Recent results on multi-adjoint concept lattices

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Outline

1. **Introduction**
2. Multi-adjoint concept lattices
3. Multi-adjoint t-concept lattices
4. Dual multi-adjoint concept lattices
5. Multi-adjoint *-oriented concept lattices
6. MACL and heterogeneous conjunctors
7. An example

Abbreviated history of formal concept analysis

- Since its introduction in the eighties by Ganter and Wille, formal concept analysis has become an appealing research topic both from theoretical and applicative perspectives
- FCA is a theory of data analysis which identifies conceptual structures among data sets. It has been applied to linguistic databases, library and information science, software re-engineering, . . .
- Handling uncertainty, imprecise data or incomplete information has become an important research topic in the recent years.

Abbreviated history of formal concept analysis

- Soon after its introduction, a number of different approaches for its generalization were introduced:
  - Concept lattices by Pollandt.
  - Fuzzy concept lattices by A. Burusco and R. Fuentes-González.
  - Generalized concept lattices by Krajči.
  - (L-equality) fuzzy concept lattices by Bělohlávek.
- Nowadays, there are works which extend the theory with:
  - fuzzy set theory
  - fuzzy logic reasoning
  - rough set theory
  - integrated approaches such as fuzzy and rough, or rough and domain theory

Abbreviated history of formal concept analysis

- The multi-adjoint framework originated as a generalisation of several non-classical logic programming frameworks whose semantic structure is the multi-adjoint lattice.
- Applying the philosophy of the multi-adjoint framework to formal concept analysis.
  - Non-commutative conjunctors have been used.
  - Several adjoint triples: different degrees of preference can be easily established.

On the multi-adjoint framework

Step by step

Classical logic programming *[Kowalski & van Emden]*:

$$paper\_accepted \leftarrow good\_work, good\_referees$$

Quantitative logic programming *[van Emden]*:

$$paper\_accepted \xleftarrow{0.9} good\_work \ \& \ good\_referees$$

Fuzzy logic programming *[Vojtáš & Paulík]*:

$$paper\_accepted \xleftarrow{0.9}_{\text{Ł}} \min(good\_work, good\_referees)$$

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On the multi-adjoint framework

Step by step

Probabilistic deductive databases *[Lakshmanan & Sadri]*:

$$(paper\_accepted \xleftarrow{\langle[0.7,0.95],[0.03,0.2]\rangle} good\_work, good\_referees \\ ; ind, pc)$$

Hybrid probabilistic logic programs *[Dekhtyar & Subrahmanian]*:

$$(paper\_accepted \vee_{pc} go\_conference) : [0.85, 0.98] \longleftarrow \\ (good\_work \wedge_{ind} good\_referees) : [0.7, 0.9] \;\&\; \\ have\_money : [0.9, 1.0]$$

On the multi-adjoint framework

Step by step

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On the multi-adjoint framework

Common features to the previous approaches

- Different types of weights, confidence values, truth-values, or degrees.
- Implications symbols with weighs associated to rules.
- Bodies built with monotone functions.

The paradigm of multi-adjoint logic programming abstracts the particular details of each of the previous approaches, and keeps the deductive engine.

On the multi-adjoint framework

Towards multi-adjoint lattices

Definition

A *residuated lattice* is a tuple $\langle L, \preceq, \&, \rightarrow, \top \rangle$ such that:

1. $\langle L, \preceq \rangle$ is a bounded lattice with $\top$ being the maximum element
2. $\langle L, \&, \top \rangle$ is a commutative monoid
3. The pair $\langle \&, \rightarrow \rangle$ is *adjoint* in L: for all $x, y, z \in L$

$$x \preceq (z \rightarrow y) \quad \Longleftrightarrow \quad (x \,\&\, z) \preceq y$$

Allowing different implications in one program leads to considering algebraic structures in which several adjoint pairs coexist within a lattice (similar motivation to $L\Pi$ logics, yesterday).

On the multi-adjoint framework

Multi-adjoint lattices

Definition

A multi-adjoint lattice is a tuple $(L, \preceq, \leftarrow_1, \&_1, \dots, \leftarrow_n, \&_n)$ such as the following conditions hold:

1. $\langle L, \preceq \rangle$ is a bounded lattice;
2. $(\leftarrow_i, \&_i)$ is an adjoint pair in $\langle L, \preceq \rangle$ for $i = 1, \dots, n$;
3. $\top \&_i \vartheta = \vartheta \&_i \top = \vartheta$ for all $\vartheta \in L$ and all $i = 1, \dots, n$.

Multi-adjoint logic programs

Syntax

Definition

A *multi-adjoint logic program* is a set $\mathbb{P}$ of rules of the form $\langle (A \leftarrow_i \mathcal{B}), \vartheta \rangle$ such that:

1. The *weight* ϑ is an element of L (a truth-value);
2. The *head* A is a propositional symbol in Π.
3. The *body* $\mathcal{B}$ is a wff built from propositional symbols and monotonic operators.
4. *Facts* are rules with body $\top$.

