An analysis of the fuzzyness of the Codd's Relational Model

Manuel Enciso
Universidad de Málaga

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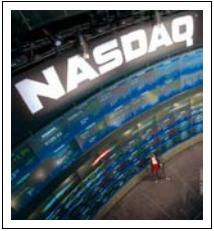
Imprecise data in the relational model

Abiteboul et al. 2005

"Traditional DBMSs were applied to business data processing, which typically focused on numbers and character strings. ... When one leaves business data processing, essentially all data is uncertain or imprecise"

- The authors asked for a way to store imprecise data
- but also a way to express imprecise queries and get imprecise answers.

Fuzzyness in the data processing



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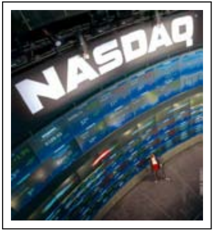

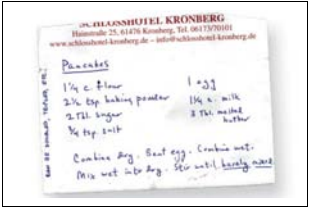

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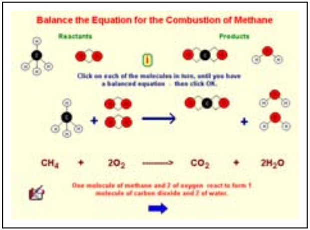

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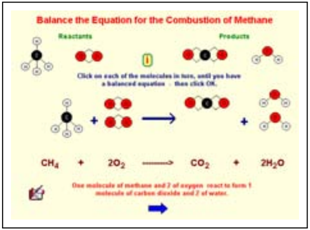
Fuzzyness in the data processing



The original relational model

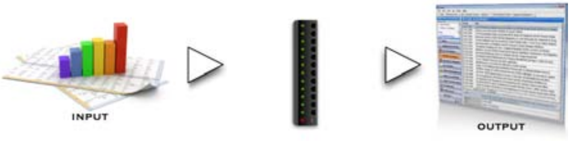
Schema

Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ a finite set of domains, $Y = \{A_1, \ldots, A_k\}$ a set of attributes where $k \geq n$. A schema is a set of pairs $\{(A_i, D_i) \mid D_i \in \mathcal{D}\}_{i \in \{1,\ldots,k\}}$

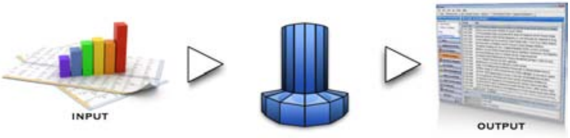
Body

Let $\mathcal{S}$ a relational schema, a body is a set of tuples $\{[(A_1, v_1^j), \ldots, (A_k, v_k^j)] \mid v_i \in D_i \; \forall i \in 1, \ldots, k\}_{j \in J}$

Databases and Software Engineering



Databases and Software Engineering



Database as a function

The functional dependency

Functional Dependency

A functional dependency is an expression $X \Rightarrow Y$ where $X, Y \subseteq \Omega$.
A relation R satisfies $X \Rightarrow Y$ if, for all tuples $t_1, t_2 \in R$,
$t_{1/X} = t_{2/X}$ implies that $t_{1/Y} = t_{2/Y}$.

Key concept to consider database as a function

Keys

Key

A set of attributes $\mathcal{K} \subseteq \Omega$ is a key iff the functional dependency $\mathcal{K} \Rightarrow \Omega$ holds.

Keys are used to identify each row of a table

Relational Normalization

1FN

A relation $\mathcal{R}$ is in First Normal Form is none of its domains is a Relation.

Skin	Hair	Name	Sex
dark	black	John	Male
		Albert	Male
light	quasi red	Mary	Female
		Dave	Male
light	black	Noa	Female

Relational Normalization

BCNF

A relation $\mathcal{R}$ is in Boyce-Codd Normal Form is all the determinants of its non-trivial functional dependencies are keys.

We look for tables which are "fully described" using functional dependencies $\mathcal{K} \Rightarrow \Omega - \mathcal{K}$.

Imprecise Data

NULL values

Let $\mathcal{D}$ be a domain. Its is used in the relational model by adding the `NULL` value.

The imprecision and vagueness is labelled by using NULL values.
The query languages area based on a three-valued logic.

Imprecise Data

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The imprecision and vagueness is labelled by using NULL values. The query languages area based on a three-valued logic.

Imprecise Data

- Inapplicability or Unknown.
- Different NULL values (NULL variables).
- Different degrees of knowledge.

Questions

- Does the relational model consider fuzzy domains?
- May we introduce the notion of fuzzy table?
- Which is a fuzzy primary key?
- There exists a common notion of fuzzy functional dependencies?

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Fuzzyness in the relational model

Approach	Domains	Table	Functional Dependency	Logic

Fuzzyness in the relational model

Approach	Domains	Table	Functional Dependency	Logic
Codd Rel. Model				SLFD

Fuzzy sets

- Fuzzy sets $\mathcal{U} \to [0, 1]$.
- L-fuzzy sets $\mathcal{U} \to L$ where L is a complete lattice.
- Complete residuated lattices: $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1 \rangle$ where:
    - $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice.
    - $\langle L, \otimes, 1 \rangle$ is a commutative monoid.
    - $\otimes$ and $\to$ satisfy $x \otimes y \leq z$ if and only if $x \leq y \to z$
- A truth-stressing hedge *: for all $x, y \in L$,

$$1^* = 1, \ x^* \leq x, \ (x \to y)^* \leq x^* \to y^*, \quad \text{and} \quad x^{**} = x^*$$

