Recent Results on Commutative and Non-Commutative Effect Algebras

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- for classical mechanics

$$\inf_s(\sigma_s(x)\sigma_s(y)) = 0.$$

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- G. Birkhoff and J. von Neumann, 1936 quantum logic

Quantum structures

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- MV-algebras - compatibility

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- **MV-algebra** $(M; \oplus, ^*, 0)$ of type $\langle 2, 1, 0\rangle$ such that

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- 1. $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
  2. $x \oplus y = y \oplus x$
  3. $x \oplus 0 = x$
  4. $x^{**} = x$
  5. $x \oplus 0^* = 0^*$
  6. $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x.$

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- $(\Gamma(G, u), \oplus, ^*, 0)$ - prototypical example of MV-algebras

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- Only countably many of subvarieties

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- **GMV-algebra** is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z;$

(A2) $x \oplus 0 = 0 \oplus x = x;$

(A3) $x \oplus 1 = 1 \oplus x = 1;$

(A4) $1^{\sim} = 0; 1^{-} = 0;$

(A5) $(x^{-} \oplus y^{-})^{\sim} = (x^{\sim} \oplus y^{\sim})^{-};$

(A6) $x \oplus (x^{\sim} \odot y) = y \oplus (y^{\sim} \odot x) = (x \odot y^{-}) \oplus y = (y \odot x^{-}) \oplus x;$

(A7) $x \odot (x^{-} \oplus y) = (x \oplus y^{\sim}) \odot y;$

(A8) $(x^{-})^{\sim} = x.$

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- $$x \leq y \quad \text{iff} \quad x^- \oplus y = 1$$
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- $x \vee y = x \oplus (x^\sim \odot y)$ and $x \wedge y = x \odot (x^- \oplus y)$.
- GMV-algebra M is an MV-algebra iff $x \oplus y = y \oplus x$ for all $x, y \in M$.
- a partial operation $+$ on M: $a + b = a \oplus b$ iff $a \odot b = 0$ iff $a \leq b^-$ iff $b \leq a^\sim$.

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$$\begin{aligned} x \oplus y &:= (x + y) \wedge u, \\ x^{-} &:= u - x, \\ x^{\sim} &:= -x + u, \\ x \odot y &:= (x - u + y) \vee 0, \end{aligned}$$

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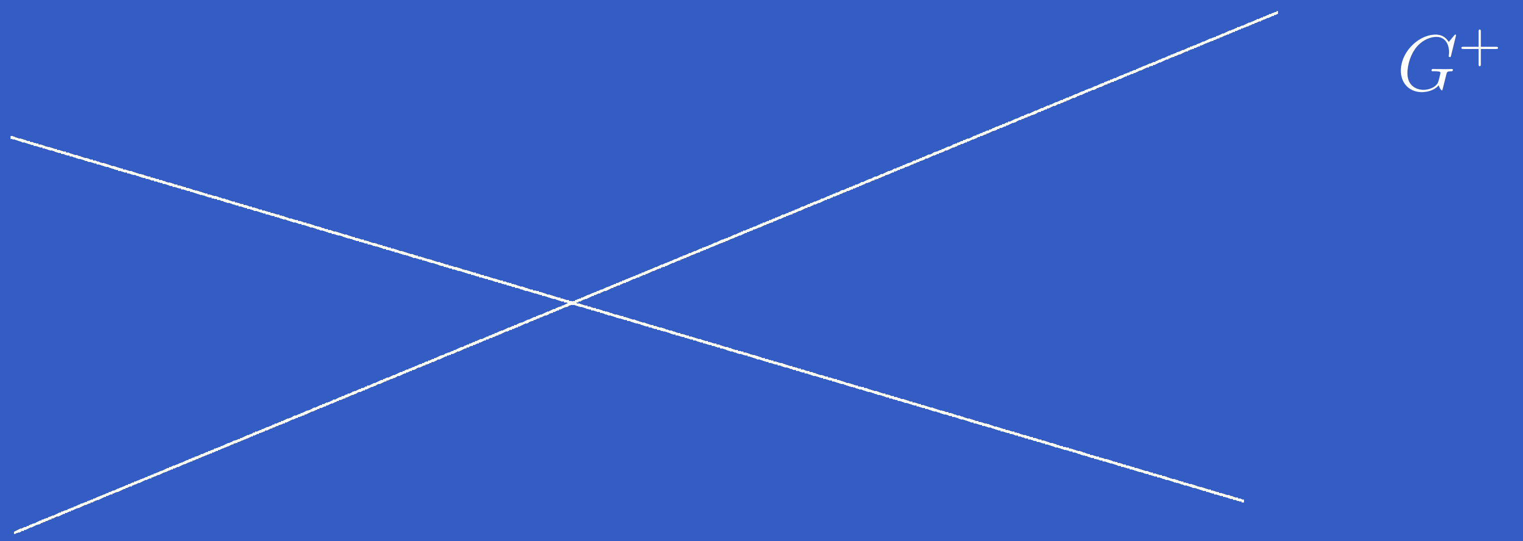
$(\Gamma(G, u); \oplus, ^-, ^\sim, 0, u)$ is a GMV-algebra.