Multi-adjoint logic programs

Semantics

Definitions

- An *interpretation* is a mapping $I\colon \Pi \to L$.
  Each interpretation can be extended to any wff in the language by homomorphic extension.
- An interpretation $I \in \mathcal{I}_{\mathfrak{L}}$ *satisfies* a rule $\langle A \leftarrow_i \mathcal{B}, \vartheta \rangle$ if and only if $\vartheta \preceq \hat{I}\,(A \leftarrow_i \mathcal{B})$.
- An interpretation $I \in \mathcal{I}_{\mathfrak{L}}$ is a *model* of program $\mathbb{P}$ if and only if satisfies all the rules in $\mathbb{P}$.

Multi-adjoint logic programs

Fixed point semantics

The immediate consequences operator introduced originally by van Emden and Kowalski can be generalized to a multi-adjoint framework as follows:

Definition

Let $\mathbb{P}$ be a multi-adjoint program. The *immediate consequence operator* $T_{\mathbb{P}}\colon \mathcal{I} \to \mathcal{I}$ is defined, given an interpretation I and an atom A, as follows

$$T_{\mathbb{P}}(I)(A) = \sup\{\vartheta \,\&_i\, I(\mathcal{B}) \mid A \xrightarrow{\vartheta}_i \mathcal{B} \in \mathbb{P}\}$$

Note that all the suprema exist, since we are working on a complete lattice.

Multi-adjoint logic programs

Models and fixed-point

Lemma

The operator $T_{\mathbb{P}}$ *is non-decreasing.*

Theorem

An interpretation I *is a model of a multi-adjoint program* $\mathbb{P}$ *if and only if* $T_{\mathbb{P}}(I) \sqsubseteq I$*.*

This theorem, together with Knaster-Tarski's theorem, implies that every program has a least model whih can be obtained by transfinite iteration from the bottom interpretation.

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Multi-adjoint concept lattice

Adjoint triples

Definition

Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ be mappings, then $(\&, \swarrow, \nwarrow)$ is an *adjoint triple* with respect to P_1, P_2, P_3 if:

1. $\&$ is order-preserving in both arguments.
2. $\swarrow$ and $\nwarrow$ are order-preserving in the succedent and order-reversing in the antecedent.
3. $x \leq_1 z \swarrow y \quad \text{iff} \quad x \,\&\, y \leq_3 z \quad \text{iff} \quad y \leq_2 z \nwarrow x$, with $x \in P_1$, $y \in P_2$ and $z \in P_3$.

Multi-adjoint concept lattice

Multi-adjoint frames and contexts

Definition

A *multi-adjoint frame* $\mathcal{L}$ is a tuple

$$(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$$

where $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are complete lattices, $(P, \leq)$ is a poset and, for all $i = 1, \dots, n$, $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple with respect to L_1, L_2, P.

Definition

Let $(L_1, L_2, P, \&_1, \dots, \&_n)$ be a multi-adjoint frame, a *context* is a tuple (A, B, R, σ) such that A and B are non-empty sets, R is a P-fuzzy relation $R \colon A \times B \to P$ and $\sigma \colon B \to \{1, \dots, n\}$ is a mapping which associates any element in B with some particular adjoint triple in the frame.

Multi-adjoint concept lattice

Galois connections: the construction

We can define the mappings ${}^{\uparrow_\sigma}\colon L_2^B \longrightarrow L_1^A$ and ${}^{\downarrow^\sigma}\colon L_1^A \longrightarrow L_2^B$:

$$\begin{aligned} g^{\uparrow_\sigma}(a) &= \inf\{R(a,b) \swarrow^{\sigma(b)} g(b) \mid b \in B\} \\ f^{\downarrow^\sigma}(b) &= \inf\{R(a,b) \nwarrow_{\sigma(b)} f(a) \mid a \in A\} \end{aligned}$$

These two arrows, $({}^{\uparrow_\sigma}, {}^{\downarrow^\sigma})$, generate a Galois connection.

Definition

The *multi-adjoint concept lattice* associated to a multi-adjoint frame $(L_1, L_2, P, \&_1, \ldots, \&_n)$ and a context (A, B, R, σ) is the set

$$\mathcal{M} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow_\sigma} = f, f^{\downarrow^\sigma} = g\}$$

where $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ is defined by $g_1 \preceq_2 g_2$.

Multi-adjoint concept lattice

The basic theorem

Theorem

A multi-adjoint concept lattice $(\mathcal{M}, \preceq)$, *defined on a fixed frame* $(L_1, L_2, P, \&_1, \dots, \&_n)$, *and a context* (A, B, R, σ), *is isomorphic to a complete lattice* $(V, \sqsubseteq)$ *if and only if there exists a pair of mappings* $\alpha \colon A \times L_1 \to V$ *and* $\beta \colon B \times L_2 \to V$ *such that:*

- $\alpha[A \times L_1]$ *is infimum-dense;*
- $\beta[B \times L_2]$ *is supremum-dense; and*
- *For each* $a \in A$, $b \in B$, $x \in L_1$ *and* $y \in L_2$*:*

$$\beta(b, y) \sqsubseteq \alpha(a, x) \quad \text{if and only if} \quad x \,\&_b\, y \leq R(a, b)$$