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Graded (fuzzy) sets

Having **L**, we define the usual notions:

- An **L**-set A in universe $\mathcal{U}$ is a mapping $A\colon \mathcal{U} \to L$ where

  $A(u)$ is "the degree in which u belongs to A".
- $\mathbf{L}^{\mathcal{U}}$ denotes the set of fuzzy sets in universe $\mathcal{U}$.
- Let $A, B \in \mathbf{L}^{\mathcal{U}}$ and $c \in L$.
  - The degree of inclusion of A in B is defined as:

    $$S(A, B) = \bigwedge_{u \in \mathcal{U}} (A(u) \to B(u))$$

    Note that $S(A, B) = 1$ iff $A(u) \leq B(u)$ for all $u \in \mathcal{U}$. In this case, we will write $A \subseteq B$.
  - $A \cup B$ is defined as $(A \cup B)(u) = A(u) \vee B(u)$ for all $u \in \mathcal{U}$.
  - $A \cap B$ is defined as $(A \cap B)(u) = A(u) \wedge B(u)$ for all $u \in \mathcal{U}$.
  - $c \otimes A$ is defined as $(c \otimes A)(u) = c \otimes A(u)$ for all $u \in \mathcal{U}$.

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Fuzzy relations

- A similarity relation in a non-empty set $\mathcal{U}$ is a mapping $\approx : \mathcal{U} \times \mathcal{U} \to L$ that satisfies:
    - Reflexivity: $(a \approx a) = 1$ for all $a \in \mathcal{U}$.
    - Symmetry: $(a \approx b) = (b \approx a)$ for all $a, b \in \mathcal{U}$.
- A similarity relation is a fuzzy equivalence if it also satisfies:
    - $\otimes$-transitivity: $(a \approx b) \otimes (b \approx c) \leq (a \approx c)$ for all $a, b, c \in \mathcal{U}$.
- A fuzzy equality is a fuzzy equivalence in which $(a \approx b) = 1$ implies $a = b$.

First approach

"The same job and the same experience imply the same salary"

Job	Experience	Name	Salary
Sales	4	John	40.000
Sales	5	Albert	35.000
Sales	5	Mary	40.000
IT	5	Dave	60.000
DBA	5	Noa	60.000

"Similar job and Similar experience imply similar salary"

$$\texttt{job,experience} \Rightarrow \texttt{salary}$$

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First approach

Let $\{(D_y, \approx_y) \mid y \in Y\}$ be a family of domains with similarity relations.
These relations can be extended to $D_A = \prod_{y \in A} D_y$, for all $A \subseteq Y$, as follows

$$(t_1[A] \approx t_2[A]) = \bigwedge_{y \in A} (t_1[y] \approx_y t_2[y])$$

Definition

A data table $\mathcal{R}$ satisfies the functional dependency $A \Rightarrow B$ if, for all $t_1, t_2 \in \mathcal{R}$, $(t_1[A] \approx t_2[A]) \leq (t_1[B] \approx t_2[B])$

First approach

- The functional dependency is not an extension.
- Armstrong's axioms are sound and complete.
- Simplification logic and its automated deduction method can be used.

First extension [?]

Approach	Domains	Table	Functional Dependency	Logic
Raju & Majumdar	✓	✗	✗	SLFD

Second approach

"Similar job and similar experience imply similar salary"

Job	Experience	Name	Salary
IT	5	John	40.000
IT	4	Albert	35.000
DBA	5	Mary	40.000
IT	5	Dave	38.000
DBA	5	Noa	60.000

"Similar job and similar experience more or less imply similar salary"

$$\texttt{job,experience} \overset{c}{\Rightarrow} \texttt{salary}$$

Second approach

“Similar job and similar experience imply similar salary”

Job	Experience	Name	Salary
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“Similar job and similar experience more or less imply similar salary”

$$\texttt{job,experience} \overset{c}{\Rightarrow} \texttt{salary}$$

Second approach

Now, a functional dependency is a formula $A \Rightarrow B$ endowed with a grade of certainty $c \in L$.

A fuzzy theory is a fuzzy set in the language $\mathcal{L}$ (i.e. a map $T \in L^{\mathcal{L}}$) such that $T(A \Rightarrow B) = c \in L$.

Definition

A datatable $\mathcal{R}$ satisfies $A \stackrel{c}{\Rightarrow} B$ if, for all $t_1, t_2 \in \mathcal{R}$,

$$c \leq (t_1[A] \approx t_2[A]) \rightarrow (t_1[B] \approx t_2[B])$$

Or, equivalently, if

$$c \leq \bigwedge_{t_1, t_2 \in \mathcal{R}} (t_1[A] \approx t_2[A]) \rightarrow (t_1[B] \approx t_2[B])$$

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Or, equivalently, if

$$c \leq \bigwedge_{t_1, t_2 \in \mathcal{R}} (t_1[A] \approx t_2[A]) \rightarrow (t_1[B] \approx t_2[B])$$

Second approach

$\mathcal{R}$ satisfies $A \overset{c}{\Rightarrow} B$ if, for all two tuples $t_1, t_2 \in \mathcal{R}$,

$$c \leq \bigwedge_{t_1, t_2 \in \mathcal{R}} (t_1[A] \approx t_2[A]) \to (t_1[B] \approx t_2[B])$$

We can define the grade in which $\mathcal{R}$ satisfies $A \Rightarrow B$ as follows

$$||A \Rightarrow B||_{\mathcal{R}} = \bigwedge_{t_1, t_2 \in \mathcal{R}} (t_1[A] \approx t_2[A]) \to (t_1[B] \approx t_2[B])$$

