Do we know how to integrate?

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WIUI 2012
Olomouc, June 2012

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Short history

1850 B.C., Egypt (Moscow Mathematical Papyrus, problem 14) volume of a frustrum of a square pyramide

370 B.C., Greece, EUDOXUS, exhaustion method

3^{rd} century B.C., Greece, ARCHIMEDES, parabols, circle

3^{rd} century A.C., China, LIU HUI, circle

5^{th} century A.C., China, ZU CHONG ZHI and ZU GENG, sphere

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What is difference in Riemann and Lebesgue approach?

3 x 100 € 5 x 50 € 6 x 20 €	2 x 100 € 7 x 50 € 3 x 20 €	4 x 100 € 2 x 50 € 9 x 20 €

RIEMANN: (300 + 250 + 120) + (200 + 350 + 60) + (400 + 100 + 180) = 670 + 610 + 680 = 1960 €

LEBESGUE: (3 + 2 + 4)*100 + (5 + 7 + 2)*50 + (6 + 3 + 4)*20 = 900 + 700 + 360 = 1960 €

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Workers a, b, c

performance per hour in units
$m(\emptyset) = 0,\ m(a) = 2,\ m(b) = 3,\ m(c) = 4,\ m(a,b) = 7,\ m(b,c) = 5,\ m(a,c) = 4,\ m(a,b,c) = 8,$
capacity in hours $f(a) = 5,\ f(b) = 4,\ f(c) = 3.$
Determine the optimal total performance!

α) only one group can work a fixed time period
β) several disjoint groups can work (fixed time in each group may differ)
γ) one group starts to work, once a worker stop to work, he cannot start again
δ) there are no constraints

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$$V_\alpha = \max\{c \cdot m(A) \mid c \cdot 1_A \le f\} = 4 \cdot 7 = 28$$

SHILKRET 1971

$$V_\beta = \max\left\{\sum c_i \cdot m(A_i) \mid \sum c_i \cdot 1_{A_i} \le f, (A_i) \text{ system disj. }\right\} =$$

$$= 4 \cdot 7 + 3 \cdot 4 = 40$$

YANG 1983 (PAN–integral)

$$V_\gamma = \max\left\{\sum c_i \cdot m(A_i) \mid \sum c_i \cdot 1_{A_i} \le f, (A_i) \text{ chain }\right\} =$$

$$= 3 \cdot 8 + 1 \cdot 7 + 1 \cdot 2 = 33$$

CHOQUET 1953

$$V_\delta = \max\left\{\sum c_i \cdot m(A_i) \mid \sum c_i \cdot 1_{A_i} \le f\right\} = 4 \cdot 7 + 1 \cdot 2 + 3 \cdot 4 = 42$$

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Universal integrals

$$I : \bigcup_{(X,\mathcal{A})\in\mathcal{S}} \left(\mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}\right) \to [0,\infty]$$

$\mathcal{S}$ the class of all measurable spaces
$\mathcal{M}_{(X,\mathcal{A})}$ all monotone measures on $(X,\mathcal{A})$
$\mathcal{F}_{(X,\mathcal{A})}$ all non–negative measurable functions $f : X \to \mathcal{A}$

(U/1) I is increasing in both coordinates

(U/2) there is pseudo–multiplication $\otimes : [0, \infty]^2 \to [0, \infty]$ such that $I(m, c1_E) = c \otimes m(E)$

(U/3) $I(m_1, f_1) = I(m_2, f_2)$ whenever $m_1(f_1 \geq x) = m_2(f_2 \geq x)$ for all $x \in (0, \infty]$

If we constraint to $[0, 1]$ case ("fuzzy"), we suppose $\otimes : [0, 1]^2 \to [0, 1]$ is a semicopula, and always $m(X) = 1$

History starts with a geometrical approach to integration, intuitive Riemann integral

Riemann 1854

X special subset R^n

$\mathcal{A} = \mathcal{B}(X)$, m Lebesgue measure ("rectangles")

