Local and Relative Local Finiteness in t-Norm Based Structures

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Starting point:
The local finiteness of bimonoids is an interesting property for weighted automata.
An actual reference is
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Problem: Which t-norm based bimonoids have this property?

Definition

An algebraic structure $\mathfrak{A}$ is *locally finite* iff each of its finite subsets G generates a finite subalgebra $\langle G \rangle_{\mathfrak{A}}$ only.

Proposition

A t-conorm monoid $([0,1], S_T, 0)$ *is locally finite iff its corresponding t-norm monoid* $([0,1], T, 1)$ *is.*

Proof: Let $\mathfrak{A} = ([0,1], T, 1)$ be a t-norm monoid and $G \subseteq [0,1]$. For each $a \in \langle G \rangle_{\mathfrak{A}}$ its dual $a^d = 1 - a$ is an element of $a \in \langle G^d \rangle_{\mathfrak{A}^d}$ for $G^d = \{a^d \mid a \in G\}$.

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Proof: In $([0,1], S_Ł, 0)$, each finite $G \subseteq [0,1]$ generates only finitely many elements: all the sums $k_1 a_1 + \cdots + k_n a_n$, $k_i \in \mathbb{N}$, of $S_Ł$-multiples of $a_1, \ldots, a_n \in G$, including 1.

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Proposition

The product monoid $([0,1], T_\mathrm{P}, 1)$ *is not locally finite.*

Proof: Any $a \in (0,1)$ generates an infinite submonoid $\langle a \rangle = \{a^n \mid n \in \mathbb{N}\}$.

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Corollary

Any algebraic structure which has the product monoid as a reduct is not locally finite.

Some general results

Theorem

A t-norm monoid $([0,1], T, 1)$ *with a continuous t-norm* T *is locally finite iff* T *does only have locally finite summands in its representation as ordinal sum of archimedean summands.*

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Corollary

A t-norm monoid $([0,1], T, 1)$ *with a continuous t-norm* T *is locally finite iff* T *does not have a product-norm isomorphic summand in its representation as ordinal sum of archimedean summands.*

t-Norm Bimonoids

Definition

A bimonoid is an algebraic structure $\mathfrak{A} = (A, *_1, *_2, e_1, e_2)$ *such that both* $(A, *_1, e_1)$ *and* $(A, *_2, e_2)$ *are monoids.*

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$\alpha_1 = (k_0 + 1) \cdot \alpha_0 - 1$.
Obviously α_1 is irrational, and also $\alpha_1 < \alpha_0$.
Iteration of this construction yields an infinite descending sequence of irrationals from $[0,1]$. Hence $\langle \alpha_0 \rangle$ is infinite.

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The rational Łukasiewicz-bimonoid $([0,1] \cap \mathbb{Q}, T_{\mathsf{L}}, S_{\mathsf{L}}, 1, 0)$ *is locally finite.*

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- the simultaneous availability of the operations T_{L} and S_{L}.

Proposition

The rational Łukasiewicz-bimonoid $([0,1] \cap \mathbb{Q}, T_{\mathrm{L}}, S_{\mathrm{L}}, 1, 0)$ *is locally finite.*

Proof: Any finite set $G \subseteq [0,1] \cap \mathbb{Q}$ is a subset of a suitable finite truth degree set of a finitely-valued Łukasiewicz system L_m.

Rational local finiteness

Definition

A t-norm based algebraic structure $\mathfrak{A}$ *over the unit interval is* **rationally locally finite** *iff each finite set* $G \subseteq [0,1] \cap \mathbb{Q}$ *generates only a finite substructure of* $\mathfrak{A}$*.*

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Corollary

The Łukasiewicz-bimonoid $([0,1], T_{\mathsf{L}}, S_{\mathsf{L}}, 1, 0)$ *is rationally locally finite.*

t-Norm Bimonoids

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Example

The t-norm bimonoid $([0,1], T^*, S_{T^*}, 1, 0)$ *with the continuous t-norm*

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$$T^* = \sum_{i \in \{1\}} ([\frac{1}{2}, 1], T_{\mathrm{L}}, \varphi^*)$$

and the order isomorphism $\varphi^* : [\frac{1}{2}, 1] \to [0,1]$ *given by* $\varphi^*(x) = 2x - 1$ *is locally finite.*

Here T^* acts on the square $u_r = [\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$ as (an isomorphic copy of) T_{L}, and acts as the min-operation otherwise.