Theorem 0.1 [Dvu 2002] *For any GMV-algebra* M*, there exists a unique* $($*up to isomorphism*$)$ *unital* ℓ*-group* G *with a strong unit* u *such that* $M \cong \Gamma(G, u)$*.*
The functor Γ *defines a categorical equivalence between the category of GMV-algebras and the category of unital* ℓ*-groups.*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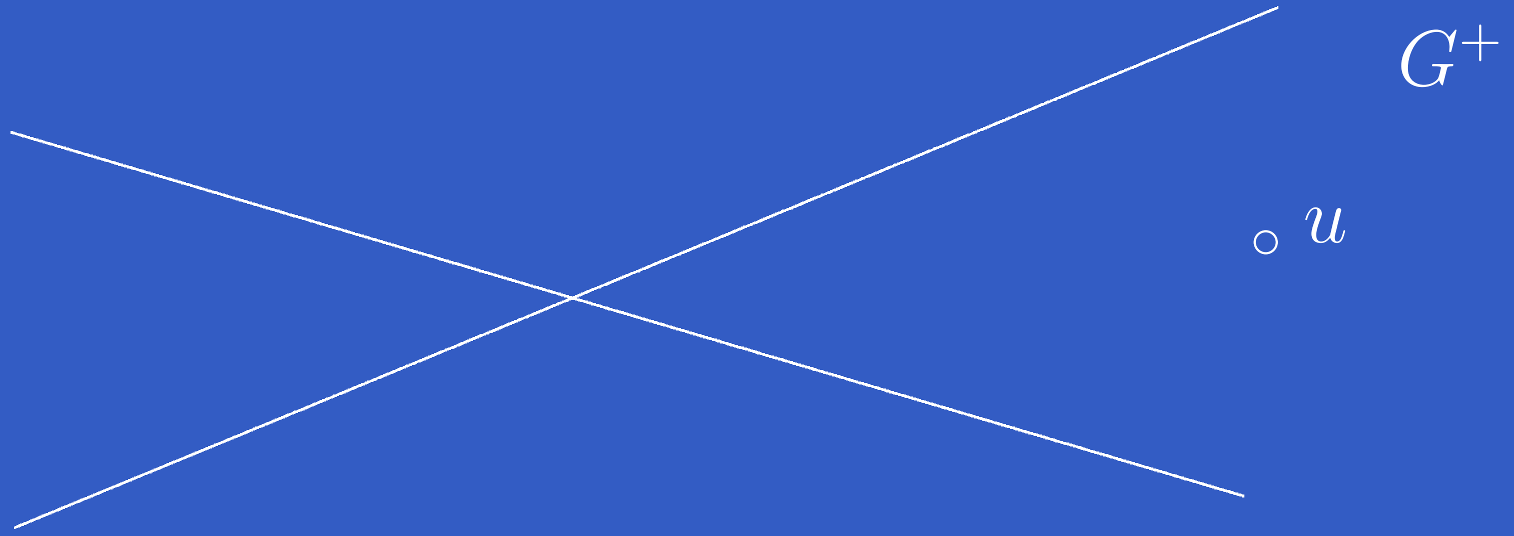
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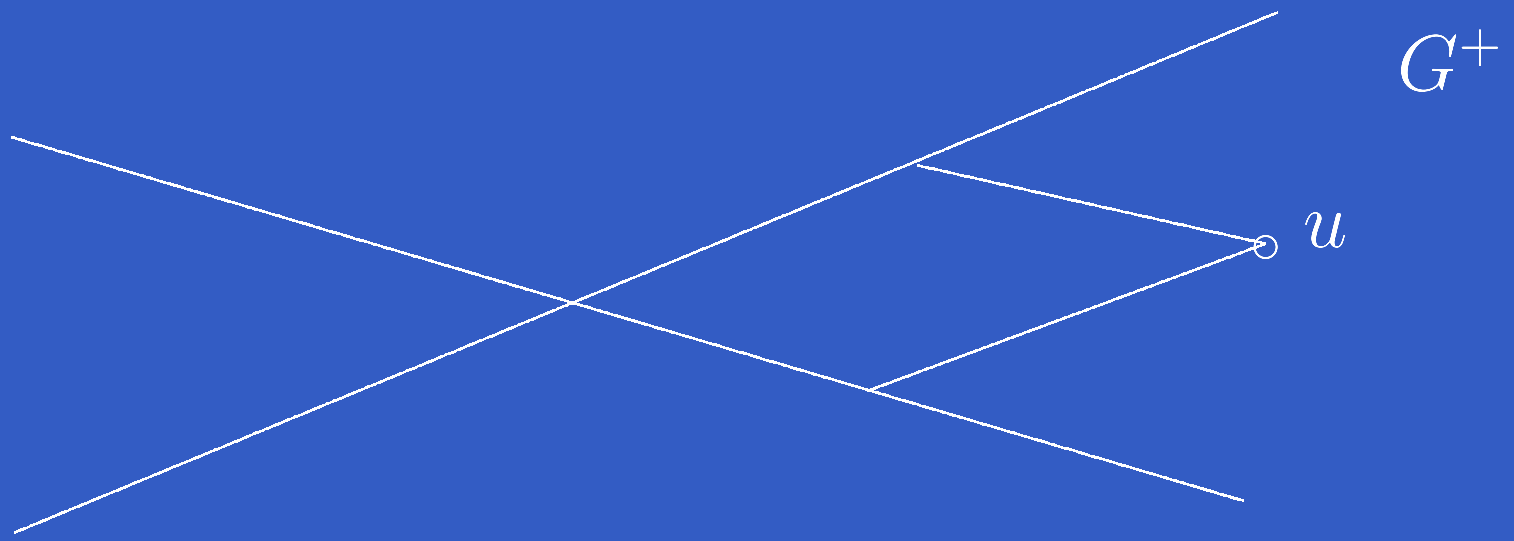
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- Boolean algebra, OML: $a + b\ \exists$ iff $a \leq b'$, $a + b := a \vee b$