Variants of multi-adjoint concept lattices

- multi-adjoint t-concept lattices
- multi-adjoint dual concept-lattices
- multi-adjoint property-oriented concept lattices
- multi-adjoint object-oriented concept lattices
- multi-adjointness and heterogeneous conjunctors

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t-concept lattices

- The t-concept lattice is introduced as a set of triples associated to graded tabular information in a non-commutative fuzzy logic.
- Following the general techniques of formal concept analysis, given a non-commutative conjunctor it is possible to provide generalizations of the mappings for the intension and the extension in two different ways, and this generates a pair of concept lattices.
- The information common to both concept lattices can be seen as a sublattice of the Cartesian product of both concept lattices.

t-concept lattices

- The t-concepts are introduced as a generalisation of the approach given by Georgescu-Popescu for non-commutative conjunctors.
- The basic structure we will work with is that of multi-adjoint frame, where the complete lattices $\langle L_1, \preceq_1 \rangle, \langle L_2, \preceq_2 \rangle$ coincide.
- Given a context (A, B, R, σ), besides the Galois connection $(^\uparrow, ^\downarrow)$ defined for the multi-adjoint concept lattice, it is possible to define an alternative version as follows:

$$\begin{aligned} g^{\uparrow_{op}}(a) &= \inf\{R(a,b) \nwarrow_b g(b) \mid b \in B\} \\ f^{\downarrow^{op}}(b) &= \inf\{R(a,b) \swarrow^b f(a) \mid a \in A\} \end{aligned}$$

- We have two Galois connections on which two different multi-adjoint concept lattices $(\mathcal{M}, \preceq)$, $(\mathcal{M}^{op}, \preceq)$ can be defined, which are different if at least one conjunctor $\&_i$ is non-commutative.

t-concept lattices

This suggests to consider the following subsets of $\mathcal{M} \times \mathcal{M}^{op}$

$$\mathcal{N}_1 = \{(\langle g, f_1\rangle, \langle g, f_2\rangle) \mid \langle g, f_1\rangle \in \mathcal{M}, \langle g, f_2\rangle \in \mathcal{M}^{op}\}$$
$$\mathcal{N}_2 = \{(\langle g_1, f\rangle, \langle g_2, f\rangle) \mid \langle g_1, f\rangle \in \mathcal{M}, \langle g_2, f\rangle \in \mathcal{M}^{op}\}$$

which are sublattices of $\mathcal{M} \times \mathcal{M}^{op}$ and, thus, are complete lattices.

Fact

The approach by Georgescu-Popescu is a particular instance of this approach.

Their construction explicitly assumes that $(L, \&, \top)$ should be a commutative monoid, but $\mathcal{N}_i$ can be defined directly from an adjoint triple which, obviously needs not be either commutative or associative.

t-concept lattices

The basic theorem

Theorem

Given two complete lattices $(V_1, \sqsubseteq_1)$, $(V_2, \sqsubseteq_2)$ *which represent the multi-adjoint concept lattices* $(\mathcal{M}, \preceq)$, $(\mathcal{M}^{op}, \preceq)$, *respectively, the sublattice* $\mathcal{V}$ *of* $V_1 \times V_2$ *is defined as:*

$$\mathcal{V} = \left\{ \left(\inf_{(a,x)\in K_1} \alpha_1(a,x), \inf_{(a,x)\in K_2} \alpha_2(a,x) \right) \mid (K_1, K_2) \in \mathcal{K} \right\}$$

where $\mathcal{K} = \{(K_1, K_2) \mid K_1, K_2 \subseteq A \times L$ *and* $\inf_{(a,x)\in K_1} @_a^{x\downarrow} = \inf_{(a,x)\in K_2} @_a^{x\downarrow^{op}}\}$ *and* α_1, α_2 *are the maps associated to* $(\mathcal{M}, \preceq)$, $(\mathcal{M}^{op}, \preceq)$ *respectively.*

The sublattice $\mathcal{V}$ *above is isomorphic to the complete sublattice of t-concepts* $\mathcal{N}_1$ *of* $\mathcal{M} \times \mathcal{M}^{op}$.

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Dual multi-adjoint concept lattices

Given a poset $(P, \leq)$, the *dual* ordering of $\leq$ is the relation $\leq^\partial$, defined as $x_1 \leq^\partial x_2$ if and only if $x_2 \leq x_1$, for all $x_1, x_2 \in P$. Usually, we will write P^∂ instead of $(P, \leq^\partial)$, and we will say that P^∂ is the *dual* of P.

Definition

A *dual multi-adjoint frame*, denoted $(L_1, L_2, P, \&_1, \dots, \&_n)^\partial$, is a (standard) frame $(L_1^\partial, L_2^\partial, P, \preceq_1, \preceq_2, \leq, \&_1, \dots, \&_n)$.
The notion of context is exactly that given previously.