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The reconstruction of the proof idea for the Łukasiewicz bimonoid becomes impossible, $([0, 1], T^*, S_{T^*}, 1, 0)$ remains locally finite.

NB: The particular choice of the order isomorphism φ^* is unimportant here.

A general result

Theorem

Suppose that T is a continuous t-norm such that

- T *has an ordinal sum representation* $T = \sum_{i \in I}([l_i, r_i], T_i, \varphi_i)$ *without product-isomorphic summands,*

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- *T has an ordinal sum representation $T = \sum_{i \in I}([l_i, r_i], T_i, \varphi_i)$ without product-isomorphic summands,*
- *for each Łukasiewicz summand $([l_k, r_k], T_{\mathrm{L}}, \varphi_k)$ the interval $[1 - r_k, 1 - l_k]$ does* **not overlap** *with any domain interval $[l_i, r_i]$ for a Łukasiewicz summand $([l_i, r_i], T_{\mathrm{L}}, \varphi_i)$, $i \in I$.*

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Then the t-norm bimonoid $([0, 1], T, S_T, 1, 0)$ is locally finite.

Relativized local finiteness

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Definition

*Let $\mathfrak{A}$ be an algebraic structure and $M \subseteq |\mathfrak{A}|$. Then $\mathfrak{A}$ is M-**locally finite** iff for each finite $G \subseteq M$ the substructure $\langle G \rangle_{\mathfrak{A}}$ has a finite carrier.*

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Actually it is not clear what will be the importance of this more general notion. However, it seems particularly with respect to computer science topics that rational local finiteness might be important: internally all numbers used in a computer are rational ones.

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The carriers of the subalgebras of an algebraic structure $\mathfrak{A}$ are just those subsets of the carrier $|\mathfrak{A}|$ which are closed under all the operations of $\mathfrak{A}$. For the case of t-norm based structures this remark specializes to the next result.

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Proposition

Let $\mathfrak{A} = ([0,1], T, (\mathrm{op}_i)_{i \in I})$*, with* T *a t-norm and* $(\mathrm{op}_i)_{i \in I}$ *a family of finitary operations in* $[0,1]$*, be a t-norm based structure. Assume that* $\mathfrak{A}$ *is rationally locally finite. If* T *and all the operations* op_i *map rationals to rationals, then* $\mathfrak{A}{\upharpoonright}\mathbb{Q} = (\mathbb{Q} \cap [0,1], T, (\mathrm{op}_i)_{i \in I})$ *is a locally finite algebraic structure.*

Enriched t-norm monoids

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If G is a finite set of rationals, argue as previously.

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Theorem

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$$I_T(u, v) = \begin{cases} \varphi_k^{-1}(I_{T_k}(\varphi_k(u), \varphi_k(v))), & \text{if } u > v \text{ and } u, v \in [l_k, r_k] \\ I_G(u, v), & \text{otherwise.} \end{cases}$$

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Hence one has in each one of the ordinal summands for T isomorphic copies of a t-norm and its residuation operation.

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Theorem

A residuation-extended t-norm-monoid $([0,1], T, I_T, 1)$ *with a continuous t-norm* T *is locally finite iff* T *does only have locally finite summands in its representation as ordinal sum of archimedean summands.*

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A residuation-extended t-norm-monoid $([0,1], T, I_T, 1)$ *with a continuous t-norm* T *is locally finite iff it is based upon the Gödel monoid, i.e. iff* $T = T_{\mathsf{G}}$.

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The more interesting case, hence, is that of rational local finiteness. It needs some more care: we have to consider the order isomorphisms $\varphi_i : [l_i, r_i] \to [0, 1]$ of the Łukasiewicz summands $([l_i, r_i], T_i, \varphi_i)$.

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Of course, only Łukasiewicz-isomorphic summands should be allowed for non-trivial results.

Relative local finiteness

Theorem

Let T be continuous and $T = \sum_{i \in I}([l_i, r_i], T_{\mathrm{L}}, \varphi_i)$ with only Łukasiewicz-isomorphic summands. Then the residuation-extended t-norm-monoid $([0,1], T, I_T, 1)$ is M-locally finite for $M = \big(([0,1] \setminus \bigcup_{i \in I}[l_i, r_i]) \cup \bigcup_{i \in I} \varphi_k^{-1}\langle \mathbb{Q} \rangle\big)$.