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- (RDP): If $c \leq a + b$ $\exists$ $a_1, b_1 \in M$ such that $a_1 \leq a$, $b_1 \leq b$ and $c = a_1 + b_1$.
- $a_1 + a_2 = b_1 + b_2$, $\exists$ $c_{11}, c_{12}, c_{21}, c_{22} \in M$ s.t. $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$, and $b_2 = c_{21} + c_{22}$.

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- s state on (G, u): $s(u) = 1$, $s(G^+) \subseteq \mathbb{R}^+$, $s(g + h) = s(g) + (h)$. $\mathcal{S}(\Gamma(G, u)) \cong \mathcal{S}(G, u)$

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- $(M;+,0,1)$ is an effect algebra with RDP
- perfect MV-algebras, $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (1,0))$
- Di Nola-Lettieri: the variety of perfect MV-algebras is categorically equivalent with the category of Abelian ℓ-groups

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- if $1+a$ or $a+1$ exists, then $a=0$.
- RDP if $a_1+a_2 = b_1+b_2$ $\exists\ d_1, d_2, d_3, d_4 \in E$ s.t. $d_1+d_2 = a_1,\ d_3+d_4 = a_2,\ d_1+d_3 = b_1$, $d_2+d_4 = b_2$.

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- RDP$_2$: RDP + $d_2 \wedge d_3 = 0$ - pseudo MV-algebra
- **Theorem 0.4** *If E is a pseudo effect algebra with* (RDP)$_1$*, $E = \Gamma(G, u)$ for some unital po-group with (RDP_1).*
  If E satisfies (RDP_2), E is a GMV-algebra.

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- two sequences of atoms $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$, similar $\exists$ permutation s.t. $a_i = b_{p_i}$

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- $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_n)$ such that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, then $m = n$ and the sequences $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are similar.

