Residuated multilattices

Between pocrims and residuated lattices

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Outline

Generalized fuzzy sets

- Fuzzy sets [Zadeh'65]:

$$A\colon \mathcal{U} \to [0,1]$$

- L-fuzzy sets [Goguen'67]:

$$A\colon \mathcal{U} \to L, \text{ where } L \text{ is a complete lattice}$$

- M-fuzzy sets:

$$A\colon \mathcal{U} \to M, \text{ where } M \text{ is a complete multilattice}$$

- Applications [Ojeda-Aciego et al'07]:

Fuzzy Logic Programming based on Multilattices

Outline

Multilattices: ordering-based definition

- Ordered multilattice and multisemilattice. [Benado'54].
- Multisupremum ($Multisup$): minimal element of the set of upper bounds.

Definition

A poset, $(M, \leq)$, is a **join-multisemilattice** if, for all $a, b, x \in M$,

$$a \leq x \text{ and } b \leq x \text{ implies that there exists } z \in Multisup(\{a, b\}) \text{ such that } z \leq x$$

Dual property defines the concept of **meet-multisemilatice**.

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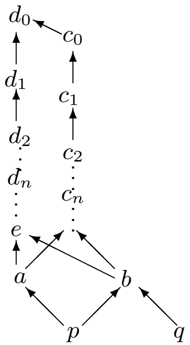

- The poset in the figure is a meet-multisemilattice but not a join-multisemilattice.
- Any finite poset is a multilattice.
- The set of chains with the substring relation is a multilattice.
- The set of circles in the euclidean plane is a multilattice.

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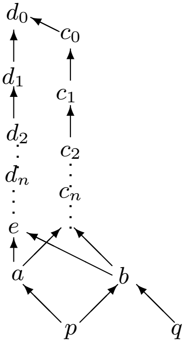

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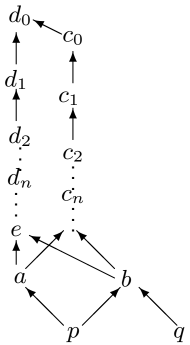

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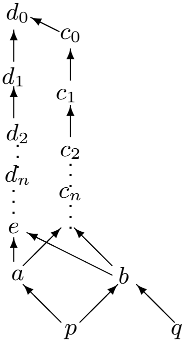

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Algebraic counterpart

- M. Benado. *Les ensembles partiellement ordonnés et le théorème de raffinement de Schreier. I*, Čehoslovack. Mat. Ž., 4(79):105–129, 1954.
- D.J. Hansen. *An axiomatic characterization of multilattices*. Discrete Mathematics, 33(1): 99–101, 1981.
- I.J. Johnston. *Some results involving multilattice ideals and distributivity*, Discrete Mathematics, 83(1):27–35, 1990.
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Hypergroupoid and nd-groupoid

- Hyperstructure theory [Marty, 1934]:
  It considers hyperoperations that are mappings $\star\colon A \times A \to 2^A \smallsetminus \{\varnothing\}$.
- Hypergroups, join-spaces, hyperrings ...
- nd-operations [Cordero et al, 2001]: $\star\colon A \times A \to 2^A$.
- We named this kind of operations *non-deterministic operations* (briefly, nd-operations) due to its relationship with the notion of non-deterministic automata.
- As usually, if $X \subseteq A$ and $a \in A$ then

$$a \star X = \bigcup_{x \in X} a \star x \qquad X \star a = \bigcup_{x \in X} x \star a$$

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Properties on nd-groupoids I

Assume that $(A, \cdot)$ is an nd-groupoid:

- **Idempotency**: for all $a \in A$, $\quad aa = a$.
- **Commutativity**: for all $a, b \in A$, $\quad ab = ba$.
- **Associativity**: for all $a, b, c \in A$, $\quad (ab)c = a(bc)$.
- **Left m-associativity**: for all $a, b, c \in A$, $\quad$ if $ab = b$, then $(ab)c \subseteq a(bc)$.
- **Right m-associativity**: for all $a, b, c \in A$, $\quad$ if $bc = c$, then $a(bc) \subseteq (ab)c$.
- **m-associativity**: if it is left and right m-associative.

