Generation of Indistinguishability Operators

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Indistinguishability Operator

Definition. *Let X be a universe and T a t-norm. A T-indistinguishability operator E on X is fuzzy relation $E : X \times X \to [0,1]$ on X satisfying for all $x, y, z \in X$*

1. $E(x,x) = 1$ (*Reflexivity*)
2. $E(x,y) = E(y,x)$ (*Symmetry*)
3. $T(E(x,y), E(y,z)) \leq E(x,z)$ (*T-Transitivity*)

E *separates points if and only if* $E(x,y) = 1$ *implies* $x = y$.

$E(x,y)$ is interpreted as the degree of indistinguishability (or similarity) between x and y.

Generation of Indistinguishability Operators

One of the most interesting issues related to indistinguishability operators is their generation, which depends on the way in which the data are given and the use we want to make of them. The four most common ways are:

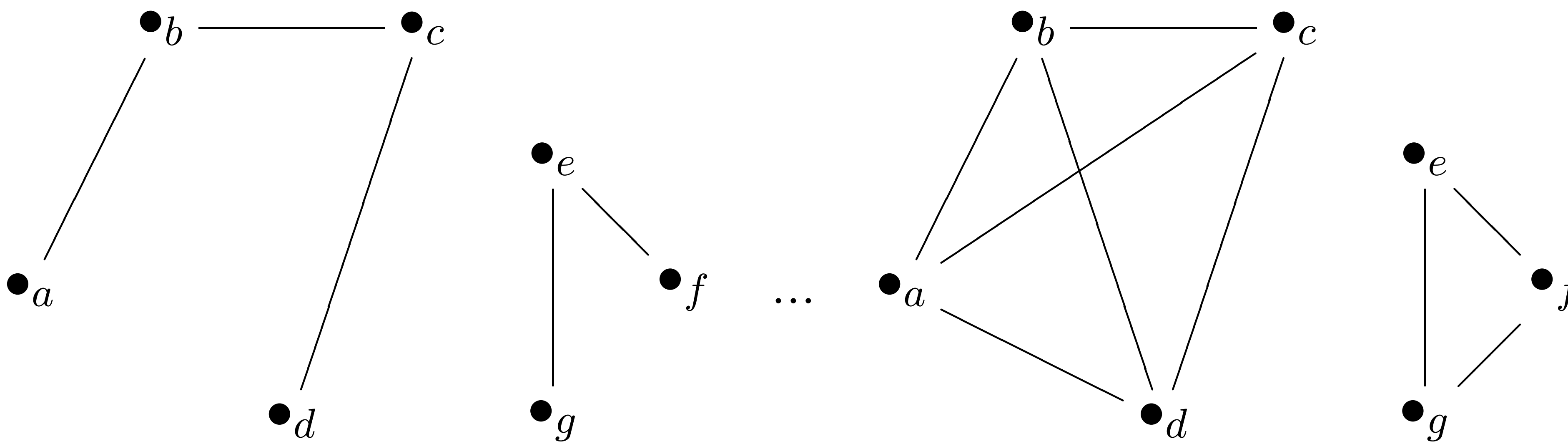
- By calculating the T-transitive closure of a reflexive and symmetric fuzzy relation (a proximity or tolerance relation).
- By using the Representation Theorem.
- By calculating a decomposable operator from a fuzzy subset.
- By obtaining a transitive opening of a proximity relation.

Transitive Closure

Given a t-norm T, the transitive closure of a reflexive and symmetric fuzzy relation R on a set X is the smallest T-indistinguishability operator relation $\overline{R}$ on X greater than or equal to R.

The Crisp Case

In the crisp case, if R is represented by a graph, its transitive closure is the smallest graph that contains R and with all its connected components complete subgraphs. This produce the well known chain effect or chaining.



$\sup -T$ Product

Definition. *Let R and S be two fuzzy relations on X and T a t-norm. The $\sup -T$ product of R and S is the fuzzy relation $R \circ S$ on X defined for all $x, y \in X$ by*