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- **Theorem 0.7** *Let G be an Abelian po-group G with a generative unit u fulfilling UARP and let for any $x \in G^+[0, u]$, there exist a finite sequence of atoms $a_1, \ldots, a_n$ in $G^+[0, u]$ such that $x = a_1 + \cdots + a_n$. Then the following statements hold:*

- (i) $G^+[0,u]$ satisfies RDP.
  (ii) For any natural $n \geqslant 1,$ the effect algebra $G^+[0,nu]$ satisfies RDP.
  (iii) $G^+[0,nu] = \underbrace{G^+[0,u] + \cdots + G^+[0,u]}_{n-times}$.
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- The equation $G^+[0,nu] = \underbrace{G^+[0,u] + \cdots + G^+[0,u]}_{n-times}$ does not hold, in general.

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- EA is *orthocomplete* if an arbitrary orthogonal family has a sum.
- EA is σ-*orthocomplete* if every countable orthogonal system has a sum.
- E satisfies the chain finite condition, if every chain in E is a finite set

- **Theorem 0.8** *If an effect algebra E with RDP satisfies the chain finite condition, then*

  (i) E *is a finite set.*

  (ii) E *is an MV-effect algebra.*

- **Theorem 0.10** *If an effect algebra E with RDP satisfies the chain finite condition, then*
  (i) *E is a finite set.*
  (ii) *E is an MV-effect algebra.*

- **Theorem 0.11** *Let E be a σ-orthocomplete atomic effect algebra with RDP and let $A(E)$ be at most countable. Then the following statements hold:*
  (i) If $a_i, a_j \in A(E)$ with $a_i \neq a_j$, then $a_i + a_j$ and $a_i \vee a_j$ exist and $a_i + a_j = a_i \vee a_j$.
  (ii) For any natural number $n \geqslant 2$, the finite set of mutually different atoms

- $\{a_1, \dots, a_n\} \subseteq A(E)$ is orthogonal in E and $\sum_{i=1}^n a_i = \bigvee_{i=1}^n a_i$.

- $\{a_1, \ldots, a_n\} \subseteq A(E)$ is orthogonal in E and $\sum_{i=1}^n a_i = \bigvee_{i=1}^n a_i$.

(iii) The set $A(E)$ is an orthogonal family, and $\sum A(E) = \bigvee A(E)$.

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(iii) The set $A(E)$ is an orthogonal family, and $\sum A(E) = \bigvee A(E)$.

- **Theorem 0.14** *Let E be a σ-orthocomplete atomic effect algebra with RDP and let $A(E)$ be at most countable. Let $\imath_i$ be the isotropic index of $a_i \in A(E)$. The following statements hold.*

- $\{a_1, \ldots, a_n\} \subseteq A(E)$ is orthogonal in E and $\sum_{i=1}^n a_i = \bigvee_{i=1}^n a_i$.

(iii) The set $A(E)$ is an orthogonal family, and $\sum A(E) = \bigvee A(E)$.

- **Theorem 0.15** *Let E be a σ-orthocomplete atomic effect algebra with RDP and let $A(E)$ be at most countable. Let $\imath_i$ be the isotropic index of $a_i \in A(E)$. The following statements hold.*

(i) For any $a_i \in A(E)$, the isotropic index $\imath_i$ is finite, $i \in I$.

(ii) For any $a_i \in A(E)$, the interval $E[0, \imath_i a_i] = \{x \in E \mid 0 \leqslant x \leqslant \imath_i a_i\}$ equals $\{0, a_i, \ldots, \imath_i a_i\}$.

(ii) For any $a_i \in A(E)$, the interval $E[0, \imath_i a_i] = \{x \in E \mid 0 \leqslant x \leqslant \imath_i a_i\}$ equals $\{0, a_i, \ldots, \imath_i a_i\}$.

(iii) For any distinct elements $a_i, a_j \in A(E)$, $(\imath_i a_i) \wedge (\imath_j a_j)$, and $(\imath_i a_i) \vee (\imath_j a_j)$ exist, and $(\imath_i a_i) \wedge (\imath_j a_j) = 0$ and $(\imath_i a_i) \wedge (\imath_j a_j) = (\imath_i a_i) + (\imath_j a_j)$.

(ii) For any $a_i \in A(E)$, the interval $E[0, \imath_i a_i] = \{x \in E \mid 0 \leqslant x \leqslant \imath_i a_i\}$ equals $\{0, a_i, \ldots, \imath_i a_i\}$.