$n = 1, \quad X = [a, b], \quad I_R \approx$ area

$I_R(f) =$

$$= \sup \left\{ \sum c_i \lambda (E_i) \mid \sum c_i \cdot 1_{E_i} \leq f, \ (E_i) \text{ disjoint system of intervals} \right\}$$

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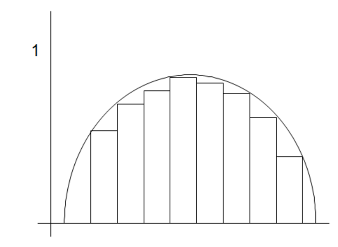
Lebesgue 1902
$(X, \mathcal{A}) \in \mathcal{S}$, but m is σ–additive
(if $m(X) = 1$, m is a probability measure)

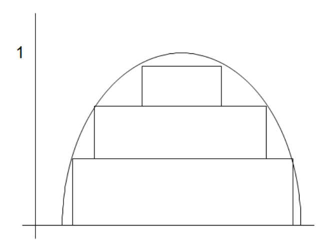
$$I_L(m, f) = \sup \left\{ \sum c_i m(E_i) \mid \sum c_i \cdot 1_{E_i} \leq f,\ (E_i) \text{ disjoint system} \right\}$$



Choquet 1953 (Šipoš 1979, but also Vitali 1925)

$$I_{Ch}(m,f) = \sup\left\{\sum c_i m(E_i) \mid \sum c_i \cdot 1_{E_i} \le f,\ (E_i)\ \text{chain}\right\}$$

Universal integral



PAN–integral Yang 1983
á la Lebesgue, m need not be σ–additive

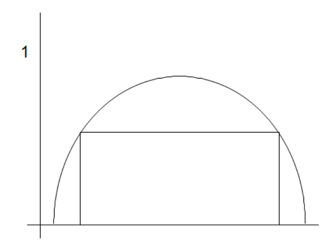
$$I_Y(m,f) = \sup\left\{\sum c_i m(E_i) \mid \sum c_i \cdot 1_{E_i} \leq f,\ (E_i) \text{ disjoint}\right\}$$

NOT a universal integral

Shilkret 1971

$$I_{Sh}(m,f) = \sup\{c \cdot m(E) \mid c \cdot 1_E \le f\}$$

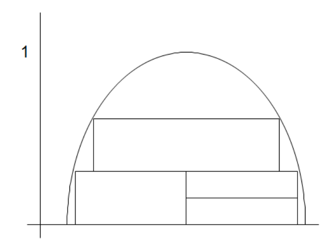
Universal integral



Lehrer 2009 (concave integral)

$$I_L(m, f) = \sup \left\{ \sum c_i m(E_i) \mid \sum c_i \cdot 1_{E_i} \leq f \right\}$$

Not a universal integral



Defect of some integrals – they do not allow to recover original measure m!

PAN–integral

$$I_Y(m, 1_E) = m(E) \text{ for } \forall E \in \mathcal{A}$$

only if m is superadditive, $m(E \cup F) \geq m(E) + m(F), \quad E \cap F = \emptyset$

concave integral

$$I_L(m, 1_E) = m(E) \text{ for } \forall E \in \mathcal{A}$$

only if m is TB (totally balanced);
(supermodularity $m(A \cup B) + m(A \cap B) \geq m(A) + m(B)$ of m is enough!)

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+ and $\cdot$ can be modified into $\oplus$ and $\odot$