$$(R \circ S)(x, y) = \sup_{z \in X} T(R(x, z), S(z, y)).$$

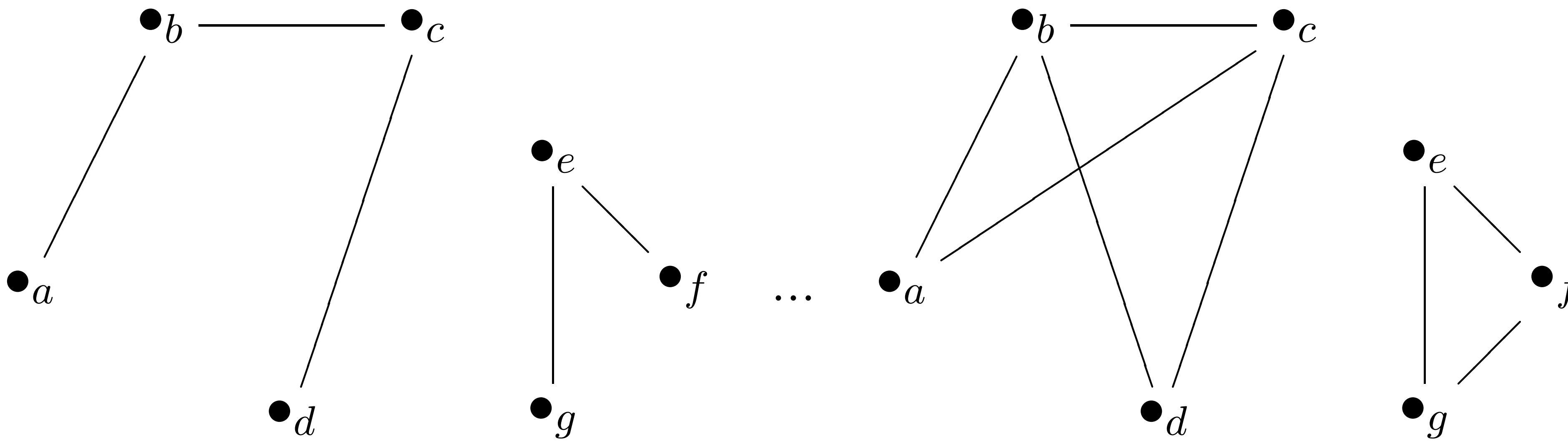
The n^{th} power R^n of a fuzzy relation R is

$$R^n = \overbrace{R \circ \ldots \circ R}^{n\ times}.$$

The crisp case:

$$xR^2y \text{ if and only if } \exists z \text{ such that } xRz \text{ and } zRy.$$

Graphs of R and R^2



Transitive Closure and $\sup -T$ Product

Theorem. *Let R be a reflexive and symmetric fuzzy relation on a set X and T a continuous t-norm. Then the fuzzy relation $\sup_{n\in\mathbb{N}} R^n$ on X is the T-transitive closure of R.*

Proposition. *The transitive closure of a reflexive and symmetric fuzzy relation R is the intersection of all T-indistinguishability operators greater than or equal to R.*

The Representation Theorem

Definition. *The residuation* $\overrightarrow{T}$ *of a t-norm* T *is the map* $\overrightarrow{T}: [0,1] \times [0,1] \to [0,1]$ *defined for all* $x, y \in [0,1]$ *by*

$$\overrightarrow{T}(x|y) = \sup\{\alpha \in [0,1] \mid T(x,\alpha) \leq y\}.$$

Definition. *The biresiduation* $\overleftrightarrow{T}$ *of a t-norm* T *is the map* $\overleftrightarrow{T}: [0,1] \times [0,1] \to [0,1]$ *defined for all* $x, y \in [0,1]$ *by*

$$\overleftrightarrow{T}(x,y) = T(\overrightarrow{T}(x|y), \overrightarrow{T}(y|x)) = \min(\overrightarrow{T}(x|y), \overrightarrow{T}(y|x)).$$

The biresiduation is also known as the natural T-indistinguishability operator associated to T and is also notated by E_T.

- If T is a continuous Archimedean t-norm with additive generator t, then
  $E_T(x,y) = t^{-1}(|t(x) - t(y)|)$ for all $x, y \in [0,1]$.
  As special cases,
  - If T is the Łukasiewicz t-norm, then
    $E_T(x,y) = \overleftrightarrow{T}(x,y) = 1 - |x-y|$ for all $x, y \in [0,1]$.
  - If T is the Product t-norm, then
    $E_T(x,y) = \overleftrightarrow{T}(x,y) = \min(\frac{x}{y}, \frac{y}{x})$ for all $x, y \in [0,1]$ where $\frac{z}{0} = 1$.
- If T is the minimum t-norm, then

$$E_T(x,y) = \overleftrightarrow{T}(x,y) = \begin{cases} \min(x,y) & \text{if } x \neq y \\ 1 & \text{otherwise.} \end{cases}$$

E_μ

Proposition. *Let μ be a fuzzy subset of X and T a continuous t-norm. The fuzzy relation E_μ on X defined for all $x, y \in X$ by*

$$E_\mu(x, y) = E_T(\mu(x), \mu(y))$$

is a T-indistinguishability operator.

In the crisp case, when $\mu = A$ is a crisp subset of X, E_A generates a partition of X into A and its complementary set $X - A$, since in this case $E_A(x, y) = 1$ if and only if x and y both belong to A or to $X - A$.