Properties on nd-groupoids II

- A join multisemilattice M satisfies idempotency, commutativity and m-associativity laws with the nd-operation given by $a \sqcup b = Multisup\{a, b\}$
- Conversely, for an nd-groupoid $(M, \cdot)$ the so-called **natural ordering**

$$a \le b \text{ if and only if } ab = b$$

is indeed an ordering relation on M if $(M, \cdot)$ is idempotent, commutative and m-associative.

Moreover, if $a \le x$ and $b \le x$ then there exists $z \in ab$ with $z \le x$.

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Multisemilattices: algebraic definition

An nd-groupoid $(A, \cdot)$ satisfies the **comparability laws** if, for all $a, b, c, d \in A$:

$(\mathbf{C_1})$ $c \in ab$ implies that $ac = c$ and $bc = c$.

$(\mathbf{C_2})$ $c, d \in ab$ and $cd = d$ imply that $c = d$.

Definition

An **algebraic multisemilattice** is an nd-groupoid that satisfies idempotency, commutativity, m-associativity and comparability laws.

Ordered and algebraic definitions of multisemilattice are equivalent.

$a \sqcup b = Multisup\{a, b\}$ and $\leq$ being the natural ordering

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Multilattices: algebraic definition

Definition

Let $\sqcup$ and $\sqcap$ be nd-operations in M, the pair $(\sqcup, \sqcap)$ is said to have the property of **absorption** if for all $a, b \in M$ the following conditions hold:

(i) $a \sqcap c = a$ for all $c \in a \sqcup b$ (Therefore, $a \sqcap (a \sqcup b) = a$

(ii) $a \sqcup c = a$ for all $c \in a \sqcap b$ and $a \sqcup (a \sqcap b) = a$)

Definition

An **algebraic multilattice** $(M, \sqcup, \sqcap)$, is a set M with two nd-operations $\sqcup$ and $\sqcap$ satisfying the absorption property and such that $(M, \sqcup)$ and $(M, \sqcap)$ are multisemilattices.

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Full multisemilattices and multilattices

Definition

A multisemilattice $(M, \sqcup)$ is **full** if, for all $a, b \in M$, $a \sqcup b \neq \varnothing$.
A multilattice is full if both multisemilattices are full.

In this case, they are hyperalgebras.

Proposition

Any bounded multilattice is full.

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Full and associatives multilattices

Proposition

Let $(M, \sqcup)$ *be a multisemilattice.*

If $\sqcup$ *is associative then, for all* $a, b \in M$, $|a \sqcup b| \leq 1$.

Proposition

Let $(M, \sqcup, \sqcap)$ *be a full multilattice. The following conditions are equivalent:*

(i) $\sqcup$ *is associative.*

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(iii) $(M, \sqcup, \sqcap)$ *is a lattice.*

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Related works

- Congruence relations on multilattices [FLINS, 2008]
- On congruences, ideals and homomorphisms over multilattices [EUROFUSE, 2009]
- Congruence relations on some hyperstructures [Ann. Math. Artif. Intell. 2009]
- Fuzzy congruence relations on nd-groupoids [Int. J. Computer Mathematics, 2009]
- A coalgebraic approach to non-determinism: applications to multilattices [Information Science, 2010]
- Finitary coalgebraic multisemilattices and multilattices. [Applied Mathematics and Computation, 2012]

Outline

POCRIMS

Definition

A tuple $(A, *, \to, 1, \leq)$ is said to be a *partially ordered commutative residuated integral monoid*, briefly a **pocrim**, if the following properties hold:

- $(A, *, 1)$ is a commutative monoid.
- $(A, \leq)$ is a partially ordered set in which 1 is the maximum.
- The *residuum property* holds. That is, for every $a, b, c \in A$,

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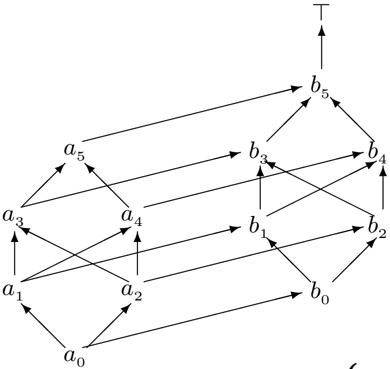
Example

