{x_3\}) = 3/4,\ P_1(\{x_4\}) = 0,$$

$$P_2(\{x_1\}) = 0,\ P_2(\{x_2\}) = 1/2,\ P_2(\{x_3\}) = 0,\ P_2(\{x_4\}) = 1/2,$$

i.e. $\mu(A) = \min_{i=1,2} P_i(A)$. Then

$$\underbrace{\mu(\{x_1, x_4\})}_{1/4} + \underbrace{\mu(\{x_3, x_4\})}_{1/2} > \underbrace{\mu(\{x_4\})}_{0} + \underbrace{\mu(\{x_1, x_3, x_4\})}_{1/2}.$$

Therefore, μ is a coherent lower probability, but it is not 2-monotone.

k-monotone measures

A monotone measure is k-monotone iff for any system of sets $C_1, ..., C_m \in 2^X$, $m \leq k$:

$$\mu\left(\bigcup_{i=1}^{m} C_i\right) + \sum_{B\subseteq\{1,...,m\},\, B\neq\emptyset} (-1)^{|B|}\mu\left(\bigcap_{i\in B} C_i\right) \geq 0.$$

The partial cases of the last inequality are

$$\mu(C_1 \cup C_2) - \mu(C_1) - \mu(C_2) + \mu(C_1 \cap C_2) \geqslant 0$$

(2-monotonicity, $m = 2$);

$$\mu(C_1 \cup C_2 \cup C_3) - \mu(C_1) - \mu(C_2) + \mu(C_1 \cap C_2) + \mu(C_1 \cap C_3) + \mu(C_2 \cap C_3) - \mu(C_1 \cap C_2 \cap C_2) \geqslant 0.$$

Belief and plausibility measures

Belief and plausibility measures are defined by means of a basic probability assignment. A basic probability assignment m is a non-negative set function on 2^X such that

1. $m(\emptyset) = 0$;
2. $\sum_{A \in 2^X} m(A) = 1$ (norming).

Then

$$Bel(A) = \sum_{B \subseteq A} m(B) \text{ and } Pl(B) = \sum_{B \cap A \neq \emptyset} m(A).$$

The set A is called focal if $m(A) > 0$.

Some times, it is useful to represent belief functions using $\{0,1\}$-valued measures:

$$\eta_{\langle B\rangle}(A) = \begin{cases} 1, & B \subseteq A, \\ 0, & otherwise. \end{cases}$$

Then

$$Bel(A) = \sum_{B \in 2^X} m(B)\eta_{\langle B\rangle}(A).$$

The sense of $\eta_{\langle B\rangle}$ is the following. It describes the situation when we know that the random variable definitely takes values from the set B, but we don't know any additional information.
Clearly, $Pl = Bel^d$.

Möbius transform

The set of all set functions on 2^X is a linear space and the system of set functions $\left\{\eta_{\langle B\rangle}\right\}_{B\in 2^X}$ is the basis of it. We can find the representation

$$\mu = \sum_{B\in 2^X} m(B)\eta_{\langle B\rangle}$$

of any $\mu : 2^X \to \mathbb{R}$ using the Möbius transform:

$$m(B) = \sum_{A\subseteq B} (-1)^{|B\setminus A|}\mu(A).$$

Notation:

M_{mon} is the set of all monotone measures on 2^X;
M_{gcoh} is the set of all generalized coherent lower probabilities on 2^X;
M_{coh} is the set of all generalized coherent lower probabilities on 2^X;
M_{k-mon}, $k = 2, 3, ...$, is the set of all k-monotone measures on 2^X;
M_{bel} is the set of all belief measures on 2^X;
M_{pr} is the set of all probability measures on 2^X.

Embeddings:

$$M_{mon} \supset M_{gcoh} \supset M_{coh}$$
$$\supset M_{2-mon} \supset ... \supset M_{bel} \supset M_{pr}.$$

Aggregation of probability measures

Let us consider the following construction. Given a finite probability space X with a probability measure P on algebra 2^X and $\{B_1, B_2, ..., B_m\}$ is a partition of X. Then P can be represented as

$$P(A) = \sum_{i=1}^{m} P(A|B_i)P(B_i).$$

Let us introduce into consideration probability measures:

$$P_i(A) = P(A|B_i), i = 1, ..., n,$$

and a linear function:

$$\varphi(x_1, ..., x_m) = P(B_1)x_1 + P(B_2)x_2 + ... + P(B_m)x_m.$$

Then P can be represented as

$$P(A) = \varphi(P_1(A), ..., P_m(A)). \quad (1)$$

In this lecture we investigate representation (1) in the theory of generalized measures.