Fuzzy Simplification Logic

- **Axioms**: for all $B \subseteq A$, $\vdash A \stackrel{1}{\Rightarrow} B$.
- **Decomposition rule**: if $C \subseteq B$ and $c_2 \leq c_1$, $A \stackrel{c_1}{\Rightarrow} B \vdash A \stackrel{c_2}{\Rightarrow} C$.
- **Composition rule**: $A \stackrel{c_1}{\Rightarrow} B,\ C \stackrel{c_2}{\Rightarrow} D \ \vdash AC \stackrel{c_1 \wedge c_2}{\Rightarrow} BD$.
- **Simplification rule**: if $A \subseteq C$ and $A \cap B = \varnothing$
  $A \stackrel{c_1}{\Rightarrow} B,\ C \stackrel{c_2}{\Rightarrow} D \ \vdash C \smallsetminus B \stackrel{c_1 \otimes c_2}{\Rightarrow} D \smallsetminus B$.

Second approach

- Yazici & Sozat use the Gödel product in $[0, 1]$.
- Ben Yahia uses the Łukasiewicz product in $[0, 1]$.

Approach	Domains	Table	Functional Dependency	Logic
Y&S, BY	✓	✗	✓	FSLFD

Third approach

$$\{\texttt{job}, {}^{0.8}/\texttt{experience}\} \stackrel{0.9}{\Rightarrow} \{{}^{0.6}/\texttt{salary}\}$$

The following assertion is true to degree at least 0.9:

> *"Same job and similar experience to degree at least* 0.8 *imply similar salary to degree at least* 0.6*"*

Third approach

A functional dependency is an expression $A \Rightarrow B$ where A and B are fuzzy sets

$$(t_1[A] \approx t_2[A]) = \bigwedge_{y \in Y} A(y) \to (t_1[y] \approx_y t_2[y])$$

Definition

The grade in which a data table $\mathcal{R}$ satisfies $A \Rightarrow B$ is

$$||A \Rightarrow B||_{\mathcal{R}} = \bigwedge_{t_1, t_2 \in \mathcal{R}} (t_1[A] \approx t_2[A])^* \to (t_1[B] \approx t_2[B])$$

Third approach

Lemma

Let $A, B \in L^Y$, $c \in L$ and $\mathcal{R}$ be a data table.

$$c \leq ||A \Rightarrow B||_{\mathcal{R}} \text{ if and only if } ||A \Rightarrow c \otimes B||_{\mathcal{R}} = 1$$

and, therefore, any fuzzy theory T is equivalent to the following crisp theory

$$c(T) = \{A \Rightarrow T(A \Rightarrow B) \otimes B \mid A, B \in L^Y \text{ and } T(A \Rightarrow B) \otimes B \neq \varnothing\}$$

$\{\texttt{job}, {}^{0.8}/\texttt{experience}\} \overset{0.9}{\Rightarrow} \{{}^{0.6}/\texttt{salary}\}$ is equivalent to
$\{\texttt{job}, {}^{0.8}/\texttt{experience}\} \Rightarrow \{{}^{0.9 \otimes 0.6}/\texttt{salary}\}$

Third approach

Axiomatic system

Let $A, B, C, D \in L^Y$ and $c \in L$.

- Axioms: $\vdash AB \Rightarrow A.$
- Cut rule: $A \Rightarrow B, BC \Rightarrow D \vdash AC \Rightarrow D.$
- Multiplication rule: $A \Rightarrow B \vdash c^* \otimes A \Rightarrow c^* \otimes B.$

Third approach

Approach	Domains	Table	Functional Dependency	Logic
Belohlávek & Vychodil				Fuzzy Logic

Third approach

		Fuzzyness on data			Executable logic
		Classical data table	Table of fuzzy sets	Ranked data table	
Fuzzyness on functional dependencies	Functional dependency	[Codd, 1970] [Armstrong, 1974]			Simplification Logic [Mora et al, 2006]
	F.D. over domains with similarities	[Raju & Majumdar, 1988]			Simplification Logic [Mora et al, 2006]
	Graded F.D. over dom. with similarities	[Yazici & Sozat, 1996] [Ben Yahia et al, 1999]			Fuzzy Simplification Logic [Cordero et al, 2010]
	Fuzzy functional dependency	[Belohlávek & Vychodil, 2006]			

Tables of fuzzy sets

A fuzzy data table over a family of domains $\{D_y \mid y \in Y\}$ is defined as a subset

$$R \subseteq \prod_{y \in Y} L^{D_y}$$

The elements in each tuple are named "possibility distributions".

name	hair	skin	age	eyes	factor
John	black	dark	[30,40]	dark	10
Albert	clear	light	about-30	$\{^1/\text{blue}, ^{0.8}/\text{green}\}$	[40,50]
Mary	auburn	lightint	$\{26, ^{0.9}/27\}$	blue	50
Dave	quasi red	light	young	blue	about-50
Noa	White	dark	about-32	green	[25,35]

Tables of fuzzy sets vs (crisp) data tables

- Obviously, any (crisp) data table is a particular case of fuzzy data table.
- From the point of view of the theory of functional dependencies, fuzzy data tables can be considered particular cases of (crisp) data tables.
  If we provide a way to extend a similarity relation $\approx$ on a domain D to another similarity relation $\widehat{\approx}$ on L^D, then, by replacing the family of domains (with similarities)

$$\{(D_y, \approx_y) \mid y \in Y\} \quad \text{by} \quad \{(L^{D_y}, \widehat{\approx}_y) \mid y \in Y\}$$

  all the previous definitions of fuzzy functional dependencies can be extended.