Let $A = \{a_i \mid 0 \leq i \leq 5\}$, $B = \{b_i \mid 0 \leq i \leq 5\}$ and $C = \{b_i \mid 2 \leq i \leq 5\}$



$$x * y = \begin{cases} x & \text{if } y = \top \\ y & \text{if } x = \top \\ b_2 & \text{if } x, y \in C \\ b_0 & \text{if } x \in B \smallsetminus C, y \in B \\ b_0 & \text{if } x \in B, y \in B \smallsetminus C \\ a_0 & \text{otherwise.} \end{cases} \qquad x \to y = \begin{cases} \top & \text{if } x \leq y \\ y & \text{if } x = \top \\ a_5 & \text{if } x \in B, y \in A \\ b_1 & \text{if } x \in C \\ & \text{and } y \in B \smallsetminus C \\ b_5 & \text{otherwise.} \end{cases}$$

Residuated lattices and Heyting algebras

A pocrim is said to be a residuated lattice if the poset is a lattice.

Residuated lattice

A tuple $(A, \vee, \wedge, *, \rightarrow, 1)$ is said to be a **residuated lattice** if the following properties hold:

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A residuated lattice in which $*$ coincides with the meet operation is said to be a **Heyting algebra**.

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A pocrim is said to be a residuated multilattice if the underlying poset is a bounded multilattice.

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A tuple $(A, \sqcup, \sqcap, *, \rightarrow, 1)$ is said to be a **residuated multilattice** if the following properties hold:

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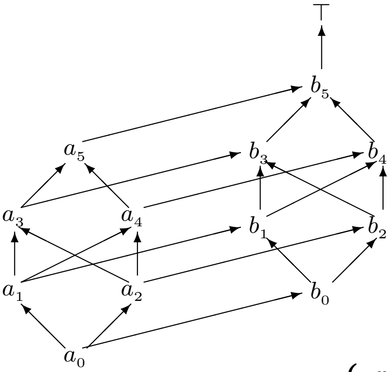

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Contextualizing

$$\text{Heyting algebras} \subsetneq \text{Residuated lattices} \subsetneq \text{Residuated multilattices} \subsetneq \text{Pocrims}$$

Remark

- *Any finite pocrim is a residuated multilattice.*
- *Any pocrim is m-associative iff it is a residuated multilattice.*
- *Any residuated multilattice is associative iff it is a residuated lattice.*
- *Any residuated lattice is idempotent iff it is a Heyting algebra.*

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From residuated multilattices to Heyting algebras

Lemma

Let $(A, \sqcup, \sqcap, *, \to, 1)$ *be a residuated multilattice and* $a, b \in A$.

- *The underlying multilattice is full.*
- $a * b$ *is a lower bound of* a *and* b.
- *If* $a * b \in a \sqcap b$ *then* $a * b$ *is the infimum of* a *and* b.
- *If* $*$ *is idempotent then* $a * b \in a \sqcap b$, *for all* $a, b \in A$.

Theorem

A residuated multilattice has idempotent product if and only if it is a Heyting algebra.

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However, a pocrim with idempotent product is not necessarily a Heyting algebra.

Outline

Homomorphisms between residuated multilattices

Definition

Let $h\colon M \to M'$ be a map between residuated multilattices, h is said to be a **homomorphism** if, for all $a, b \in M$,

- $h(a * b) = h(a) * h(b)$, $h(a \to b) = h(a) \to h(b)$,
- $h(a \sqcup b) \subseteq h(a) \sqcup h(b)$, and $h(a \sqcap b) \subseteq h(a) \sqcap h(b)$,

As a consequence the following conditions hold:

- $h(1) = 1$,
- $h(a \sqcup b) = \big(h(a) \sqcup h(b)\big) \cap h(M)$ and $h(a \sqcap b) = \big(h(a) \sqcap h(b)\big) \cap h(M)$.