Lemma. *Let* $(E_i)_{i\in I}$ *be a family of* T*-indistinguishability operators on a set* X*. The relation* E *on* X *defined for all* $x, y \in X$ *by*

$$E(x, y) = \inf_{i\in I} E_i(x, y)$$

is a T*-indistinguishability operator.*

Representation Theorem

Theorem. *Representation Theorem. Let R be a fuzzy relation on a set X and T a continuous t-norm. R is a T-indistinguishability operator if and only if there exists a family $(\mu_i)_{i\in I}$ of fuzzy subsets of X such that for all $x, y \in X$*

$$R(x,y) = \inf_{i\in I} E_{\mu_i}(x,y).$$

$(\mu_i)_{i\in I}$ is called a generating family of R. A fuzzy subset belonging to a generating family of R is called a generator of R. A generating family of R with minimal cardinality is called a basis of E and the cardinality of the corresponding set of indexes its dimension.

Lemma. *μ is a generator of E if and only if $E_\mu \geq E$.*

Generalization to T-transitive Fuzzy Relations

Theorem. *Let R be a fuzzy relation on a set X and T a continuous t-norm. R is T-transitive if and only if there exist two families $(\mu_i)_{i\in I}$ and $(\nu_i)_{i\in I}$ of fuzzy subsets of X with $\mu_i \geq \nu_i \ \forall i \in I$ such that for all $x, y \in X$*

$$R(x,y) = \inf_{i\in I} \overrightarrow{T}(\mu_i(x)|\nu_i(y)).$$

Decomposable Indistinguishability Operators

Definition. *Let T be a t-norm. The decomposable T-indistinguishability operator E^{μ} generated by a fuzzy subset μ of X is defined for all $x, y \in X$ by*

$$E^{\mu}(x,y) = \begin{cases} T(\mu(x), \mu(y)) & \text{if } x \neq y \\ 1 & \text{otherwise.} \end{cases}$$

Tetrahedric relation

Proposition. *Let T be a continuous Archimedean t-norm with additive generator t and μ a fuzzy subset of X. If the decomposable T-indistinguishability operator E^μ on X generated by μ satisfies $E^\mu(x,y) \neq 0$ for all $x, y \in X$, then it generates the following tetrahedric relation on X: Given four different elements $x, y, z, t \in X$,*

$$T(E^\mu(x,y), E^\mu(z,t)) = T(E^\mu(x,z), E^\mu(y,t)).$$

Transitive Openings

Definition. *Let R be a proximity relation on a set X and T a t-norm. A T-indistinguishability operator $\underline{R}$ on X is a T-transitive opening of R if and only if*

- $\underline{R} \leq R$
- *If E is another T-indistinguishability operator on X satisfying $E \leq R$, then $E \leq \underline{R}$.*

Complete linkage

In the complete linkage, the entries of a proximity relation $R = (a_{ij})$ on a finite set X are modified according to the next algorithm to obtain a $\min$-transitive opening. Given two disjoint subsets C_i C_j of X its similarity degree S is defined by $S(C_i, C_j) = \min_{i \in C_i, j \in C_j}(a_{ij})$.

1. Initially a cluster C_i is assigned to every element x_i of X (i.e. the clusters of the first partition are singletons).

2. In each new step two clusters are merged in the following way. If $\{C_1, C_2, ..., C_k\}$ is the actual partition, then we must select the two clusters C_i and C_j for which the similarity degree $S(C_i, C_j)$ is maximal. (If there are several such maximal pairs, one pair is picked at random). The new cluster $C_i \cup C_j$ replaces the two clusters C_i and C_j, and all entries of a_{mn} and a_{nm} of R with $m \in C_i$ and $n \in C_j$ are lowered to $S(C_i, C_j)$.

3. Step 2 is repeated until there remains one single cluster containing all the elements of X.

Example. *Let us consider the proximity R on $X = \{x_1, x_2, x_3, x_4\}$ with matrix*

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0.1 & 0.7 & 0.4 \\ 0.1 & 1 & 0.4 & 0.3 \\ 0.7 & 0.4 & 1 & 0.5 \\ 0.4 & 0.3 & 0.5 & 1 \end{pmatrix} \end{array}.$$

The first partition is $C_1 = \{x_1\}$, $C_2 = \{x_2\}$, $C_3 = \{x_3\}$, $C_4 = \{x_4\}$. The greatest similarity degree between clusters is $S(C_1, C_3) = 0.7$. These two clusters are merged to form $C_{13} = \{x_1, x_3\}$. The matrix does not change in this step.
The new partition is C_{13}, C_2, C_4. The similarity degrees are

$$\begin{aligned} S(C_{13}, C_2) &= \min(a_{12}, a_{32}) = \min(0.1, 0.4) = 0.1 \\ S(C_{13}, C_4) &= \min(a_{14}, a_{34}) = \min(0.4, 0.5) = 0.4 \\ S(C_2, C_4) &= a_{24} = 0.3. \end{aligned}$$