Tables of fuzzy sets

		Fuzzyness on data			
		Classical data table	Table of fuzzy sets	Ranked data table	Executable logic
Fuzzyness on functional dependencies	Functional dependency	[Codd, 1970] [Armstrong, 1974]	[Buckles & Petri, 1982] [Prade & Testemale, 1984]		Simplification Logic [Mora et al, 2006]
	F.D. over domains with similarities	[Raju & Majumdar, 1988]	[Liu, 1994] [Saxena & Tyagi, 1995]		Simplification Logic [Mora et al, 2006]
	Graded F.D. over dom. with similarities	[Yazici & Sozat, 1996] [Ben Yahia et al, 1999]	[Chen, 1991]		Fuzzy Simplification Logic [Cordero et al, 2010]
	Fuzzy functional dependency	[Belohlávek & Vychodil, 2006]	[Cubero & Vila, 1994]		

Ranked data table

It is possible to provide an extension of the notion of datatable over $\{D_y \mid y \in Y\}$ as a fuzzy subset of the product

$$\mathcal{D}\colon \prod_{y \in Y} D_y \to L$$

Recently, [Belohlávek and Vychodil, 2006] have given a reasonable semantic for this kind of data tables.

$\mathcal{D}$(t)	name	hair	skin	age	eyes	factor
1.0	John	Black	dark	34	Brown	10
0.8	Albert	Brown	light	32	Blue	50
0.6	Mary	Auburn	lig-int	29	Blue	50
0.4	Dave	Red	light	26	Blue	50
0.1	Noa	White	dark	44	Green	30

It may be seen as an answer to a similarity query "show all persons with age approximately 34".

Fuzzy functional dependencies on ranked data tables

Given a family of domains with similarities $\{(D_y, \approx_y) \mid y \in Y\}$ and a ranked data table

$$\mathcal{D}\colon \prod_{y \in Y} D_y \to L$$

the relative similarity relation is defined as follows

$$(t_1[A] \approx_{\mathcal{D}} t_2[A]) = (\mathcal{D}(t_1) \otimes \mathcal{D}(t_2)) \to \bigwedge_{y \in Y} (A(y) \to (t_1[y] \approx_y t_2[y]))$$

Fuzzy functional dependencies on ranked data tables

Definition

The grade in which $\mathcal{R}$ satisfies $A \Rightarrow B$ is

$$||A \Rightarrow B||_{\mathcal{R}} = \bigwedge_{t_1, t_2} (t_1[A] \approx_{\mathcal{D}} t_2[A])^* \rightarrow (t_1[B] \approx_{\mathcal{D}} t_2[B])$$

Ranked data tables

		Fuzzyness on data			
		Classical data table	**Table of fuzzy sets**	**Ranked data table**	**Executable logic**
Fuzzyness on functional dependencies	Functional dependency	[Codd, 1970] [Armstrong, 1974]	[Buckles & Petri, 1982] [Prade & Testemale, 1984]	[Baldwin & Zhou, 1983] [Raju & Majumdar, 1988]	Simplification Logic [Mora et al, 2006]
	F.D. over domains with similarities	[Raju & Majumdar, 1988]	[Liu, 1994] [Saxena & Tyagi, 1995]	[Raju & Majumdar, 1988] [Tyagi et al, 2005]	Simplification Logic [Mora et al, 2006]
	Graded F.D. over dom. with similarities	[Yazici & Sozat, 1996] [Ben Yahia et al, 1999]	[Chen, 1991]	[Cordero et al, 2011]	Fuzzy Simplification Logic [Cordero et al, 2010]
	Fuzzy functional dependency	[Belohlávek & Vychodil, 2006]	[Cubero & Vila, 1994]	[Belohlávek & Vychodil, 2006]	

New axiomatic system

Our starting point is the axiomatic system proposed by R. Belohlavek and V. Vychodil.

Definition

Let $A, B, C, D \in \mathbf{L}^Y$ and $c \in L$.

- Axioms: $\vdash AB \Rightarrow A$.
- Cut rule: $\{A \Rightarrow B, BC \Rightarrow D\} \vdash AC \Rightarrow D$.
- Multiplication rule: $\{A \Rightarrow B\} \vdash c^* \otimes A \Rightarrow c^* \otimes B$.

Difference: A new operation over fuzzy sets

- The paradigm of the simplification logics is to infer implicit information via redundancy removing.
- When we work with dependencies $A \Rightarrow B$ in which A and B are (crisp) sets we use the difference of sets $A \smallsetminus B$.
- So, we need to extend this difference to fuzzy sets.
- There exist different ways to extend it. What is appropriate to reach our objective?
- It is necessary that the following equalities hold:

$$A \smallsetminus B \subseteq A \quad \text{and} \quad (A \smallsetminus B) \cup B = A \cup B \quad \text{for all } A, B \in \mathbf{L}^{\mathcal{U}}.$$

Structures of degrees

- We consider an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, \setminus, ^*, 0, 1\rangle$ such that:
  - $\langle L, \wedge, \vee, \otimes, \rightarrow, ^*, 0, 1\rangle$ is a complete residuated lattice with hedge.
  - For all $x, y, z \in L$, $x \setminus y \leq z$ if and only if $x \leq y \vee z$.
- Consequently, $\langle L, \wedge, \vee, \setminus, 0, 1\rangle$ is a Browerian algebra (dual to a Heyting algebra) and,
- so, $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice.