(iii) For any distinct elements $a_i, a_j \in A(E)$, $(\imath_i a_i) \wedge (\imath_j a_j)$, and $(\imath_i a_i) \vee (\imath_j a_j)$ exist, and $(\imath_i a_i) \wedge (\imath_j a_j) = 0$ and $(\imath_i a_i) \wedge (\imath_j a_j) = (\imath_i a_i) + (\imath_j a_j)$.

(iv) For any finite set of mutually distinct elements $a_1, \ldots, a_n \in A(E)$, $n \geq 1$, $(\imath_1 a_1) + \cdots + (\imath_n a_n)$ exists and $(\imath_1 a_1) + \cdots + (\imath_n a_n) = (\imath_1 a_1) \vee \cdots \vee (\imath_n a_n)$.

(iii) The set $\{\imath_i a_i \mid a_i \in A(E)\}$ is an orthogonal system, and $\sum\{\imath_i a_i \mid a_i \in A(E)\} = \bigvee\{\imath_i a_i \mid a_i \in A(E)\} = 1$.

(iii) The set $\{\imath_i a_i \mid a_i \in A(E)\}$ is an orthogonal system, and $\sum\{\imath_i a_i \mid a_i \in A(E)\} = \bigvee\{\imath_i a_i \mid a_i \in A(E)\} = 1$.

- **Theorem 0.17** *Let E be a σ-orthocomplete atomic effect algebra with RDP and let $A(E)$ be at most countable. Let $\imath_i$ be the isotropic index of $a_i \in A(E)$. The following statements hold.*

(iii) The set $\{\imath_i a_i \mid a_i \in A(E)\}$ is an orthogonal system, and $\sum\{\imath_i a_i \mid a_i \in A(E)\} = \bigvee\{\imath_i a_i \mid a_i \in A(E)\} = 1$.

- **Theorem 0.18** *Let E be a σ-orthocomplete atomic effect algebra with RDP and let $A(E)$ be at most countable. Let $\imath_i$ be the isotropic index of $a_i \in A(E)$. The following statements hold.*

(i) For any $a_i \in A(E)$, the isotropic index $\imath_i$ is finite, $i \in I$.

(ii) For any $a_i \in A(E)$, the interval $E[0, \imath_i a_i] = \{x \in E \mid 0 \leqslant x \leqslant \imath_i a_i\}$ equals $\{0, a_i, \ldots, \imath_i a_i\}$.

(ii) For any $a_i \in A(E)$, the interval $E[0, \imath_i a_i] = \{x \in E \mid 0 \leqslant x \leqslant \imath_i a_i\}$ equals $\{0, a_i, \ldots, \imath_i a_i\}$.

(iii) For any distinct elements $a_i, a_j \in A(E)$, $(\imath_i a_i) \wedge (\imath_j a_j)$, and $(\imath_i a_i) \vee (\imath_j a_j)$ exist, and $(\imath_i a_i) \wedge (\imath_j a_j) = 0$ and $(\imath_i a_i) \wedge (\imath_j a_j) = (\imath_i a_i) + (\imath_j a_j)$.

(ii) For any $a_i \in A(E)$, the interval $E[0, \imath_i a_i] = \{x \in E \mid 0 \leqslant x \leqslant \imath_i a_i\}$ equals $\{0, a_i, \ldots, \imath_i a_i\}$.

(iii) For any distinct elements $a_i, a_j \in A(E)$, $(\imath_i a_i) \wedge (\imath_j a_j)$, and $(\imath_i a_i) \vee (\imath_j a_j)$ exist, and $(\imath_i a_i) \wedge (\imath_j a_j) = 0$ and $(\imath_i a_i) \wedge (\imath_j a_j) = (\imath_i a_i) + (\imath_j a_j)$.

(iv) For any finite set of mutually distinct elements $a_1, \ldots, a_n \in A(E)$, $n \geq 1$, $(\imath_1 a_1) + \cdots + (\imath_n a_n)$ exists and $(\imath_1 a_1) + \cdots + (\imath_n a_n) = (\imath_1 a_1) \vee \cdots \vee (\imath_n a_n)$.

(v) The set $\{\imath_i a_i \mid a_i \in A(E)\}$ is an orthogonal system, and $\sum\{\imath_i a_i \mid a_i \in A(E)\} = \bigvee\{\imath_i a_i \mid a_i \in A(E)\} = 1$.