Congruences on residuated multilattices

Definition

Let M be a residuated multilattice, a **congruence** on M is any equivalence relation $\equiv$ such that, for all $a, b, c \in M$, if $a \equiv b$, then

- $a * c \equiv b * c$, $a \rightarrow c \equiv b \rightarrow c$, $c \rightarrow a \equiv c \rightarrow b$,
- $a \sqcup c \mathrel{\widehat{\equiv}} b \sqcup c$, and $a \sqcap c \mathrel{\widehat{\equiv}} b \sqcap c$,

Theorem

Let $h\colon M \rightarrow M'$ be a homomorphism between residuated multilattices.
The **kernel relation**, *defined as $a \equiv b$ if and only if $h(a) = h(b)$, is a congruence.*

Theorem

Let M be a residuated multilattice and $\equiv$ a congruence relation on M.
The mapping $p\colon M \rightarrow M/_{\equiv}$ such that $p(x) = [x]$ is a surjective homomorphism of residuated multilattices.

Filters in pocrims

Definition

Given $\mathcal{A} = (A, \leq, *, \rightarrow, 1)$ a pocrim, $\varnothing \neq F \subseteq A$ is said to be a **filter** if the following conditions hold:

i) if $a, b \in F$, then $a * b \in F$

ii) if $a \leq b$ and $a \in F$, then $b \in F$.

On the other hand, F is said to be a **deductive system** if

i) $1 \in F$ and

ii) $a \rightarrow b \in F$ and $a \in F$ imply $b \in F$.

It is not difficult to see that both definitions are equivalent.

In [Halaš et al, 09] congruence relations and several kinds of filters in pocrims have been introduced.

Filters in multilattices

Definition

Let $(M, \sqcup, \sqcap)$ be a multilattice. A non-empty set $F \subseteq M$ is said to be a **filter** if the following conditions hold:

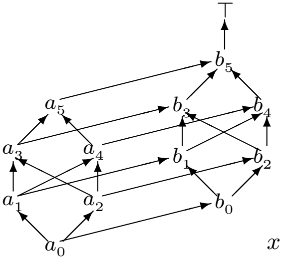
1. $i, j \in F$ implies $\varnothing \neq i \sqcap j \subseteq F$.
2. $i \in F$ implies $i \sqcup a \subseteq F$ for all $a \in M$.
3. For all $a, b \in M$, if $(a \sqcup b) \cap F \neq \varnothing$ then $a \sqcup b \subseteq F$.

Hereinafter, a non-empty set in a residuated multilattice is going to be named

- **deductive system** if it is a filter in the underlying pocrim and
- **m-filter** if it is a filter in the underlying multilattice.

Example

Let $A = \{a_i \mid 0 \leq i \leq 5\}$, $B = \{b_i \mid 0 \leq i \leq 5\}$ and $C = \{b_i \mid 2 \leq i \leq 5\}$



$$x * y = \begin{cases} x & \text{if } y = \top \\ y & \text{if } x = \top \\ b_2 & \text{if } x, y \in C \\ b_0 & \text{if } x \in B \setminus C, y \in B \\ b_0 & \text{if } x \in B, y \in B \setminus C \\ a_0 & \text{otherwise.} \end{cases} \qquad x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x = \top \\ a_5 & \text{if } x \in B, y \in A \\ b_1 & \text{if } x \in C \\ & \text{and } y \in B \setminus C \\ b_5 & \text{otherwise.} \end{cases}$$

- $C \cup \{\top\}$ is a deductive system but it is not an m-filter because $b_3 \sqcap b_4 = \{b_1, b_2\} \not\subseteq C$.
- $\{b_5, \top\}$ is an m-filter that is not a deductive system because $b_5 * b_5 = b_2 \notin \{b_5, \top\}$.
- $B \cup \{\top\}$ is both a deductive system and an m-filter.

Does a filter define a congruence?

Proposition

Let M be a residuated multilattice, 1 its maximum element and $\equiv$ a congruence. The equivalence class $[1]$ is a deductive system and an m-filter.