For example,

$$x \oplus_p y = (x^p + y^p)^{\frac{1}{p}}, \quad p > 0$$

$$x \odot y = xy$$

pseudo–concave integral (Jun Li, R. Mesiar & E. Pap, 2011)

$$I_L^{\oplus_p,\odot}(m, f) = \sup\left\{ \bigoplus_p (c_i \odot m(E_i)) \mid \bigoplus_p b(c_i, E_i) \leq f \right\}$$

where $b(c, E)(x) = \begin{cases} c & \text{if } x \in E, \\ 0 & \text{else} \end{cases}$

qualitatively nothing new up to an isomorphism

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$\oplus = \vee$ (sup)
Working on $[0, 1]$, considering $\odot$ any semicopula (1 is neutral element)

all 4 approaches are equivalent, yielding unique integral

$$I^{\oplus,\odot}(m, f) = \sup\{c \odot m(E) \mid b(c, E) \le f\} =$$

$$= \sup\{t \odot m(f \ge t) \mid t \in [0, 1]\}$$

$I^{\vee,\wedge}$ SUGENO 1974

$I^{\vee,\cdot}$ SHILKRET 1971
$I^{\vee,T}$ WEBER 1986

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WEBER 1984, on $[0,1]$
S: Continuous Archimedean t–conorm
$g : [0,1] \to [0,\infty]$ additive generator

$$S(x,y) = g^{(-1)}\left(g(x) + g(y)\right)$$

$\exists$ countable partition $\{X_n\}$ of X, $m(X_n) < 1$,
m is S–additive

$$W_m(f) = g^{(-1)}\left(\sum_m \int_{X_n} f\, dg \circ m\right) =$$

$$= \sup\left\{\sum_i g^{-1}\left(a_i g\left(m(A_i)\right)\right) \mid \sum_i b(a_i, A_i) \le f,\ (A_i) \text{ disjoint},\ m(A_i) < 1\right\}$$

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Example

$X = [0, \infty]$, λ Lebesgue measure
i) $S(x, y) = x + y - xy$
$g(x) = -\log(1 - x)$
$m(E) = g^{-1} \circ \lambda(E) = 1 - e^{-\lambda(E)}, \quad f(x) = \frac{1}{1+x^2}$

$$W_m(f) = g^{-1}\left(\int_{[0,\infty]} f \, d\lambda\right) = g^{-1}\left(\frac{\pi}{2}\right) = 1 - e^{-\frac{\pi}{2}}$$

ii) $S(x, y) = \min(x + y, 1)$
$g(x) = x, \quad g^{(-1)} = min(x, 1)$
$m(E) = \min(\lambda(E), 1)$

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MUROFUSHI & SUGENO 1991,
fuzzy t–conorm integral (on $[0,1]$)

1. type $I^{\vee,\odot}$
2. type á la distorted Choquet

$$I_m^{h,k,g}(f) = h^{(-1)}\left(Ch_{g\circ m}(k\circ f)\right)$$

$g, h, k : [0,1] \to [0,\infty]$ additive
generators of t–conorms;
$h = g \sim$ strict t–conorm S
m is S–additive $\Rightarrow$ Weber integral (1984)

MESIAR 1996,
Choquet–like integral (on $[0, \infty]$)

1. type $I^{\vee,\otimes}$
2. type $I^g_m(f) = g^{-1}\left(Ch_{g\circ m}(g \circ f)\right)$
   $g : [0, \infty] \to [0, \infty]$ automorphism

Universal integrals based on copulas, on [0, 1]
KLEMENT, MESIAR, PAP 2004, 2010

$$C : [0,1]^2 \to [0,1]$$

$$C(0,x) = C(x,0) = 0$$

$$C(1,x) = C(x,1) = x$$

$$C((x_1,y_1) \vee (x_2,y_2)) + C((x_1,y_1) \wedge (x_2,y_2)) \geq C(x_1,y_1) + C(x_2,y_2)$$

$$C \longleftrightarrow P_C \text{ on } \mathcal{B}\left([0,1]^2\right),$$

$$P_C([a,b] \times [0,1]) = P_C([0,1] \times [a,b]) = b - a$$

$$I_C(m,f) = P_C\left(\{(u,v) \in [0,1]^2 \mid v \leq m(f \geq u)\}\right)$$

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$$C((x_1, y_1) \vee (x_2, y_2)) + C((x_1, y_1) \wedge (x_2, y_2)) \geq C(x_1, y_1) + C(x_2, y_2)$$

$$C \longleftrightarrow P_C \text{ on } \mathcal{B}\left([0, 1]^2\right),$$

$$P_C([a, b] \times [0, 1]) = P_C([0, 1] \times [a, b]) = b - a$$

$$I_C(m, f) = P_C\left(\left\{(u, v) \in [0, 1]^2 \mid v \leq m(f \geq u)\right\}\right)$$