The mappings ${}^{\uparrow\triangledown} : L_2^B \to L_1^A$ and ${}^{\downarrow^\triangledown} : L_1^A \to L_2^B$ are defined, given $g \colon B \to L_2$, $f \colon A \to L_1$, as

$$
\begin{aligned}
g^{\uparrow\triangledown}(a) &= \sup_1\{R(a,b) \swarrow^{\sigma(b)} g(b) \mid b \in B\} \\
f^{\downarrow^\triangledown}(b) &= \sup_2\{R(a,b) \nwarrow_{\sigma(b)} f(a) \mid a \in A\}
\end{aligned}
$$

where $\sup_1$, $\sup_2$ are the supremum operators on L_1 and L_2, respectively.

Dual multi-adjoint concept lattices

Proposition

The mappings ${}^{\uparrow_\triangledown}\colon L_2^B \to L_1^A$ *and* ${}^{\downarrow^\triangledown}\colon L_1^A \to L_2^B$ *satisfy that* $g^{\uparrow_\triangledown} = ((g^\partial)^\uparrow)^\partial$, $f^{\downarrow^\triangledown} = ((f^\partial)^\downarrow)^\partial$, *for all* $g \in L_2^B$, $f \in L_1^A$.

Proposition

1. ${}^{\uparrow_\triangledown}\colon L_2^B \to L_1^A$ *and* ${}^{\downarrow^\triangledown}\colon L_1^A \to L_2^B$ *are order-reversing.*
2. ${}^{\uparrow_\triangledown\downarrow^\triangledown}\colon L_2^B \to L_2^B$, ${}^{\downarrow^\triangledown\uparrow_\triangledown}\colon L_1^A \to L_1^A$ *are interior operators.*
3. $g^{\uparrow_\triangledown\downarrow^\triangledown\uparrow_\triangledown} = g^{\uparrow_\triangledown}$, $f^{\downarrow^\triangledown\uparrow_\triangledown\downarrow^\triangledown} = f^{\downarrow^\triangledown}$, *for all* $g \in L_2^B$, $f \in L_1^A$.

It is worth to note that the pair $({}^{\uparrow_\triangledown}, {}^{\downarrow^\triangledown})$ is **not** a Galois connection but satisfies that ${}^{\uparrow_\triangledown\downarrow^\triangledown}$ and ${}^{\downarrow^\triangledown\uparrow_\triangledown}$ are interior operators, and this is enough to construct a concept lattice.

Dual multi-adjoint concept lattices

Definition

1. A *dual concept* is a pair $\langle g, f \rangle$ such that $g \in L_2^B$, $f \in L_1^A$ and the equations $g^{\uparrow_\nabla} = f$ and $f^{\downarrow^\nabla} = g$ hold.
2. Given a dual multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)^\partial$ and a context (A, B, R, σ), a *dual multi-adjoint concept lattice* is the pair $(\mathcal{M}^\nabla, \leq^\nabla)$, where

$$\mathcal{M}^\nabla = \{\langle g, f \rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow_\nabla} = f, f^{\downarrow^\nabla} = g\}$$

is the set of dual concepts, and $\leq^\nabla$ is the order defined $\langle g_1, f_1 \rangle \leq^\nabla \langle g_2, f_2 \rangle$ if and only if $g_1 \preceq_2 g_2$ (or, equivalently, $f_2 \preceq_1 f_1$).

As $({}^{\uparrow_\nabla}, {}^{\downarrow^\nabla})$ is not a Galois connection, the proof that $(\mathcal{M}^\nabla, \leq^\nabla)$ is indeed a complete lattice does not follow the usual approach. In order to prove this fact, we have to consider an auxiliary multi-adjoint concept lattice.

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Multi-adjoint *-oriented concept lattices (Medina)

Modal-style operators

Given the sets A, B, and a crisp relation $R\colon A \times B \to \{0,1\}$, we can consider the following mappings ${}^{\pi}\colon 2^B \to 2^A$, ${}^{N}\colon 2^B \to 2^A$, ${}^{\nabla}\colon 2^B \to 2^A$ defined, for each $X \subseteq B$, as:

$$\begin{aligned} X^{\pi} &= \{a \in A \mid \text{ there is } b \in X, \text{ such that } aRb\} \\ X^{N} &= \{a \in A \mid \text{ for all } b \in B, \text{ if } aRb, \text{ then } b \in X\} \\ X^{\nabla} &= \{a \in A \mid \text{ there exists } b \in X^{c}, \text{ such that } aR^{c}b\} \end{aligned}$$

where X^c, R^c are the complement of X and the complement relation of R, respectively. Some pairs of the derivation operators above can be shown to form (antitone or isotone) Galois connections and allow for constructing new formal concept lattices: the *classical*, the *dual*, the *object-oriented* and the *property-oriented*.