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Proof: We argue as in the proof of the last Theorem. But we have to have in mind that T becomes M-locally finite only for such M for which one has that $\varphi_i\langle M \cap [l_i, r_i]\rangle \subseteq \mathbb{Q}$ for all $i \in I$.

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Proof: We argue as in the proof of the last Theorem. But we have to have in mind that T becomes M-locally finite only for such M for which one has that $\varphi_i \langle M \cap [l_i, r_i] \rangle \subseteq \mathbb{Q}$ for all $i \in I$. And the maximal such M is just $M = \bigcup_{i \in I} \varphi_k^{-1}\langle \mathbb{Q} \rangle \cup ([0,1] \setminus \bigcup_{i \in I}[l_i, r_i])$.

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Let T be continuous and $T = \sum_{i \in I}([l_i, r_i], T_{\mathrm{L}}, \varphi_i)$ with only Łukasiewicz-isomorphic summands. Then the residuation-extended t-norm-monoid $([0,1], T, I_T, 1)$ is M-locally finite for $M = (([0,1] \setminus \bigcup_{i \in I}[l_i, r_i]) \cup \bigcup_{i \in I} \varphi_k^{-1}\langle \mathbb{Q} \rangle)$.

Proof: We argue as in the proof of the last Theorem. But we have to have in mind that T becomes M-locally finite only for such M for which one has that $\varphi_i \langle M \cap [l_i, r_i] \rangle \subseteq \mathbb{Q}$ for all $i \in I$. And the maximal such M is just $M = \bigcup_{i \in I} \varphi_k^{-1}\langle \mathbb{Q} \rangle \cup ([0,1] \setminus \bigcup_{i \in I}[l_i, r_i])$. Here $[0,1] \setminus \bigcup_{i \in I}[l_i, r_i]$ is obviously the set of all points of the main diagonal of the unit square which belong to the "min-area" of the particular ordinal sum representation of T.

Relative local finiteness

So the problem of the rational local finiteness of T is reduced to the problem to have

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Particular case: The order isomorphisms in the T-summands are rational functions with rational coefficients.

t-Algebras

Now $\wedge, \vee$ are the lattice operations related to the natural (lattice) ordering of the real unit interval, i.e. $\wedge = \min$ and $\vee = \max$.

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And the insertion of the constant 0 instead of the constant 1 also does not essentially modify the previous situations: one always has 1 available because of $I_T(0,0) = 1$.

So we obviously have the following results.

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The Gödel-algebra $([0,1], \wedge, \vee, T_G, I_G, 0)$ *is locally finite.*

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Proposition

Suppose that T is a continuous t-norm with an ordinal sum representation which has only Łukasiewicz-isomorphic summands. If all the order isomorphisms in the T-summands map rationals to rationals then the t-algebra $([0,1], \wedge, \vee, T, I_T, 0)$ rationally locally finite.

Negation enriched t-Monoids

We consider the standard negation functions $N_T(x) = I_T(x, 0)$.

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Proposition

The negation-extended Gödel monoid $([0,1], T_\mathrm{G}, N_\mathrm{G}, 1)$ *is locally finite.*

Proposition

The negation-extended product monoid $([0,1], T_\mathrm{P}, N_\mathrm{P}, 1)$ *is neither locally finite nor rationally locally finite.*

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The negation-extended t-norm monoid $([0,1], T_{\text{L}}, N_{\text{L}}, 1)$ *is not locally finite, but it is rationally locally finite.*

Proof: In this structure we can define the t-conorm S_{L}. Hence this structure is not locally finite. That it is rationally locally finite follows as previously.

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Consider now again a continuous t-norm T with ordinal sum representation $T = \sum_{i \in I}([l_i, r_i], T_i, \varphi_i)$. Then the representation theorem for I_T yields two cases.

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Consider now again a continuous t-norm T with ordinal sum representation $T = \sum_{i \in I}([l_i, r_i], T_i, \varphi_i)$. Then the representation theorem for I_T yields two cases.

(i) If there is no (first) summand of the form $([0, r_i], T_i, \varphi_i)$ then one never has the situation that $x, 0 \in [l_i, r_i]$ for some $i \in I$, and hence has $N_T(x) = I_\mathsf{G}(x, 0) = N_\mathsf{G}(x)$ for all x. So the additional operation N_G cannot change the local finiteness behavior of the t-norm based monoid $([0, 1], T, 1)$.