Universal integrals based on copulas, on [0, 1]
KLEMENT, MESIAR, PAP 2004, 2010

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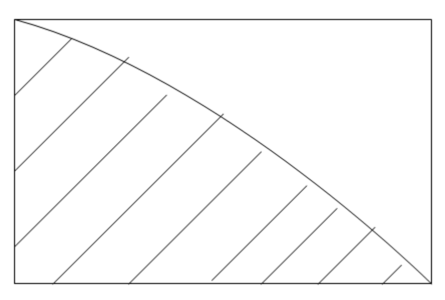
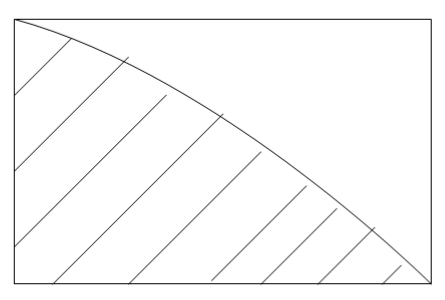


Figure: graph of the function $m(f \geq u)$

$I_{\Pi} = I_C$ Choquet

$I_{Min} = I^{\vee,\wedge}$ Sugeno

$$I_C(m, c \cdot 1_E) = C(c, m(E))$$



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Fuzzy measure–based integral, on $[0,1]$
Klement, Mesiar, Pap 2004

μ fuzzy measure on $\mathcal{B}\left([0,1]^2\right)$,
$\mu\left([0,c]\times[0,1]\right)=\mu\left([0,1]\times[0,c]\right)=c$

$$I_\mu(m,f)=\mu\left(\{(u,v)\in[0,1]^2 | v\leq m(f\leq u)\}\right)$$

$\mu(E)=\sup\left(uv | (u,v)\in E\right)\to$ SHILKRET

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on $[0, \infty]$ similar integrals

one application:
h–index, q–index

$$H = I^{\vee,\wedge}(m, f) \quad Q = I^{\vee,A}(m, \sqrt{f})$$

X all publications of an author
$m(E) = card\ E$
$f(i) = c_i$ number of citations of publication i

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1. Kolesárová A; Komorníková M: Triangular norm-based iterative compensatory operators. FUZZY SETS AND SYSTEMS 104 (1999) 109-120 (**38**)
2. Calvo T; Kolesárová A; Komorníková M; et al.: Aggregation operators: Properties, classes and construction methods. In: AGGREGATION OPERATORS: NEW TRENDS AND APPLICATIONS Book Series: Studies in Fuzziness and Soft Computing 97, pp. 3-104 Published: 2002 (**21**)
3. Komorník J.; Komorníková M.; Mesiar R.; et al.:Comparison of forecasting performance of nonlinear models of hydrological time series. PHYSICS AND CHEMISTRY OF THE EARTH 31 (2006)1127-1145 (**12**)
4. Komorníková M: Aggregation operators and additive generators. INTERNATIONAL JOURNAL OF UNCERTAINTY FUZZINESS AND KNOWLEDGE-BASED SYSTEMS 9 (2001) 205-215 (**11**)
5. Komorníková M.; Szolgay J.; Svetlíková D.; et al.: A hybrid modelling framework for forecasting monthly reservoir inflows. JOURNAL OF HYDROLOGY AND HYDROMECHANICS 56 (2008) 145-162 (**5**)
6. Hanus J; Komorníková M; Mináriková J: Influence of boxing materials on the properties of different paper items stored inside. RESTAURATOR-INTERNATIONAL JOURNAL FOR THE PRESERVATION OF LIBRARY AND ARCHIVAL MATERIAL 16 (1995) 94-208 (**4**)

h = 5, q = 3

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Thanks for your attention!