Aggregation of monotone measures

Let $\varphi : [0,1]^n \to [0,1]$ be an aggregation function. i.e.

1. $\varphi\,(0, ..., 0) = 0$, $\varphi\,(1, ..., 1) = 1$;
2. $\mathbf{x} \leqslant \mathbf{y}$ implies $\varphi(\mathbf{x}) \leqslant \varphi(\mathbf{y})$.

Let $\mu_1, ..., \mu_n$ be monotone measures on 2^X. Then a monotone measure μ defined by

$$\mu(A) = \varphi\left(\mu_1(A), ..., \mu_n(A)\right), \, A \in 2^X.$$

is called the aggregation of $\mu_1, ..., \mu_n$ by φ.

Example 1. Consider a belief measure $Bel = \sum_{i=1}^{k} m(B_i)\eta_{\langle B_i \rangle}$, where $B_1, ..., B_k$ are focal elements of Bel. Then $Bel = \varphi(\eta_{\langle B_1 \rangle}, ..., \eta_{\langle B_k \rangle})$, where

$$\varphi(x_1, ..., x_k) = \sum_{i=1}^{k} m(B_i)x_i.$$

Each $\eta_{\langle B_i \rangle}$ can be represented as

$$\eta_{\langle B_i \rangle} = \prod_{x \in B_i} \eta_{\langle \{x\} \rangle}.$$

Therefore, any belief measure can be generated with the help of a linear aggregation function and product from Dirac measures $\eta_{\langle \{x\} \rangle}$.

Example 2. Let μ be a coherent lower probability and let $P_1, ..., P_k \in M_{pr}$ be extreme points of $\mathbf{P}(\mu)$. Then

$$\mu = \min\{P_1, ..., P_k\},$$

where $\min$ is an aggregation function.

The problem of monotone measures representation

To define any monotone measure μ on 2^X we need to assign its $2^{|X|} - 2$ values. Therefore, space complexity grows exponentially w.r.t. cardinality of X. With the help of aggregation functions we can try to represent μ as

$$\mu = \varphi(\mu_1, ..., \mu_k),$$

where μ_i is a monotone measure on 2^{B_i} and $\{B_1, ..., B_k\}$ is a partition of X.
Assuming that for assigning φ, we need $2^k - 2$ variables, we can find that the space complexity is

$$2^k + \sum_{i=1}^{k} 2^{|B_i|} - 2(k+1). \quad (1)$$

In particular, if $|B_i| = k$, $i = 1, ..., k$, then (1) is transformed to

$$\left(\sqrt{|X|} + 1\right)\left(2^{\sqrt{|X|}} - 2\right).$$

Consensus requirement

When we construct a measure μ with an aggregation function, we need to guarantee some of its properties. For example, if we work with lower probabilities, then μ should be also a lower probability. This can be provided if the consensus requirement is fulfilled.

An aggregation function $\varphi : [0, 1]^n \to [0, 1]$ obeys *the consensus requirement* for lower probabilities if $\mu = \varphi(\mu_1, ..., \mu_n)$ is in M_{low} for any tuple $(\mu_0, ..., \mu_n) \in M^n_{low}$.

This definition is extended for coherent lower probabilities, 2-monotone measures, etc.

Aggregation functions for probability measures

Notation: $\tilde{M}_{mon}$ is the set of all aggregation functions.
Proposition. An aggregation function $\varphi : [0,1]^n \to [0,1]$ obeys the consensus requirement for probability measures iff

$$\varphi(x_1, ..., x_n) = \sum_{i=1}^{n} a_i x_i,$$

where $\sum_{i=1}^{n} a_i = 1$ and $a_i \geqslant 0$, $i = 1, ..., n$.

Notation: $\tilde{M}_{pr}$ is the set of all aggregation functions for probability measures.

Aggregation functions for lower probabilities

Proposition. An aggregation function $\varphi : [0,1]^n \to [0,1]$ obeys the consensus requirement for lower probabilities iff there is $\alpha \in \tilde{M}_{pr}$ such that $\varphi(\mathbf{x}) \leqslant \alpha(\mathbf{x})$ for all $\mathbf{x} \in [0,1]^n$.

Notation: $\tilde{M}_{low}$ is the set of all aggregation functions for lower probabilities.