Multi-adjoint *-oriented concept lattices

(Isotone) Galois connections

Definitions

- Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets, and ${}^{\downarrow}\colon P_1 \to P_2$, ${}^{\uparrow}\colon P_2 \to P_1$ mappings, the pair $({}^{\uparrow}, {}^{\downarrow})$ forms a *Galois connection* between P_1 and P_2 if and only if:
  - ${}^{\uparrow}$ and ${}^{\downarrow}$ are order-reversing;
  - $x \leq_1 x^{\downarrow\uparrow}$, for all $x \in P_1$, and
  - $y \leq_2 y^{\uparrow\downarrow}$, for all $y \in P_2$.
- Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets, and ${}^{\downarrow}\colon P_1 \to P_2$, ${}^{\uparrow}\colon P_2 \to P_1$ mappings, the pair $({}^{\uparrow}, {}^{\downarrow})$ forms an *isotone Galois connection* between P_1 and P_2 if and only if:
  - ${}^{\uparrow}$ and ${}^{\downarrow}$ are order-preserving;
  - $x \leq_1 x^{\downarrow\uparrow}$, for all $x \in P_1$, and
  - $y^{\uparrow\downarrow} \leq_2 y$, for all $y \in P_2$.

Multi-adjoint *-oriented concept lattices

Special types of frames

Definitions

- A *multi-adjoint* **property-oriented** *frame* $\mathcal{L}$ is a frame $(L_1, L_2, P, \&_1, \ldots, \&_n)$ such that $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple **with respect to** P, L_2, L_1, for all $i = 1, \ldots, n$.
- A *multi-adjoint* **object-oriented** *frame* $\mathcal{L}$ is a tuple $(L_1, L_2, P, \&_1, \ldots, \&_n)$ such that $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple **with respect to** L_1, P, L_2, for all $i = 1, \ldots, n$.

Multi-adjoint *-oriented concept lattices

Adjoint triples and duality

Lemma

Given the posets $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ *and an adjoint triple with respect them* $(\&, \swarrow, \nwarrow)$, *we obtain that:*

1. $(\&^{op}, \nwarrow, \swarrow)$ *is an adjoint triple with respect to* P_2, P_1, P_3.
2. $(\swarrow, \&, \nwarrow_{op})$ *is an adjoint triple with respect to* P_3^∂, P_2, P_1^∂.
3. $(\nwarrow, \&^{op}, \swarrow^{op})$ *is an adjoint triple with respect to* P_3^∂, P_1, P_2^∂.
4. $(\nwarrow_{op}, \swarrow^{op}, \&^{op})$ *is an adjoint triple with respect to* P_1, P_3^∂, P_2^∂.

Multi-adjoint *-oriented concept lattices

Fuzzy version of modal operators

We can define ${}^{\uparrow_N}\colon L_2^B \to L_1^A$, ${}^{\uparrow_\pi}\colon L_2^B \to L_1^A$:

$$
\begin{aligned}
g^{\uparrow_\pi}(a) &= \sup\{R(a,b) \mathbin{\&_b} g(b) \mid b \in B\} \\
g^{\uparrow_N}(a) &= \inf\{g(b) \nwarrow_b R(a,b) \mid b \in B\}
\end{aligned}
$$

Analogously, ${}^{\downarrow^N}\colon L_2^A \to L_1^B$, ${}^{\downarrow^\pi}\colon L_2^A \to L_1^B$:

$$
\begin{aligned}
f^{\downarrow^\pi}(b) &= \sup\{R(a,b) \mathbin{\&_b} f(a) \mid a \in A\} \\
f^{\downarrow^N}(b) &= \inf\{f(a) \nwarrow_b R(a,b) \mid a \in A\}
\end{aligned}
$$

Clearly, these definitions are generalizations of the classical and fuzzy possibility and necessity operators by Gediga-Düntsch, Georgescu-Popescu, Chen-Yao, Lai-Zhang, etc.

Multi-adjoint *-oriented concept lattices

Definition

A *multi-adjoint property-oriented concept lattice* is the set

$$\mathcal{M}_{\pi N} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow_\pi} = f, f^{\downarrow^N} = g\}$$

with ordering defined by $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ iff $g_1 \preceq_2 g_2$ (or $f_1 \preceq_1 f_2$).

Theorem

The multi-adjoint property-oriented concept lattice $(\mathcal{M}_{\pi N}, \preceq)$ *is, indeed, a complete lattice where*

$$\begin{aligned}
\inf\{\langle g_i, f_i\rangle \mid i \in I\} &= \langle \inf_2\{g_i \mid i \in I\}, (\inf_1\{f_i \mid i \in I\})^{\downarrow^N \uparrow_\pi}\rangle \\
\sup\{\langle g_i, f_i\rangle \mid i \in I\} &= \langle (\sup_2\{g_i \mid i \in I\})^{\uparrow_\pi \downarrow^N}, \sup_1\{f_i \mid i \in I\}\rangle
\end{aligned}$$

The mappings ${}^{\uparrow_N} : L_2^B \to L_1^A$, ${}^{\downarrow^\pi} : L_2^A \to L_1^B$ are used, together with multi-adjoint object-oriented frames, to obtain the so-called multi-adjoint object-oriented concept lattice.

Multi-adjoint *-oriented concept lattices

Definition

A *multi-adjoint property-oriented concept lattice* is the set

$$\mathcal{M}_{\pi N} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow_\pi} = f, f^{\downarrow^N} = g\}$$

with ordering defined by $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ iff $g_1 \preceq_2 g_2$ (or $f_1 \preceq_1 f_2$).