Structures of degrees

Example

Let us consider the subset of the unit interval $\{0, 0.1, 0.2, \ldots, 0.9, 1\}$ with the natural ordering, the Łukasiewiz adjoint par, the difference and the hedge given by

$$x \otimes y = \max\{x + y - 1, 0\}$$

$$x \to y = \min\{1 - x + y, 1\}$$

$$x \setminus y = \begin{cases} x & \text{if } x > y, \\ 0 & \text{otherwise.} \end{cases}$$

$$x^* = \begin{cases} 1 & \text{if } x = 1, \\ 0.5 & \text{if } 0.5 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

New axiomatic system

In the new syntactico-semantically complete axiomatic system, rule *Cut* is replace by a new rule named *rule of simplification*.

Definition

Let $A, B, C, D \in \mathbf{L}^Y$ and $c \in L$.

- Axioms: $\vdash AB \Rightarrow A.$
- Simplification rule: $\{A \Rightarrow B, C \Rightarrow D\} \vdash A(C \setminus B) \Rightarrow D.$
- Multiplication rule: $\{A \Rightarrow B\} \vdash c^* \otimes A \Rightarrow c^* \otimes B.$

The new system is called FASL (Fuzzy Attribute Simplification Logic)

New axiomatic system

Lemma

The following inference rules are derived: Let $A, B, C, D \in \mathbf{L}^Y$.

- *Decomposition rule:* $\{A \Rightarrow BC\} \vdash A \Rightarrow B.$
- *Composition rule:* $\{A \Rightarrow B, C \Rightarrow D\} \vdash AC \Rightarrow BD.$

Some derived equivalences

The importance of these inference rules is that they can be easily extended to obtain a set of equivalencies that are focussed on removing redundant information in the theories.

Theorem

Let $A, B, C, D \in \mathbf{L}^Y$.

1. *Decomposition Eq.:* $\{A \Rightarrow B\} \equiv \{A \Rightarrow B \setminus A\}$
2. *Union Eq.:* $\{A \Rightarrow B, A \Rightarrow C\} \equiv \{A \Rightarrow BC\}$
3. *Simplification Eq.: If* $A \subseteq C$ *then* $\{A \Rightarrow B, C \Rightarrow D\} \equiv \{A \Rightarrow B, A(C \setminus B) \Rightarrow D \setminus B\}$

The automated reasoning method

Theorem

Let $A, B \in \mathbf{L}^Y$, $c \in L$ *and* $T \in \mathbf{L}^{\mathcal{L}}$.

$$T \vdash A \stackrel{c}{\Rightarrow} B \text{ if and only if } \{\varnothing \Rightarrow A\} \cup c(T) \vdash \varnothing \Rightarrow c \otimes B$$

Theorem

For all $A, B, C \in \mathbf{L}^Y$, *if* $A' = A(S(B, A)^* \otimes C)$ *then*

$$\{\varnothing \Rightarrow A, B \Rightarrow C\} \equiv \{\varnothing \Rightarrow A', B - A' \Rightarrow C - A'\}$$

Particularly,

1. *if* $B \setminus A' = \varnothing$ *then* $\{\varnothing \Rightarrow A, B \Rightarrow C\} \equiv \{\varnothing \Rightarrow A'C\}$.
2. *if* $C \setminus A' = \varnothing$ *then* $\{\varnothing \Rightarrow A, B \Rightarrow C\} \equiv \{\varnothing \Rightarrow A'\}$.

Example

We consider the truthfulness structure described in Example 7. Let T be the following fuzzy theory.

$$
\begin{array}{llll}
T = \{ & \{0.4/a, 0.6/c\} & \stackrel{0.6}{\Rightarrow} \{0.8/c, 0.5/d, 0.6/e, 0.7/f\}, & \boxed{1} \\
& \{0.2/d, 0.3/f\} & \stackrel{0.9}{\Rightarrow} \{1/d,\ 0.6/e,\ 0.9/g\}, & \boxed{2} \\
& \{0.4/d, 0.5/e\} & \stackrel{0.8}{\Rightarrow} \{0.6/h, 0.2/d\}, & \boxed{3} \\
& \{0.6/d, 0.4/i\} & \stackrel{1}{\Rightarrow} \{0.7/a, 0.7/d\}, & \boxed{4} \\
& \{0.3/c, 0.4/e\} & \stackrel{1}{\Rightarrow} \{0.2/h\}, & \boxed{5} \\
& \{0.4/c, 0.6/h\} & \stackrel{0.6}{\Rightarrow} \{0.3/b, 0.7/e, 0.8/i\}, & \boxed{6} \\
& \{0.2/g\} & \stackrel{0.6}{\Rightarrow} \{0.7/a, 0.4/d\}, & \boxed{7} \\
& \{0.6/c, 0.5/d\} & \stackrel{0.8}{\Rightarrow} \{0.4/e\}\} & \boxed{8}
\end{array}
$$

and we want to check if

$$T \vdash \{0.2/c, 0.6/f\} \stackrel{0.8}{\Rightarrow} \{0.5/a, 0.5/d, 0.6/g, 0.6/h\}$$

Example

By Theorem 11, this problem is equivalent to the following:

$$\{\varnothing \Rightarrow A\} \cup c(T) \vdash \{\varnothing \Rightarrow \{0.3/a, 0.3/d, 0.4/g, 0.4/h\}\}$$

being $\{\varnothing \Rightarrow A\}$ and $c(T)$ the following formula and (crisp) theory

$$\{\varnothing \Rightarrow A\} = \{\ \varnothing \Rightarrow \{0.2/c, 0.6/f\}\}$$