It is well known that a filter (deductive system) F in a **pocrim** defines a congruence that can be characterized as follows

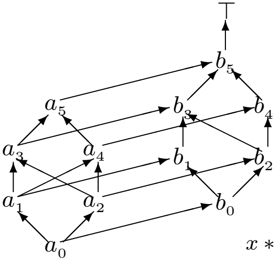
$$a \equiv b \quad \text{if and only if} \quad a \to b,\ b \to a \in F$$

It is also true that, in a **residuated lattice**, given a filter F the relation defined as follows is a congruence

$$a \equiv b \quad \text{iff} \quad a \to b,\ b \to a \in F \quad \text{iff} \quad (a \to b) \wedge (b \to a) \in F$$

Example

Let $A_0 = \{a_0, a_1\}, A_1 = \{a_i \mid 2 \leq i \leq 5\}, B_0 = \{b_0, b_1\}, B_1 = \{b_i \mid 2 \leq i \leq 5\}$ such that $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$.



$$x * y = \begin{cases} x & \text{if } y = \top \\ y & \text{if } x = \top \\ a_0 & \text{if } x \in A \cup B \text{ and } y \in A_0 \\ & \quad \text{or } x \in B_0 \text{ and } y \in A \\ a_2 & \text{if } x \in A_1 \cup B_1 \text{ and } y \in A_1 \\ b_0 & \text{if } x \in B \text{ and } y \in B_0 \\ b_2 & \text{if } x, y \in B_1 \end{cases} \qquad x \to y = \begin{cases} \top & \text{if } x \leq y \\ y & \text{if } x = \top \\ a_1 & \text{if } x \in B_1 \text{ and } y \in A_0 \\ a_5 & \text{if } x \in B_0 \text{ and } y \in A \\ & \quad \text{or } x \in B_1 \text{ and } y \in A_1 \\ b_1 & \text{if } x \in A_1, y \in A_0 \cup B_0 \\ b_5 & \text{otherwise.} \end{cases}$$

$B_1 \cup \{\top\}$ is a deductive system and an m-filter.
But there not exists a congruence such that $[\top] = B_1 \cup \{\top\}$.

Filters in residuated multilattices

Definition

Let M be a residuated multilattice. $F \subseteq M$ is said to be a **filter** if the following conditions hold:

- F is a deductive system and
- $a \to b \in F$ implies $a \sqcup b \to b \subseteq F$ and $a \to a \sqcap b \subseteq F$.

Theorem

Let M be a residuated multilattice and F a deductive system. F is a filter if and only if the following conditions hold:

1. *F is an m-filter,*
2. *for all $x, y \in a \sqcup b$, if $x \to y \in F$ then $y \to x \in F$.*
3. *for all $x, y \in a \sqcap b$, if $x \to y \in F$ then $y \to x \in F$.*

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Filters in residuated multilattices

Theorem

Let $h\colon M \to M'$ be a homomorphism between residuated multilattices.

- $h^{-1}(1) = \{x \in M \mid h(x) = 1\}$ *is a filter of M, the* **kernel filter**.

Theorem

Let M be a residuated multilattice and F be a filter. Then, the relation

$$a \equiv_F b \quad \textit{iff} \quad a \to b,\ b \to a \in F \quad \textit{iff} \quad (a \to b) \sqcap (b \to a) \subseteq F$$

defines a congruence.

Fine filters in residuated multilattices

Definition

Let M be a residuated multilattice. A deductive system F is said to be **fine** if for all $a, b, c \in M$ the following conditions hold:

1. If $a \to c, b \to c \in F$, then $(a \sqcup b) \to c \subseteq F$
2. If $c \to a, c \to b \in F$, then $c \to (a \sqcap b) \subseteq F$

Proposition

Every fine deductive system is a filter.

Theorem

Let $h\colon M \to M'$ be a homomorphism between residuated multilattices. Then,

$$h(M) \text{ is a lattice} \quad \text{if and only if} \quad h^{-1}(1) \text{ is a fine filter.}$$

Outline

Future work

- Studying applications of this residuated structure.
- Extending this structure by replacing the operations $*$ and $\longrightarrow$ by hyperoperations.
- Looking for an adequate definition of distributivity on multilattices and deepening in the relationship between boolean multilattices and hyperings.

Residuated multilattices

Between pocrims and residuated lattices

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