Theorem

The multi-adjoint property-oriented concept lattice $(\mathcal{M}_{\pi N}, \leq)$ *is, indeed, a complete lattice where*

$$\begin{aligned}\inf\{\langle g_i, f_i\rangle \mid i \in I\} &= \langle \inf_2\{g_i \mid i \in I\}, (\inf_1\{f_i \mid i \in I\})^{\downarrow^N\uparrow_\pi}\rangle \\ \sup\{\langle g_i, f_i\rangle \mid i \in I\} &= \langle (\sup_2\{g_i \mid i \in I\})^{\uparrow_\pi\downarrow^N}, \sup_1\{f_i \mid i \in I\}\rangle\end{aligned}$$

The mappings ${}^{\uparrow_N} : L_2^B \to L_1^A$, ${}^{\downarrow^\pi} : L_2^A \to L_1^B$ are used, together with multi-adjoint object-oriented frames, to obtain the so-called multi-adjoint object-oriented concept lattice.

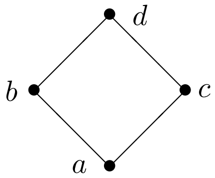
Outline

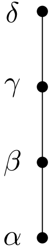
1. Introduction
2. Multi-adjoint concept lattices
3. Multi-adjoint t-concept lattices
4. Dual multi-adjoint concept lattices
5. Multi-adjoint *-oriented concept lattices
6. **MACL and heterogeneous conjunctors**
7. An example

Use of heterogeneous conjunctors

Motivation

Two experts use two different lattices to evaluate a given task.





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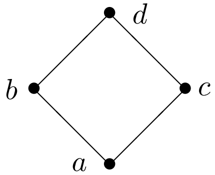

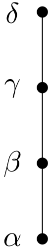

In order to unify both evaluations, we want to embed the lattices above into another one.

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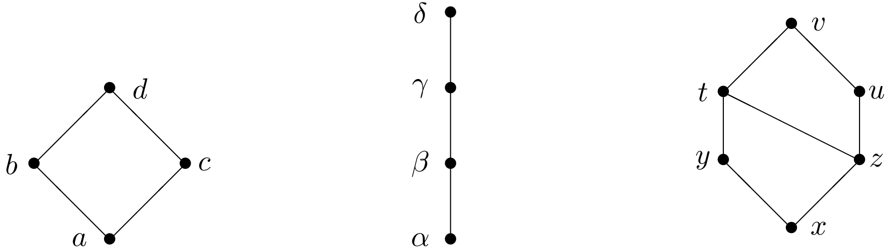

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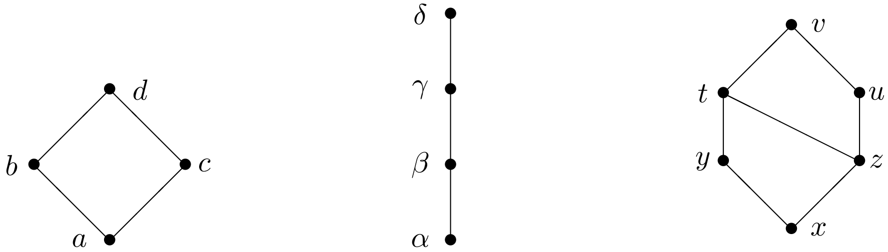

In order to unify both evaluations, we want to embed the lattices above into another one. But, how can we formalize this idea?

Use of heterogeneous conjunctors

P-connection

Definition

Given the posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P, \leq)$, we say that P_1 and P_2 are *P-connected* if there exist increasing mappings $i_1 \colon P_1 \to P$, $\phi_1 \colon P \to P_1$, $i_2 \colon P_2 \to P$ and $\phi_2 \colon P \to P_2$ verifying that $\phi_1(i_1(x)) = x$, and $\phi_2(i_2(y)) = y$, for all $x \in P_1$, $y \in P_2$

In the example above we can define two mappings $i_1 \colon L_1 \to P$, $i_2 \colon L_2 \to P$.

	a	b	c	d
i_1	x	y	u	v

	α	β	γ	δ
i_2	x	y	t	v

and mappings $\phi_1 \colon P \to L_1$, $\phi_2 \colon P \to L_2$ that satisfy the properties in the Definition.

	x	y	z	t	u	v
ϕ_1	a	b	c	d	c	d

	x	y	z	t	u	v
ϕ_2	α	β	γ	γ	δ	δ

Therefore, L_1 and L_2 are P-connected.

Use of heterogeneous conjunctors

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Therefore, L_1 and L_2 are P-connected.