Aggregation functions for generalized coherent lower probabilities

Notation: $\mathbf{z} = \mathbf{xy}$ for $\mathbf{x} = (x_1, x_2, ..., x_n)$, $\mathbf{y} = (y_1, y_2, ..., y_n)$, $\mathbf{z} = (z_1, z_2, ..., z_n)$ if $z_i = x_i y_i$, $i = 1, ..., n$.

Proposition. An aggregation function $\varphi : [0,1]^n \to [0,1]$ obeys the consensus requirement for generalized coherent lower probabilities iff for any $\mathbf{y} \in [0,1]^n$ there is $\alpha \in \tilde{M}_{pr}$ such that $\varphi(\mathbf{x}) \leqslant \alpha(\mathbf{x})\varphi(\mathbf{y})$ for all $\mathbf{x} \in [0,1]^n$.

Notation: $\tilde{M}_{gcoh}$ is the set of all aggregation functions for generalized coherent lower probabilities.

Aggregation functions for coherent lower probabilities

Notation: $\mathbf{1} = (1, 1, ..., 1)$.

Proposition. An aggregation function $\varphi : [0,1]^n \to [0,1]$ obeys the consensus requirement for coherent lower probabilities iff for any $\mathbf{y} \in [0,1]^n$ there are $\alpha, \beta \in \tilde{M}_{pr}$ such that

$$\varphi(\mathbf{xy} + \mathbf{z}(\mathbf{1} - \mathbf{y})) \leqslant \alpha(\mathbf{x})\varphi(\mathbf{y}) + \beta(\mathbf{z})(1 - \varphi(\mathbf{y}))$$

for all $\mathbf{x}, \mathbf{z} \in [0,1]^n$.

Notation: $\tilde{M}_{coh}$ is the set of all aggregation functions for coherent lower probabilities.

Aggregation functions for 2-monotone measures

Proposition. An aggregation function $\varphi : [0,1]^n \to [0,1]$ obeys the consensus requirement for 2-monotone measures iff

$$\varphi(\mathbf{x} + \Delta\mathbf{y} + \Delta\mathbf{z}) \geqslant \varphi(\mathbf{x} + \Delta\mathbf{y}) + \varphi(\mathbf{x} + \Delta\mathbf{z}) - \varphi(\mathbf{x})$$

for any $\mathbf{x}, \Delta\mathbf{y}, \Delta\mathbf{z}, \mathbf{x} + \Delta\mathbf{y} + \Delta\mathbf{z} \in [0,1]^n$.

Corollary. If $\varphi : [0,1]^n \to [0,1]$ is 2 times differentiable on $[0,1]^n$ and $\frac{\partial\varphi(\mathbf{x})}{\partial x_i} \geqslant 0$, $\frac{\partial^2\varphi(\mathbf{x})}{\partial x_i \partial x_j} \geqslant 0$, $i, j \in \{1, ..., n\}$ for any point $\mathbf{x} \in [0,1]^n$. Then φ obeys the consensus requirement for 2-monotone measures.

Notation: $\tilde{M}_{2-mon}$ is the set of all aggregation functions for 2-monotone measures.

Aggregation functions for k-monotone measures

Proposition. An aggregation function $\varphi : [0,1]^n \to [0,1]$ obeys the consensus requirement for k-monotone measures iff

$$\sum_{A \subseteq \{1,\dots,m\}} (-1)^{m-|A|} \varphi\left(\mathbf{x} + \textstyle\sum_{i\in A} \Delta\mathbf{x}_i\right) \geqslant 0$$

for any $\mathbf{x}, \Delta\mathbf{x}_1, \dots, \Delta\mathbf{x}_m, \mathbf{x} + \Delta\mathbf{x}_1 + \dots$ $+\Delta\mathbf{x}_m \in [0,1]^n$, $m \in \{1, \dots, k\}$.

Corollary. If an aggregation function

1. $\varphi : [0,1]^n \to [0,1]$ is k times differentiable on $[0,1]^n$;
2. $\dfrac{\partial^m \varphi(\mathbf{x})}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_m}} \geqslant 0$ for any point $\mathbf{x} \in [0,1]^n$ and for any $i_1, i_2, \ldots, i_m \in \{1, 2, \ldots, n\}$, $m \leqslant k$.

Then φ obeys the consensus requirement for k-monotone measures.

Notation: $\tilde{M}_{k-mon}$ is the set of all aggregation functions for k-monotone measures.