Guide

$$
\begin{aligned}
c(T) = \{\ & \{0.4/a, 0.6/c\} \Rightarrow \{0.4/c, 0.1/d, 0.2/e, 0.3/f\} && 1\\
& \{0.2/d, 0.3/f\} \Rightarrow \{0.9/d, 0.5/e, 0.8/g\} && 2\\
& \{0.4/d, 0.5/e\} \Rightarrow \{0.4/h\} && 3\\
& \{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\} && 4\\
& \{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\} && 5\\
& \{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\} && 6\\
& \{0.2/g\} \Rightarrow \{0.3/a\} && 7\\
& \{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\}\ \} && 8
\end{aligned}
$$

Example

By Theorem 11, this problem is equivalent to the following:

$$\{\varnothing \Rightarrow A\} \cup c(T) \;\vdash\; \{\varnothing \Rightarrow \{0.3/a, 0.3/d, 0.4/g, 0.4/h\}\}$$

being $\{\varnothing \Rightarrow A\}$ and $c(T)$ the following formula and (crisp) theory

$\{\varnothing \Rightarrow A\} = \{$	$\varnothing$	$\Rightarrow$	$\{0.2/c, 0.6/f\}\}$	Guide
$c(T) = \{$	$\{0.4/a, 0.6/c\}$	$\Rightarrow$	$\{0.4/c, 0.1/d, 0.2/e, 0.3/f\}$	1
	$\{0.2/d, 0.3/f\}$	$\Rightarrow$	$\{0.9/d, 0.5/e, 0.8/g\}$	2
	$\{0.4/d, 0.5/e\}$	$\Rightarrow$	$\{0.4/h\}$	3
	$\{0.6/d, 0.4/i\}$	$\Rightarrow$	$\{0.7/a, 0.7/d\}$	4
	$\{0.3/c, 0.4/e\}$	$\Rightarrow$	$\{0.2/h\}$	5
	$\{0.4/c, 0.6/h\}$	$\Rightarrow$	$\{0.3/e, 0.4/i\}$	6
	$\{0.2/g\}$	$\Rightarrow$	$\{0.3/a\}$	7
	$\{0.6/c, 0.5/d\}$	$\Rightarrow$	$\{0.2/e\}\ \}$	8

Applying to every formula: (**DeEq**) : $\{A \Rightarrow B\} \equiv \{A \Rightarrow B \setminus A\}$

Example

By Theorem 11, this problem is equivalent to the following:

$$\{\varnothing \Rightarrow A\} \cup c(T) \vdash \{\varnothing \Rightarrow \{0.3/a, 0.3/d, 0.4/g, 0.4/h\}\}$$

being $\{\varnothing \Rightarrow A\}$ and $c(T)$ the following formula and (crisp) theory

$\{\varnothing \Rightarrow A\} = \{$	$\varnothing$	$\Rightarrow \{0.2/c, 0.6/f\}\}$		Guide
$c(T) = \{$	$\{0.4/a, 0.6/c\}$	$\Rightarrow \{$	$0.1/d, 0.2/e, 0.3/f\}$	1
	$\{0.2/d, 0.3/f\}$	$\Rightarrow \{0.9/d, 0.5/e, 0.8/g\}$		2
	$\{0.4/d, 0.5/e\}$	$\Rightarrow \{0.4/h\}$		3
	$\{0.6/d, 0.4/i\}$	$\Rightarrow \{0.7/a, 0.7/d\}$		4
	$\{0.3/c, 0.4/e\}$	$\Rightarrow \{0.2/h\}$		5
	$\{0.4/c, 0.6/h\}$	$\Rightarrow \{0.3/e, 0.4/i\}$		6
	$\{0.2/g\}$	$\Rightarrow \{0.3/a\}$		7
	$\{0.6/c, 0.5/d\}$	$\Rightarrow \{0.2/e\}\ \}$		8

Applying to every formula: (**DeEq**) : $\{A \Rightarrow B\} \equiv \{A \Rightarrow B \setminus A\}$

Example

$$\{\emptyset \Rightarrow A\} = \{ \quad \emptyset \Rightarrow \{0.2/c, 0.6/f\}\}$$ Guide

$T = \{$	$\{0.4/a, 0.6/c\}$	$\Rightarrow \{0.1/d, 0.2/e, 0.3/f\}$	1
	$\{0.2/d, 0.3/f\}$	$\Rightarrow \{0.9/d, 0.5/e, 0.8/g\}$	2
	$\{0.4/d, 0.5/e\}$	$\Rightarrow \{0.4/h\}$	3
	$\{0.6/d, 0.4/i\}$	$\Rightarrow \{0.7/a, 0.7/d\}$	4
	$\{0.3/c, 0.4/e\}$	$\Rightarrow \{0.2/h\}$	5
	$\{0.4/c, 0.6/h\}$	$\Rightarrow \{0.3/e, 0.4/i\}$	6
	$\{0.2/g\}$	$\Rightarrow \{0.3/a\}$	7
	$\{0.6/c, 0.5/d\}$	$\Rightarrow \{0.2/e\} \}$	8

(gSiEq):

$$\{\emptyset \Rightarrow \{0.2/c, 0.6/f\}, \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e, 0.3/f\}\} \equiv$$

$$\equiv \{\emptyset \Rightarrow \{0.2/c, 0.6/f\}, \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\}\}$$

Example

$$\{\emptyset \Rightarrow A\} = \{\ \emptyset \Rightarrow \{0.2/c, 0.6/f\}\}$$ Guide

$$T = \{\ \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\ \}$$ 1

$$\{0.2/d, 0.3/f\} \Rightarrow \{0.9/d, 0.5/e, 0.8/g\}$$ 2

$$\{0.4/d, 0.5/e\} \Rightarrow \{0.4/h\}$$ 3

$$\{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\}$$ 4

$$\{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\}$$ 5

$$\{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\}$$ 6

$$\{0.2/g\} \Rightarrow \{0.3/a\}$$ 7

$$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\}\ \}$$ 8

(gSiEq):

$$\{\emptyset \Rightarrow \{0.2/c, 0.6/f\}, \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e, 0.3/f\}\} \equiv$$

$$\equiv \{\emptyset \Rightarrow \{0.2/c, 0.6/f\}, \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\}\}$$