(v) The set $\{\imath_i a_i \mid a_i \in A(E)\}$ is an orthogonal system, and $\sum\{\imath_i a_i \mid a_i \in A(E)\} = \bigvee\{\imath_i a_i \mid a_i \in A(E)\} = 1$.

- **Theorem 0.20** *Let E be a σ-orthocomplete atomic effect algebra with RDP and let $A(E)$ be at most countable. Let $\imath_i$ be the isotropic index of $a_i \in A(E)$. The following statements hold.*

(v) The set $\{\imath_i a_i \mid a_i \in A(E)\}$ is an orthogonal system, and $\sum\{\imath_i a_i \mid a_i \in A(E)\} = \bigvee\{\imath_i a_i \mid a_i \in A(E)\} = 1$.

- **Theorem 0.21** *Let E be a σ-orthocomplete atomic effect algebra with RDP and let $A(E)$ be at most countable. Let $\imath_i$ be the isotropic index of $a_i \in A(E)$. The following statements hold.*

(i) For any $a_i \in A(E)$, the isotropic index $\imath_i$ is finite, $i \in I$.

(ii) For any $a_i \in A(E)$, the interval $E[0, \imath_i a_i] = \{x \in E \mid 0 \leqslant x \leqslant \imath_i a_i\}$ equals $\{0, a_i, \ldots, \imath_i a_i\}$.

(ii) For any $a_i \in A(E)$, the interval $E[0, \imath_i a_i] = \{x \in E \mid 0 \leqslant x \leqslant \imath_i a_i\}$ equals $\{0, a_i, \ldots, \imath_i a_i\}$.

(iii) For any distinct elements $a_i, a_j \in A(E)$, $(\imath_i a_i) \wedge (\imath_j a_j)$, and $(\imath_i a_i) \vee (\imath_j a_j)$ exist, and $(\imath_i a_i) \wedge (\imath_j a_j) = 0$ and $(\imath_i a_i) \wedge (\imath_j a_j) = (\imath_i a_i) + (\imath_j a_j)$.

(ii) For any $a_i \in A(E)$, the interval $E[0, \imath_i a_i] = \{x \in E \mid 0 \leqslant x \leqslant \imath_i a_i\}$ equals $\{0, a_i, \ldots, \imath_i a_i\}$.

(iii) For any distinct elements $a_i, a_j \in A(E)$, $(\imath_i a_i) \wedge (\imath_j a_j)$, and $(\imath_i a_i) \vee (\imath_j a_j)$ exist, and $(\imath_i a_i) \wedge (\imath_j a_j) = 0$ and $(\imath_i a_i) \wedge (\imath_j a_j) = (\imath_i a_i) + (\imath_j a_j)$.

(iv) For any finite set of mutually distinct elements $a_1, \ldots, a_n \in A(E)$, $n \geq 1$, $(\imath_1 a_1) + \cdots + (\imath_n a_n)$ exists and $(\imath_1 a_1) + \cdots + (\imath_n a_n) = (\imath_1 a_1) \vee \cdots \vee (\imath_n a_n)$.

(v) The set $\{\imath_i a_i \mid a_i \in A(E)\}$ is an orthogonal system, and $\sum\{\imath_i a_i \mid a_i \in A(E)\} = \bigvee\{\imath_i a_i \mid a_i \in A(E)\} = 1$.

Applications

- PEA E is *monotone σ-complete* provided that every ascending sequence $x_1 \leqslant x_2 \leqslant \cdots$ of elements in E has a supremum $x = \bigvee_n x_n$.

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- **Theorem 0.23** *Let E be a monotone σ-complete atomic pseudo-effect algebra with RDP and let $A(E)$ be at most countable. Then E is a commutative PEA, i.e., E is an effect algebra.*

Applications

- PEA E is *monotone σ-complete* provided that every ascending sequence $x_1 \leqslant x_2 \leqslant \cdots$ of elements in E has a supremum $x = \bigvee_n x_n$.
- **Theorem 0.24** *Let E be a monotone σ-complete atomic pseudo-effect algebra with RDP and let $A(E)$ be at most countable. Then E is a commutative PEA, i.e., E is an effect algebra.*
- A state s is σ-additive if, for any monotone sequence $\{a_i\}$ s.t. $\bigvee_i a_i = a$, we have $s(a) = \lim_i s(a_i)$. Or, if $a = \sum_n a_n$, then $s(a) = \sum_n s(a_n)$.