Aggregation functions for belief measures

Proposition. An aggregation function $\varphi : [0,1]^n \to [0,1]$ obeys the consensus requirement for belief measures iff

$$\sum_{A \subseteq \{1,\ldots,m\}} (-1)^{m-|A|} \varphi\left(\mathbf{x} + \sum_{i \in A} \Delta\mathbf{x}_i\right) \geqslant 0$$

for any $\mathbf{x}, \Delta\mathbf{x}_1, \ldots, \Delta\mathbf{x}_m, \mathbf{x} + \Delta\mathbf{x}_1 + \ldots$
$+\Delta\mathbf{x}_m \in [0,1]^n$, $m = 1, 2, \ldots$.

Corollary. If an aggregation function

1. $\varphi : [0,1]^n \to [0,1]$ is infinitely differentiable on $[0,1]^n$;
2. $\dfrac{\partial^m \varphi(\mathbf{x})}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_m}} \geqslant 0$ for any point $\mathbf{x} \in [0,1]^n$ and for any $i_1, i_2, ..., i_m \in \{1, 2, ..., n\}$, $m = 1, 2, ...$

Then φ obeys the consensus requirement for belief measures.

Notation: $\tilde{M}_{bel}$ is the set of all aggregation functions for k-monotone measures.

Composition of aggregation functions

Let $\varphi_i : [0,1]^n \to [0,1], i = 1, ..., m$,
$\varphi : [0,1]^m \to [0,1]$.
Then their *composition* $\psi : [0,1]^n \to [0,1]$ is defined by

$$\psi(x_1, ..., x_n) = \varphi(\varphi_1(x_1, ..., x_n), ..., \varphi_m(x_1, ..., x_n)).$$

Proposition. Let $\psi = \varphi(\varphi_1, ..., \varphi_m)$ be a composition of aggregation functions $\varphi, \varphi_1, ..., \varphi_m$. Then

1. $\varphi, \varphi_1, ..., \varphi_m \in \tilde{M}_{low}$ implies $\psi \in \tilde{M}_{low}$;
2. $\varphi, \varphi_1, ..., \varphi_m \in \tilde{M}_{gcoh}$ implies $\psi \in \tilde{M}_{gcoh}$;
3. $\varphi, \varphi_1, ..., \varphi_m \in \tilde{M}_{coh}$ implies $\psi \in \tilde{M}_{coh}$;
4. $\varphi, \varphi_1, ..., \varphi_m \in \tilde{M}_{k-mon}$, $k = 2, 3, ...$ implies $\psi \in \tilde{M}_{k-mon}$;
5. $\varphi, \varphi_1, ..., \varphi_m \in \tilde{M}_{bel}$ implies $\psi \in \tilde{M}_{bel}$;
6. $\varphi, \varphi_1, ..., \varphi_m \in \tilde{M}_{pr}$ implies $\psi \in \tilde{M}_{pr}$.

Monotone measures of fuzzy sets

Any aggregation function $\varphi : [0,1]^n \to [0,1]$ can be interpreted as a monotone measure of a fuzzy subset of $\{1, ..., n\}$.

For this purpose, any fuzzy subset $A : \{1, ..., n\} \to [0,1]$, we consider as a vector $\mathbf{x}_A = (A(1), ..., A(n))$.

Clearly, introduced families of aggregation functions $\tilde{M}_{low}$, $\tilde{M}_{gcoh}$, $\tilde{M}_{coh}$, $\tilde{M}_{k-mon}$, $\tilde{M}_{bel}$, $\tilde{M}_{pr}$ are generalizations of corresponding families of usual monotone measures.

Operations on fuzzy sets

We can interpret properties of monotone measures of fuzzy sets through the following operations:

1. $\bar{A}$ is the complement of A if $\bar{A}(i) = 1 - A(i)$, $i = 1, .., n$;
2. $C = A \cap B$ if $C(i) = A(i)B(i)$, $i = 1, .., n$;
3. $C = A \cup B$ for sets $A \cap B = \emptyset$ if $C(i) = A(i) + B(i)$, $i = 1, .., n$.