Example

$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.6/f\}\}$ Guide

$T = \{$	$\{0.4/a, 0.6/c\}$	$\Rightarrow \{0.1/d, 0.2/e \quad\}$	1
	$\{0.2/d, 0.3/f\}$	$\Rightarrow \{0.9/d, 0.5/e, 0.8/g\}$	2
	$\{0.4/d, 0.5/e\}$	$\Rightarrow \{0.4/h\}$	3
	$\{0.6/d, 0.4/i\}$	$\Rightarrow \{0.7/a, 0.7/d\}$	4
	$\{0.3/c, 0.4/e\}$	$\Rightarrow \{0.2/h\}$	5
	$\{0.4/c, 0.6/h\}$	$\Rightarrow \{0.3/e, 0.4/i\}$	6
	$\{0.2/g\}$	$\Rightarrow \{0.3/a\}$	7
	$\{0.6/c, 0.5/d\}$	$\Rightarrow \{0.2/e\} \}$	8

(gSiUnEq):

$$\{\varnothing \Rightarrow \{0.2/c, 0.6/f\}, \{0.2/d, 0.3/f\} \Rightarrow \{0.9/d, 0.5/e, 0.8/g\}\} \equiv$$
$$\equiv \{\varnothing \Rightarrow \{0.2/c, 0.4/d, 0.6/f, 0.3/g\}, \varnothing \Rightarrow \{0.9/d, 0.5/e, 0.8/g\}\} \equiv$$
$$\equiv \{\varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g\}\}$$

Example

$$\{\varnothing \Rightarrow A\} = \{\quad \varnothing \quad \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g\}\}$$ Guide

$T = \{$	$\{0.4/a, 0.6/c\}$	$\Rightarrow \{0.1/d, 0.2/e \quad \}$	1
			2
	$\{0.4/d, 0.5/e\}$	$\Rightarrow \{0.4/h\}$	3
	$\{0.6/d, 0.4/i\}$	$\Rightarrow \{0.7/a, 0.7/d\}$	4
	$\{0.3/c, 0.4/e\}$	$\Rightarrow \{0.2/h\}$	5
	$\{0.4/c, 0.6/h\}$	$\Rightarrow \{0.3/e, 0.4/i\}$	6
	$\{0.2/g\}$	$\Rightarrow \{0.3/a\}$	7
	$\{0.6/c, 0.5/d\}$	$\Rightarrow \{0.2/e\}\ \}$	8

(gSiUnEq):

$$\{\varnothing \Rightarrow \{0.2/c, 0.6/f\}, \{0.2/d, 0.3/f\} \Rightarrow \{0.9/d, 0.5/e, 0.8/g\}\} \equiv$$
$$\equiv \{\varnothing \Rightarrow \{0.2/c, 0.4/d, 0.6/f, 0.3/g\}, \varnothing \Rightarrow \{0.9/d, 0.5/e, 0.8/g\}\} \equiv$$
$$\equiv \{\varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g\}\}$$

Example

$$\{\varnothing \Rightarrow A\} = \{\ \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g\}\}$$ Guide

$$T = \{\ \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\ \}$$ 1

2

$$\{0.4/d, 0.5/e\} \Rightarrow \{0.4/h\}$$ 3

$$\{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\}$$ 4

$$\{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\}$$ 5

$$\{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\}$$ 6

$$\{0.2/g\} \Rightarrow \{0.3/a\}$$ 7

$$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\}\ \}$$ 8

(gSiAxEq):

$$\{\varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g\}, \{0.4/d, 0.5/e\} \Rightarrow \{0.4/h\}\} \equiv$$
$$\equiv \{\varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \varnothing \Rightarrow \varnothing\} \equiv$$
$$\equiv \{\varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Example

$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$ Guide

$T = \{ \quad \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e \quad \}$ 1

2

3

$\{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\}$ 4

$\{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\}$ 5

$\{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\}$ 6

$\{0.2/g\} \Rightarrow \{0.3/a\}$ 7

$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\} \}$ 8

(gSiAxEq):

$$\{\varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g\}, \{0.4/d, 0.5/e\} \Rightarrow \{0.4/h\}\} \equiv$$
$$\equiv \{\varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \varnothing \Rightarrow \varnothing\} \equiv$$
$$\equiv \{\varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Example

$\{\emptyset \Rightarrow A\} = \{\ \emptyset \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$

Guide

$T = \{\ \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\}$

$\{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\}$

$\{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\}$

$\{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\}$

$\{0.2/g\} \Rightarrow \{0.3/a\}$

$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\}\ \}$

1 2 3 4 5 6 7 8

(gSiEq):

$$\{\emptyset \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\}\} \equiv$$
$$\equiv \{\emptyset \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \{0.4/i\} \Rightarrow \{0.7/a\}\}$$

Example

$$\{\emptyset \Rightarrow A\} = \{\quad \emptyset \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Guide

$$T = \{\quad \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\}$$

$$\{0.4/i\} \Rightarrow \{0.7/a\}$$
$$\{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\}$$
$$\{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\}$$
$$\{0.2/g\} \Rightarrow \{0.3/a\}$$
$$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\}\ \}$$