Derivation operators and L-connection

Given a complete lattice $(L, \preceq)$ such that L_1 and L_2 are L-connected, a multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)$, and a context (A, B, R, σ), we can define the following mappings: ${}^{\uparrow_{c\sigma}} : L^B \to L^A$ and ${}^{\downarrow^{c\sigma}} : L^A \to L^B$:

$$g^{\uparrow_{c\sigma}}(a) = i_1(\inf\{R(a,b) \swarrow^{\sigma(b)} \phi_2(g(b)) \mid b \in B\})$$

$$f^{\downarrow^{c\sigma}}(b) = i_2(\inf\{R(a,b) \nwarrow_{\sigma(b)} \phi_1(f(a)) \mid a \in A\})$$

Fact

$({}^{\uparrow_{c\sigma}}, {}^{\downarrow^{c\sigma}})$ is **not** a Galois connection since ${}^{\downarrow^{c\sigma}\uparrow_{c\sigma}}$ is not a closure operator

Proposition

Let $(L_1, L_2, P, \&_1, \dots, \&_n)$ be a multi-adjoint frame, where L_1 and L_2 are L-connected, and a context (A, B, R, σ), then $f^{\downarrow^c} = f^{\downarrow^c \uparrow_c \downarrow^c}$, $g^{\uparrow_c} = g^{\uparrow_c \downarrow^c \uparrow_c}$, for all $g \in L^B$ and $f \in L^A$.

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Multi-adjoint L-connected concept lattice

On a frame where $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are L-connected, we define a new concept lattice by following the usual construction:

Definition

Given the complete lattices $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(L, \preceq)$, where L_1 and L_2 are L-connected, the *multi-adjoint L-connected concept lattice* associated to a multi-adjoint frame $(L_1, L_2, L, \&_1, \dots, \&_n)$ and context (A, B, R, σ) is the set

$$\mathcal{M}_L = \{\langle g^*, f^* \rangle \mid (g^*)^{\uparrow_c} = f^* \text{ and } (f^*)^{\downarrow^c} = g^*\}$$

in which the ordering is defined by $\langle g_1^*, f_1^* \rangle \leq \langle g_2^*, f_2^* \rangle$ if and only if $g_1^* \preceq g_2^*$ (equivalently $f_2^* \preceq f_1^*$).

Multi-adjoint L-connected concept lattice

The following theorem defines meet and join operators which will provide $\mathcal{M}_L$ a complete lattice structure.

Theorem

Given complete lattices $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ *and* $(L, \preceq)$, *where* L_1 *and* L_2 *are* L*-connected, a context* (A, B, R, σ), *and a multi-adjoint frame* $(L_1, L_2, L, \&_1, \dots, \&_n)$, *the multi-adjoint* L*-connected concept lattice* $\mathcal{M}_L$ *is actually a complete lattice with the meet and join operators* $\curlywedge, \curlyvee \colon \mathcal{M}_L \times \mathcal{M}_L \to \mathcal{M}_L$ *defined below, for all* $\langle g_1^*, f_1^* \rangle, \langle g_2^*, f_2^* \rangle \in \mathcal{M}_L$,

$$\begin{aligned} \langle g_1^*, f_1^* \rangle \curlywedge \langle g_2^*, f_2^* \rangle &= \langle \psi_2 \circ \phi_2(g_1^* \wedge g_2^*), (f_1^* \vee f_2^*)^{\downarrow^c \uparrow_c} \rangle \\ \langle g_1^*, f_1^* \rangle \curlyvee \langle g_2^*, f_2^* \rangle &= \langle (g_1^* \vee g_2^*)^{\uparrow_c \downarrow^c}, \psi_1 \circ \phi_1(f_1^* \wedge f_2^*) \rangle \end{aligned}$$

Outline

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7 **An example**

A detailed example

- We have written a scientific paper but we still have to decide which journal it will be submitted to.
- A number of journals are considered as potential target, depending on several parameters appearing in the ISI Journal Citation Report.
- The sets of attributes and objects are the following:

$$A = \{\text{Impact}, \text{Imm Index}, \text{Cited HL}, \text{Best Pos}\}$$
$$B = \{\text{AMC}, \text{CAMWA}, \text{FSS}, \text{IEEE-FS}, \text{IJGS}, \text{IJUFKS}, \text{JIFS}\}$$

A detailed example

The frame

We consider the following frame

$$([0,1]_{20},[0,1]_8,[0,1]_{100},\leq,\leq,\leq,\&_P^*)$$

where $[0,1]_m$ denotes a regular partition of $[0,1]$ into m pieces.

- Information about the attributes, is considered in steps of 0.05, in order to appreciate a qualitative difference.
- Levels of preference of the journal are considered in eighths.
- Information taken from the JCR is rounded to the second decimal digit.

A detailed example

The frame

The corresponding residuated implications

$$\swarrow^*_P \colon [0,1]_{100} \times [0,1]_8 \longrightarrow [0,1]_{20}$$

$$\nwarrow^*_P \colon [0,1]_{100} \times [0,1]_{20} \longrightarrow [0,1]_8$$

are defined as:

$$z \swarrow^*_P y = \frac{\lfloor 20 \cdot \min\{1, z/y\} \rfloor}{20} \qquad z \nwarrow^*_P x = \frac{\lfloor 8 \cdot \min\{1, z/x\} \rfloor}{8}$$

where $\lfloor \_ \rfloor$ is the floor function.

A detailed example

The frame

Fuzzy relation between the objects and the attributes.