Proposition. Let φ be an aggregation function, $A \subseteq \{1, ..., n\}$, and $\mathbf{x}_A = (x_1, ..., x_n)$ is such that $x_i = 1$ if $i \in A$, and $x_i = 0$ otherwise. Consider a monotone measure μ defined by $\mu(A) = \varphi(\mathbf{x}_A)$. Then

1. $\varphi \in \tilde{M}_{low}$ implies $\mu \in M_{low}$;
2. $\varphi \in \tilde{M}_{gcoh}$ implies $\mu \in M_{gcoh}$;
3. $\varphi \in \tilde{M}_{coh}$ implies $\mu \in M_{coh}$;
4. $\varphi \in \tilde{M}_{k-mon}, k = 2, 3, ...$ implies $\mu \in M_{k-mon}$;
5. $\varphi \in \tilde{M}_{bel}$ implies $\mu \in M_{bel}$;
6. $\varphi \in \tilde{M}_{pr}$ implies $\mu \in M_{pr}$.

Problem of aggregation functions construction using monotone measures

Given a monotone measure μ on 2^X, where $X = \{1, ..., n\}$.

Is it possible to construct an aggregation function $\varphi : [0,1]^n \to [0,1]$ such that $\varphi(\mathbf{x}_A) = \mu(A)$ for all $A \in 2^X$ under the consensus requirement?

The straightforward way is to look at non-additive integrals w.r.t. a monotone measure μ.

It is easy to check that for Choquet integral the consensus requirement is fulfilled for lower probabilities, probability measures, but it is not for other families of monotone measures.

For example, let $\varphi(f) = (Choquet) \int f d\mu$ and $\mu \in M_{2-mon}$.

Then $\varphi \in \tilde{M}_{coh}$, but $\varphi \notin \tilde{M}_{2-mon}$ in general.

the solution of this problem is to use the multilinear extension that has remarkable properties.

Multilinear extension

Let μ a monotone measure μ on 2^X, where $X = \{1, ..., n\}$, and let m be its Möbius transform. Then *the multilinear extension* φ of μ is defined by

$$\varphi(\mathbf{x}) = \sum_{B \in 2^X} m(B) \prod_{i \in B} x_i.$$

Proposition. Let φ be a multilinear extension of μ. Then φ is an aggregation function and $\varphi(\mathbf{x}_A) = \mu(A)$, $A \subseteq \{1, ..., n\}$.

Remark. The multilinear extension can be defined as

$$\varphi(\mathbf{x}) = \sum_{B \in 2^X} \mu(B) \prod_{i \in B} x_i \prod_{i \notin B} (1 - x_i).$$

Proposition. Let μ a monotone measure μ on 2^X, where $X = \{1, ..., n\}$, and $\varphi : [0, 1]^n \to [0, 1]$ its multilinear extension. Then

1. $\mu \in M_{low}$ implies $\varphi \in \tilde{M}_{low}$;
2. $\mu \in M_{gcoh}$ implies $\varphi \in \tilde{M}_{gcoh}$;
3. $\mu \in M_{coh}$ implies $\varphi \in \tilde{M}_{coh}$;
4. $\mu \in M_{k-mon}$, $k = 2, 3, \ldots$ implies $\varphi \in \tilde{M}_{k-mon}$;
5. $\mu \in M_{bel}$ implies $\varphi \in \tilde{M}_{bel}$;
6. $\mu \in M_{pr}$ implies $\varphi \in \tilde{M}_{pr}$.

Example

Let μ on 2^Z, where $Z = \{1, 2, 3\}$, defined by
$\mu(\{1, 2, 3\}) = 1$, $\mu(\{1, 2\}) = 2/3$, $\mu(\{2, 3\}) = 2/3$;
μ is equal to zero on other sets.
μ is a generalyzed coherent lower probability.
Let us compute also the natural extension $\tilde{\mu}$ of μ:

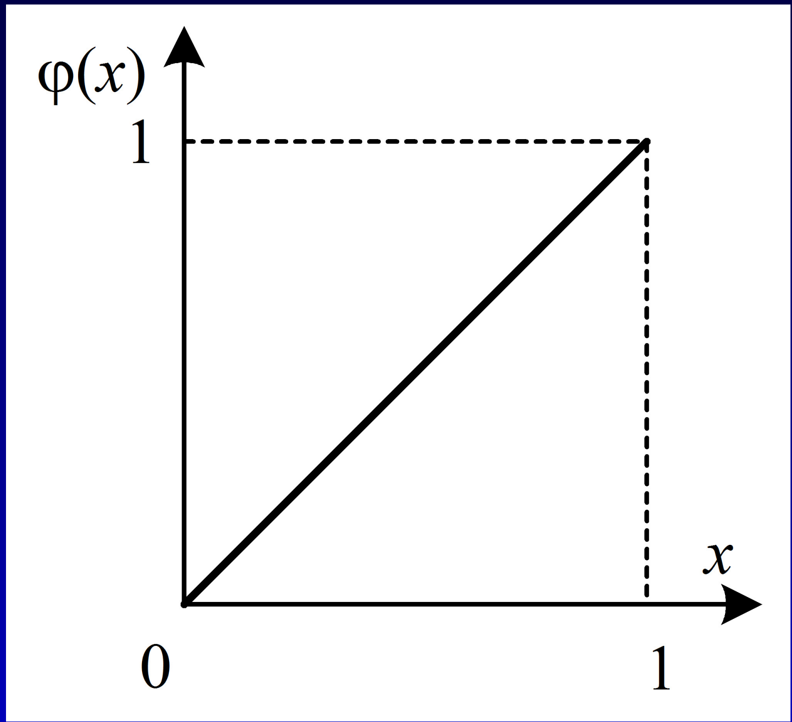
$$\tilde{\mu}(A) = \inf_{P \in \mathbf{P}(\mu)} P(A),\ A \in 2^X.$$