- **Theorem 0.25** *Let* E *be a* σ*-orthocomplete atomic effect algebra with RDP and let* $A(E)$ *be at most countable. Let* $\imath_i$ *be the isotropic index of* $a_i \in A(E)$. *For any* $i \in I$, *we define a mapping* $s_i : E \to [0,1]$ *via*

$$s_i(a) = \max\{j \mid ja_i \leqslant a \wedge \imath_i a_i\}/\imath_i, \quad a \in E.$$

Then s_i *is an extremal state on* E *which is also* σ*-additive. If* s *is a* σ*-additive state on* E, *then* $s(a) = \sum_i \lambda_i s_i(a)$, $a \in E$.

- Moreover, every extremal state that is also σ-additive is just of the form s_i for a unique i, and a state $s = s_i$ for some $i \in I$ if and only if $s(\imath_i a_i) = 1$.

Observables

- observable: $x : \mathcal{B}(\mathbb{R}) \to E$, E monotone σ-complete EA: (i) $x(\mathbb{R}) = 1$, (ii) if E and F are mutually disjoint Borel sets, then $x(E \cup F) = x(E) + x(F)$, and (iii) if $\{E_i\}$ is a sequence of Borel sets such that $E_i \subseteq E_{i+1}$ for every i and $E = \bigcup_i E_i$, then $x(E) = \bigvee_i x(E_i)$.

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- $\{a_n : n \in N\}$ be a finite or infinite sequence of summable elements, $\sum_{n\in N} a_n = 1$,
- $x(E) := \sum\{a_n : t_n \in E\},\ E \in \mathcal{B}(\mathbb{R})$. observable

- $x_t := x((-\infty, t))\ t \in \mathbb{R}$

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- A tribe is a collection $\mathcal{T} \subseteq [0,1]^{\Omega}$ s.t. (i) $1 \in \mathcal{T}$, (ii) if $f \in \mathcal{T}$, then $1 - f \in \mathcal{T}$, and (iii) if $\{f_n\}$ is a sequence from $\mathcal{T}$, then $\min\{\sum_{n=1}^{\infty} f_n, 1\} \in \mathcal{T}$.

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- monotone σ-complete EA is representable if there is $(\Omega, \mathcal{T}, h)$ such that $\mathcal{T}$ is a effect-tribe, $h : \mathcal{T} \to E$ onto
- Loomis-Sikorski Theorem [BCD] Every monotone σ-complete EA with RDP is a σ-epimorphic image of an effect-tribe with RDP
- $\mathcal{E}(H)$ is representable

- **Theorem 0.26 ([DvKu])** *If $\{x_t : t \in \mathbb{R}\}$ is a system of elements of a representable monotone σ-complete EA satisfying (1)-(3), there is a unique observable x such that $x_t = x((-\infty, t)), t \in \mathbb{R}$.*

- **Theorem 0.27 ([DvKu])** *If $\{x_t : t \in \mathbb{R}\}$ is a system of elements of a representable monotone σ-complete EA satisfying (1)-(3), there is a unique observable x such that $x_t = x((-\infty, t)), t \in \mathbb{R}$.*
- The same is true if: $E = \mathcal{E}(H)$

- **Theorem 0.28 ([DvKu])** *If* $\{x_t : t \in \mathbb{R}\}$ *is a system of elements of a representable monotone* σ*-complete EA satisfying (1)-(3), there is a unique observable* x *such that* $x_t = x((-\infty, t)), t \in \mathbb{R}$.
- The same is true if: $E = \mathcal{E}(H)$
- E is a σ-lattice EA

- **Theorem 0.29 ([DvKu])** *If* $\{x_t : t \in \mathbb{R}\}$ *is a system of elements of a representable monotone* σ*-complete EA satisfying (1)-(3), there is a unique observable* x *such that* $x_t = x((-\infty, t)), t \in \mathbb{R}$.
- The same is true if: $E = \mathcal{E}(H)$
- E is a σ-lattice EA
- E is a Boolean σ-algebra, and x preserves $\bigcup$

- **Theorem 0.30 ([DvKu])** *If* $\{x_t : t \in \mathbb{R}\}$ *is a system of elements of a representable monotone* σ*-complete EA satisfying (1)-(3), there is a unique observable* x *such that* $x_t = x((-\infty, t)), t \in \mathbb{R}$.
- The same is true if: $E = \mathcal{E}(H)$
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- E σ-orthocomplete orthomodular poset

Thank you for your attention

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