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(ii) But if there is a first summand of the form $([0, r_0], T_0, \varphi_0)$ then N_T is inside $[0, r_0]$ the suitable isomorphic form of N_{T_0}, i.e. $N_T = \varphi_0^{-1} \circ N_{T_0}$, and is N_{G} outside $[0, r_0]$.

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These considerations easily yield the following result.

Proposition

Let T be continuous without product-isomorphic summands in the ordinal sum representation $T = \sum_{i \in I}([l_i, r_i], T_i, \varphi_i)$.

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Two particular cases seem to be the most interesting ones.

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Corollary

Assume that $T = \sum_{i \in I}([l_i, r_i], T_i, \varphi_i)$ *has no product-isomorphic summands. If there is a (first) summand* $([0, r_0], T_{\mathsf{L}}, \varphi_0)$ *such that* φ_0 *maps rationals to rationals, then the extended monoid* $([0,1], T, N_T, 1)$ *is rationally locally finite, but not locally finite.*

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Decidability of t-Norm Based Residuated Logics

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If a continuous t-norm T determines a locally finite t-algebra $[0,1]_T = ([0,1], \wedge, \vee, T, I_T, 0)$ then the t-norm based residuated logic $\mathcal{L}_T$ is decidable.

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Proof: If T is continuous then $[0,1]_T$ determines a subvariety BL_T of the variety BL of all BL-algebras, and this subalgebra is determined by finitely many additional equations. Therefore $\mathcal{L}_T$ is a finite extension of the finitely axiomatizable logic BL, and the class of logically valid formulas of $\mathcal{L}_T$ is recursively enumerable.

The local finiteness assumption yields that if a formula φ of $\mathcal{L}_T$ is not logically valid then it has a finite subalgebra of $[0,1]_T$ as a countermodel, hence a finite structure of BL_T.

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Therefore, because all the finite structures of BL_T can be generated in a systematic way, i.e., because this class is recursively enumerable, one gets that also the class of all formulas of $\mathcal{L}_T$ which are not logically valid is recursively enumerable. All together one has the decidability of $\mathcal{L}_T$.

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Therefore, because all the finite structures of BL_T can be generated in a systematic way, i.e., because this class is recursively enumerable, one gets that also the class of all formulas of $\mathcal{L}_T$ which are not logically valid is recursively enumerable. All together one has the decidability of $\mathcal{L}_T$.
Remark: By the previous local finiteness results this result applies only to the infinite valued Gödel logic.

Problem: Might the rational local finiteness provide further insights?

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So one should have, on the one hand side, a rationally finite t-algebra $[0,1]_T$ such that the rational unit interval $\mathbb{Q} \cap [0,1]$ is closed under the t-norm T and the corresponding residuation operation I_T. But looking for a decidability result for $\mathcal{L}_T$ one should also have, on the other hand side, that the $\mathbb{Q}$-restriction $[0,1]_T \restriction \mathbb{Q}$ of the t-algebra $[0,1]_T$ is also a characteristic matrix for the logic $\mathcal{L}_T$.

This last mentioned condition seems to be the most restrictive here. To the best of the author's knowledge one has only the following sufficient condition.

Proposition

Assume that the t-algebra $[0,1]_T$ *has the property that* T *and* I_T *map rationals to rationals. If* T *as well as* I_T *are continuous functions then the* $\mathbb{Q}$*-restriction* $[0,1]_T \restriction \mathbb{Q}$ *of* $[0,1]_T$ *is also a characteristic matrix for the logic* $\mathcal{L}_T$.

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Remark: Again, caused by the continuity assumption, this result applies only to one case: the infinite-valued Łukasiewicz logic.

Concluding Remarks

(1) Missing is a discussion of t-norms which are only left continuous. But this problem seems to be strongly linked to the lack of a sufficiently well developed structure theory for left continuous t-norms.

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Proposition

The t-norm monoid $([0,1], T_{\mathrm{nM}}, 1)$ *based upon the nilpotent minimum*

$$T_{\mathrm{nM}}(x,y) = \begin{cases} \min\{x,y\}, & \text{if } u+v>1 \\ 0 & \text{otherwise} \end{cases}$$

is locally finite, as is the corresponding bimonoid.

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are sufficient for the local finiteness of the bimonoid $([0, 1], T, S_T, 1, 0)$.

Are they also necessary? Or does one, in the case of overlapping domain squares for Łukasiewicz t-norm and t-conorm summands, also need e.g. the **rationality** of the overlap corners?

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Thank You