$\tilde{\mu}(\{2\}) = 1/3$ and it has the same values as μ on other sets.
$\tilde{\mu}$ is a belief measure.

- The Möbius transform m_μ of μ:
  $m_\mu(\{1,2,3\}) = -1/3$, $m_\mu(\{1,2\}) = 2/3$,
  $m_\mu(\{2,3\}) = 2/3$;
  m_μ is equal to zero on other sets.
- The Choquet integral of μ:
  $\varphi_1(\mathbf{x}) = \frac{2}{3}(x_1 \wedge x_2) + \frac{2}{3}(x_2 \wedge x_3) - (x_1 \wedge x_2 \wedge x_3)$
- The multilinear extension of μ:
  $\varphi_2(\mathbf{x}) = \frac{2}{3}x_1x_2 + \frac{2}{3}x_2x_3 - x_1x_2x_3$
- $\varphi_1 \in \tilde{M}_{low}$, $\varphi_2 \in \tilde{M}_{gcoh}$.

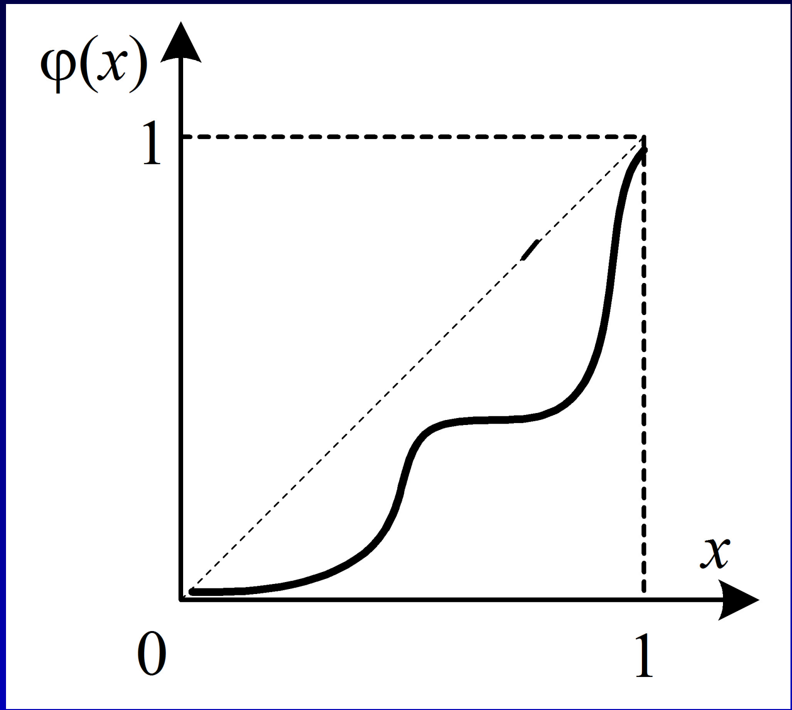
- The Möbius transform $m_{\tilde{\mu}}$ of $\tilde{\mu}$:
  $m_{\tilde{\mu}}(\{2\}) = 1/3$, $m_{\tilde{\mu}}(\{1,2\}) = 1/3$,
  $m_{\tilde{\mu}}(\{2,3\}) = 1/3$;
  $m_{\tilde{\mu}}$ is equal to zero on other sets.
- The Choquet integral of $\tilde{\mu}$:
  $\varphi_3(\mathbf{x}) = \frac{1}{3}x_2 + \frac{1}{3}(x_1 \wedge x_2) + \frac{1}{3}(x_2 \wedge x_3)$
- The multilinear extension of $\tilde{\mu}$:
  $\varphi_4(\mathbf{x}) = \frac{1}{3}x_2 + \frac{1}{3}x_1x_2 + \frac{1}{3}x_2x_3$
- $\varphi_3 \in \tilde{M}_{coh}$, $\varphi_4 \in \tilde{M}_{bel}$.