R	AMC	CAMWA	FSS	IEEE-FS	IJGS	IJUFKS	JIFS
Impact	0.34	0.21	0.52	0.85	0.43	0.21	0.09
Imm Index	0.13	0.09	0.36	0.17	0.10	0.04	0.06
Cited HL	0.31	0.71	0.92	0.65	0.89	0.47	0.93
Best Pos	0.75	0.50	1.00	1.00	0.50	0.25	0.25

A detailed example

Problem: Choosing a suitable journal to submit

Depends on the notion of "suitability" we have in mind. For example, in the context (A, B, R, σ) where $\sigma(b) = \&_P$ for every $b \in B$, by the fuzzy subset $f\colon A \to [0,1]$ below:

$$f(\text{Impact}) = 0.75, \qquad f(\text{Imm Index}) = 0.30,$$
$$f(\text{Cited HL}) = 0.55, \qquad f(\text{Best Pos}) = 0.50$$

Now, the problem consists in finding a multi-adjoint concept which represents the suitable journal as defined by the fuzzy set f.

A detailed example

Computing the concept

Any concept gets completely determined by any of its components.

$$\begin{aligned} f^{\downarrow}(\mathsf{AMC}) &= \inf\{R(a,\mathsf{AMC}) \nwarrow^{*}_{P} f(a) \colon a \in A\} \\ &= \frac{\lfloor 8 \cdot \min\{1, 0.13/0.3\} \rfloor}{8} = 0.375 \end{aligned}$$

$$\begin{array}{lll} f^{\downarrow}(\mathsf{AMC}) = 0.375 & f^{\downarrow}(\mathsf{CAMWA}) = 0.250 & f^{\downarrow}(\mathsf{FSS}) = 0.625 \\ f^{\downarrow}(\mathsf{JIFS}) = 0.000 & f^{\downarrow}(\mathsf{IJGS}) = 0.250 & f^{\downarrow}(\mathsf{IJUFKS}) = 0.125 \\ f^{\downarrow}(\mathsf{IEEE\text{-}FS}) = 0.500 & & \end{array}$$

Note that, despite being the property with the highest weight, the use of this definition for "suitability" does not directly select the one with highest impact factor, since other attributes are taken into account as well.

A detailed example

Including preferences

Consider the context (A, B, R, σ'), where $\sigma'(b) = \&_P$ for all $b \in B_1$ and $\sigma'(b) = \&_L$ for all $b \in B_2$, where $B_1 = \{\text{AMC}, \text{CAMWA}, \text{FSS}, \text{IJGS}\}$ and $B_2 = \{\text{IEEE-FS}, \text{IJUFKS}, \text{JIFS}\}$.

This particular choice of σ' allows for using the Łukasiewicz implication in order to compute the values for journals in the AI category, hence the definition of $f^{\downarrow}$ is modified considering different cases:

$$f^{\downarrow}(b) = \begin{cases} \inf\{R(a,b) \nwarrow_P^* f(a) \colon a \in A\} & \text{if } b \in B_1 \\ \inf\{R(a,b) \nwarrow_L^* f(a) \colon a \in A\} & \text{if } b \in B_2 \end{cases}$$

The final result that we obtain in this case is that the journal that better suits our needs is IEEE-FS:

$$\begin{array}{lll} f^{\downarrow}(\text{AMC}) = 0.375 & f^{\downarrow}(\text{CAMWA}) = 0.250 & f^{\downarrow}(\text{FSS}) = 0.625 \\ f^{\downarrow}(\text{JIFS}) = 0.250 & f^{\downarrow}(\text{IJGS}) = 0.250 & f^{\downarrow}(\text{IJUFKS}) = 0.375 \\ f^{\downarrow}(\text{IEEE-FS}) = 0.750 & & \end{array}$$

A detailed example

Including preferences

The mere assignment of 'greater' operators to a subset of objects does not imply that the best selection is necessarily in this subset. For instance, consider f_1 to be the following modification of the notion of suitability:

$$f_1(\text{Impact}) = 0.65, \qquad f_1(\text{Imm Index}) = 0.45,$$
$$f_1(\text{Cited HL}) = 0.55, \qquad f_1(\text{Best Pos}) = 0.5$$

The results associated to this f_1 are shown below

$$f_1^{\downarrow}(\text{AMC}) = 0.250 \quad f_1^{\downarrow}(\text{CAMWA}) = 0.125 \quad f_1^{\downarrow}(\text{FSS}) = 0.750$$
$$f_1^{\downarrow}(\text{JIFS}) = 0.375 \quad f_1^{\downarrow}(\text{IJGS}) = 0.125 \quad f_1^{\downarrow}(\text{IJUFKS}) = 0.500$$
$$f_1^{\downarrow}(\text{IEEE-FS}) = 0.625$$

Therefore, in spite of having increased the preference for journals in the AI category, for this particular definition of suitable journal FSS remains as the best journal, and IEEE-FS is the second best suited.

Current and future work

- Practical applications of these concept lattices, algorithms for computing them, etc
- MACL with heterogeneous conjunctors seems to embed the theory of fuzzy concept lattices with hedges by Bělohlávek and Vychodil.
  Initial results have already been obtained in joint work with Olomouc to export results based on hedges to the more general framework.
- General study of conditions characterising general forms of Galois connections in terms of properties of the fuzzy conjunctor and implication considered in a multi-adjoint frame.

Recent results on multi-adjoint concept lattices

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