1 2 3 4 5 6 7 8

(gSiEq):

$$\{\emptyset \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \{0.6/d, 0.4/i\} \Rightarrow \{0.7/a, 0.7/d\}\} \equiv$$
$$\equiv \{\emptyset \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \{0.4/i\} \Rightarrow \{0.7/a\}\}$$

Example

$$\{\emptyset \Rightarrow A\} = \{ \quad \emptyset \quad \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Guide

$$
\begin{aligned}
T = \{ \quad & \{0.4/a, 0.6/c\} && \Rightarrow \{0.1/d, 0.2/e \quad\quad \} && 1 \\
& && && 2 \\
& && && 3 \\
& \{ \quad\quad 0.4/i\} && \Rightarrow \{0.7/a \quad\quad \} && 4 \\
& \{0.3/c, 0.4/e\} && \Rightarrow \{0.2/h\} && 5 \\
& \{0.4/c, 0.6/h\} && \Rightarrow \{0.3/e, 0.4/i\} && 6 \\
& \{0.2/g\} && \Rightarrow \{0.3/a\} && 7 \\
& \{0.6/c, 0.5/d\} && \Rightarrow \{0.2/e\} \ \} && 8
\end{aligned}
$$

(gSiAxEq):

$$\{\emptyset \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\}\} \equiv$$
$$\equiv \{\emptyset \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Example

$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$ Guide

$T = \{ \quad \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e \quad \}$ 1

2

3

$\{ \quad 0.4/i\} \Rightarrow \{0.7/a \quad \}$ 4

5

$\{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\}$ 6

$\{0.2/g\} \Rightarrow \{0.3/a\}$ 7

$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\} \}$ 8

(gSiAxEq):

$$\{\varnothing \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \{0.3/c, 0.4/e\} \Rightarrow \{0.2/h\}\} \equiv$$
$$\equiv \{\varnothing \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Example

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \quad \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Guide

$$T = \{ \quad \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e \quad \}$$

$$\{ \quad 0.4/i\} \Rightarrow \{0.7/a \quad \}$$

$$\{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\}$$

$$\{0.2/g\} \Rightarrow \{0.3/a\}$$

$$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\} \}$$

1 2 3 4 5 6 7 8

(gSiEq):

$$\{\varnothing \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\}\} \equiv$$

$$\equiv \{\varnothing \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \{0.4/c, 0.6/h\} \Rightarrow \{0.4/i\}\}$$

Example

$\{\emptyset \Rightarrow A\} = \{\ \emptyset \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$ Guide

$T = \{\ \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\ \}$ 1

2

3

$\{\ 0.4/i\} \Rightarrow \{0.7/a\ \}$ 4

5

$\{0.4/c, 0.6/h\} \Rightarrow \{\ 0.4/i\}$ 6

$\{0.2/g\} \Rightarrow \{0.3/a\}$ 7

$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\}\ \}$ 8

(gSiEq):

$\{\emptyset \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \{0.4/c, 0.6/h\} \Rightarrow \{0.3/e, 0.4/i\}\} \equiv$

$\equiv \{\emptyset \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \{0.4/c, 0.6/h\} \Rightarrow \{0.4/i\}\}$

Example

$$\{\varnothing \Rightarrow A\} = \{ \quad \varnothing \Rightarrow \{0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Guide

$$\begin{aligned} T = \{ \quad & \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e \quad\} \\ & \{ \quad 0.4/i\} \Rightarrow \{0.7/a \quad\} \\ & \{0.4/c, 0.6/h\} \Rightarrow \{ \quad 0.4/i\} \\ & \{0.2/g\} \Rightarrow \{0.3/a\} \\ & \{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\} \} \end{aligned}$$

1 2 3 4 5 6 7 8

(gSiAxEq):

$$\{\varnothing \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \{0.2/g\} \Rightarrow \{0.3/a\}\} \equiv$$
$$\equiv \{\varnothing \Rightarrow \{0.3/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Example

$$\{\varnothing \Rightarrow A\} = \{\ \varnothing \Rightarrow \{0.3/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Guide

$$T = \{\ \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\ \}$$

$$\{\ 0.4/i\} \Rightarrow \{0.7/a\ \}$$

$$\{0.4/c, 0.6/h\} \Rightarrow \{\ 0.4/i\}$$

$$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\}\ \}$$

1 2 3 4 5 6 7 8

(gSiAxEq):

$$\{\varnothing \Rightarrow \{0.2/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}, \{0.2/g\} \Rightarrow \{0.3/a\}\} \equiv$$
$$\equiv \{\varnothing \Rightarrow \{0.3/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$$

Example

$\{\emptyset \Rightarrow A\} = \{\quad \emptyset \Rightarrow \{0.3/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}\}$ Guide

$T = \{\quad \{0.4/a, 0.6/c\} \Rightarrow \{0.1/d, 0.2/e\quad\}$

$\{\quad 0.4/i\} \Rightarrow \{0.7/a\quad\}$

$\{0.4/c, 0.6/h\} \Rightarrow \{\quad 0.4/i\}$

$\{0.6/c, 0.5/d\} \Rightarrow \{0.2/e\}\ \}$

1 2 3 4 5 6 7 8

Conclusion:

$T \vdash \{0.2/a, 0.3/f\} \overset{0.8}{\Rightarrow} \{0.5/a, 0.5/d, 0.6/g, 0.6/h\}$ because
$0.8 \otimes \{0.5/a, 0.5/d, 0.6/g, 0.6/h\} = \{0.3/a, 0.3/d, 0.4/g, 0.4/h\} \subseteq$
$\{0.3/a, 0.2/c, 0.9/d, 0.5/e, 0.6/f, 0.8/g, 0.4/h\}$

Experiment Result