Consensus for probability measures



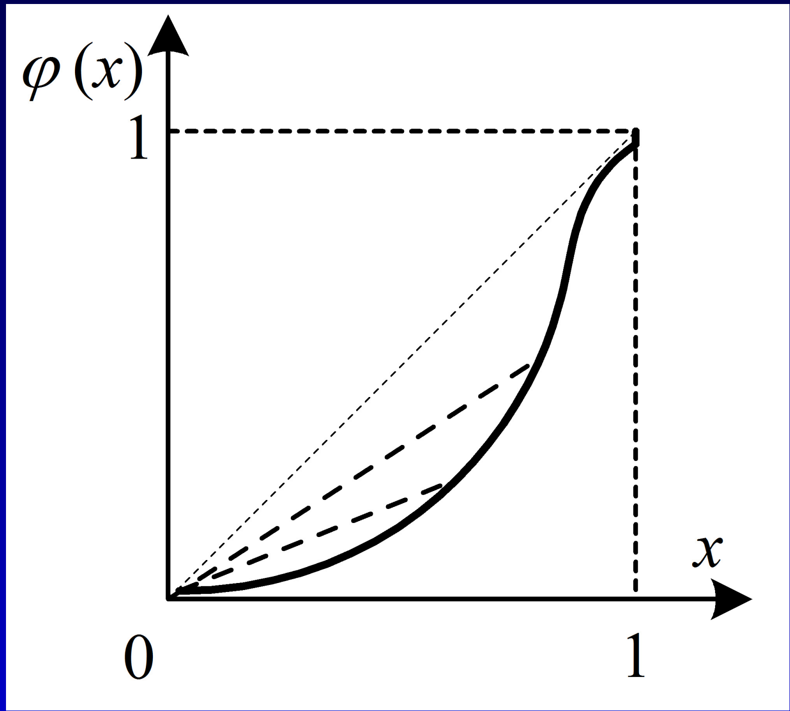
$$(\forall x \in [0,1])\varphi(x) = x.$$

Consensus for lower probabilities



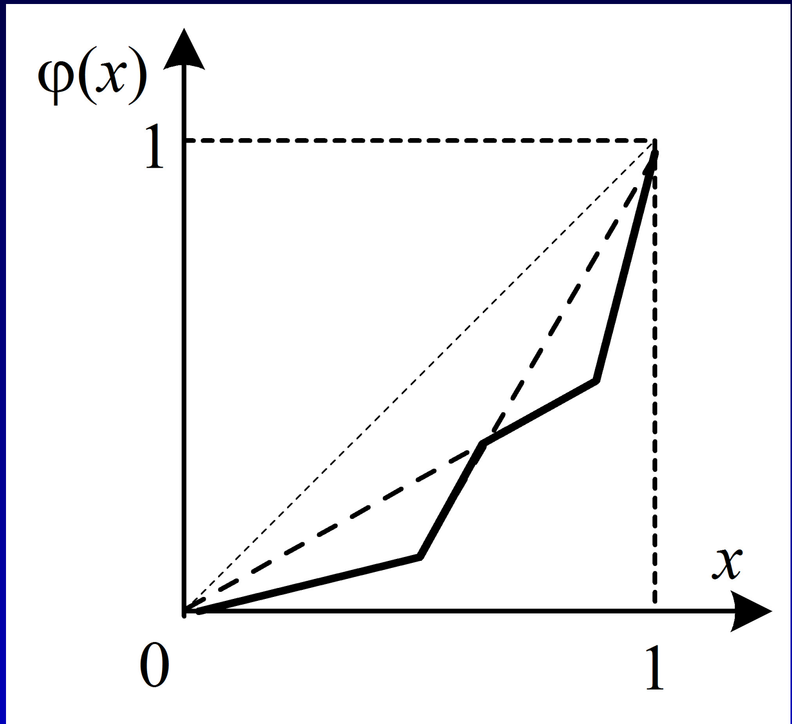
$$(\forall x \in [0,1])\varphi(x) \leqslant x.$$

Consensus for generalized coherent lower probabilities



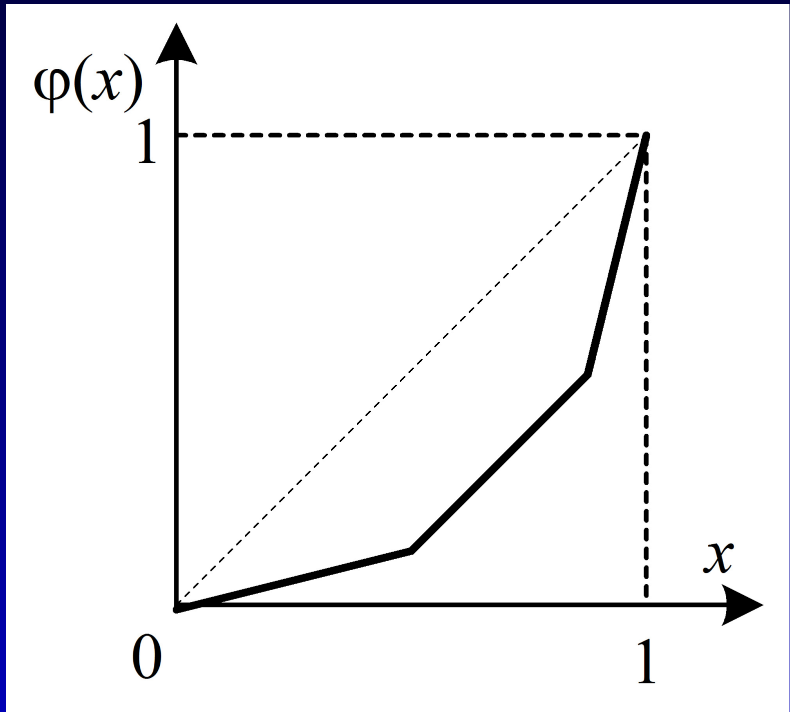
$$(\forall x, y \in [0,1])\varphi(xy) \leqslant x\varphi(y).$$

Consensus for coherent lower probabilities



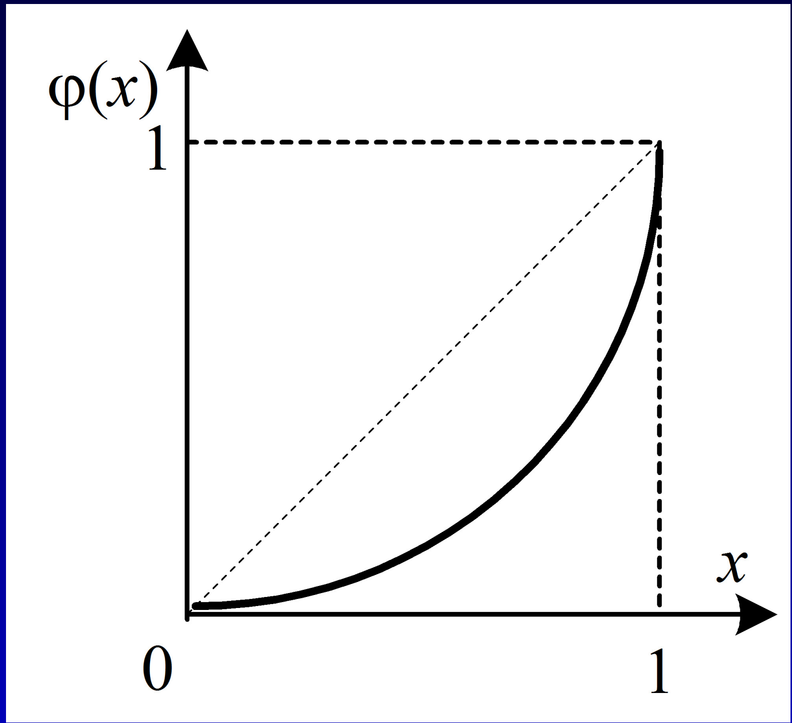
1. $(\forall x, y \in [0,1])\varphi(xy) \leqslant x\varphi(y)$;
2. $(\forall x, y \in [0,1])\varphi(xy + x(1-y)) \leqslant x\varphi(y) + x(1-\varphi(y))$.

Consensus for 2-monotone measures



φ is convex.

Consensus for belief measures



1. φ is convex;
2. $\forall x \in [0,1) \dfrac{d^k \varphi(x)}{dx^k} \geqslant 0, k = 1, 2, \ldots.$