The greatest similarity degree is 0.4 and the new partition is therefore $C_{134} = \{x_1, x_3, x_4\}$, $C_2 = \{x_2\}$. The entries a_{14}, a_{41}, a_{34}, a_{43} of the matrix R are replaced by 0.4 obtaining

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0.1 & 0.7 & 0.4 \\ 0.1 & 1 & 0.4 & 0.3 \\ 0.7 & 0.4 & 1 & 0.4 \\ 0.4 & 0.3 & 0.4 & 1 \end{pmatrix} \end{array}.$$

In the last step, we merge the two clusters C_{134}, C_2. The similarity degree is

$$S(C_{134}, C_2) = \min(a_{12}, a_{32}, a_{42}) = \min(0.1, 0.4, 0.3) = 0.1.$$

The transitive opening of R obtained by complete linkage is then

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0.1 & 0.7 & 0.4 \\ 0.1 & 1 & 0.1 & 0.1 \\ 0.7 & 0.1 & 1 & 0.4 \\ 0.4 & 0.1 & 0.4 & 1 \end{pmatrix} \end{array}.$$

The Archimedean case

Proposition. *Let R be a proximity relation on a finite set $X = \{r_1, r_2, ..., r_s\}$ of cardinality s and T a t-norm. S is a T-indistinguishability operator smaller than or equal to R if and only if its entries satisfy the following system of inequalities:*

$$
\begin{aligned}
0 \leq S(r_i, r_j) &\leq R(r_i, r_j) \\
&\text{for all } i, j = 1, 2, ..., s. \\
T(S(r_i, r_j), S(r_j, r_k)) &\leq S(r_i, r_k) \\
&\text{for all } i, j, k = 1, 2, ..., s. \\
S(r_i, r_j) &= S(r_j, r_i) \\
&\text{for all } i, j = 1, 2, ..., s.
\end{aligned}
$$

Example. *Let us consider the reflexive and symmetric fuzzy relation* $R = \begin{pmatrix} 1 & \frac{2}{3} & 0 \\ \frac{2}{3} & 1 & \frac{2}{3} \\ 0 & \frac{2}{3} & 1 \end{pmatrix}$ *on* $X = \{a, b, c\}$*. A fuzzy relation* S *on* X *with matrix* $S = \begin{pmatrix} 1 & p & q \\ p & 1 & r \\ q & r & 1 \end{pmatrix}$ *is an Ł-indistinguishability operator smaller than or equal to* R *if and only if*

$$
\begin{array}{ll}
0 \leq p \leq \frac{2}{3} & Ł(q,p) \leq r \\
0 \leq q \leq 0 & Ł(q,r) \leq p \\
0 \leq r \leq \frac{2}{3} & Ł(r,p) \leq q \\
Ł(p,q) \leq r & Ł(r,q) \leq p \\
Ł(p,r) \leq q. &
\end{array}
$$

There are 8 possible solutions:

$$p = 0, \frac{1}{3}, \quad q = 0, \quad r = 0, \frac{1}{3}, \frac{2}{3}$$
$$p = \frac{2}{3}, \quad q = 0, \quad r = 0, \frac{1}{3}.$$

Among them, there are 2 Ł-transitive openings of R. Namely

$$\begin{pmatrix} 1 & \frac{1}{3} & 0 \\ \frac{1}{3} & 1 & \frac{2}{3} \\ 0 & \frac{2}{3} & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \frac{2}{3} & 0 \\ \frac{2}{3} & 1 & \frac{1}{3} \\ 0 & \frac{1}{3} & 1 \end{pmatrix}.$$

Granularity and Extensional Sets

- Extensional fuzzy subsets
- Upper approximations
- Lower approximations
- Fuzzy Points

Granularity

According to *L.A. Zadeh*, granularity is one of the basic concepts that underlie human cognition and the elements within a granule '*have to be dealt with as a whole rather than individually*'.

> *Informally, granulation of an object A results in a collection of granules of A, with a granule being a clump of objects (or points) which are drawn together by indistinguishability, similarity, proximity or functionality.*
>
> *L.A. Zadeh*

Extensional Fuzzy Subsets

Definition. *Let E be a T-indistinguishability operator on a set X. A fuzzy subset μ of X is extensional with respect to E (or simply extensional) if and only if for all $x, y \in X$*

$$T(E(x,y), \mu(y)) \leq \mu(x).$$

H_E will be the set of extensional fuzzy subsets of X with respect to E.

This definition fuzzifies the predicate

If x and y are equivalent and $y \in \mu$, then $x \in \mu$.

Proposition. *Let E be a T-indistinguishability operator on X, μ a fuzzy subset of X and E_μ the T-indistinguishability operator generated by μ. $\mu \in H_E$ if and only if $E_\mu \geq E$.*

Hence H_E coincides with the set of generators of E.

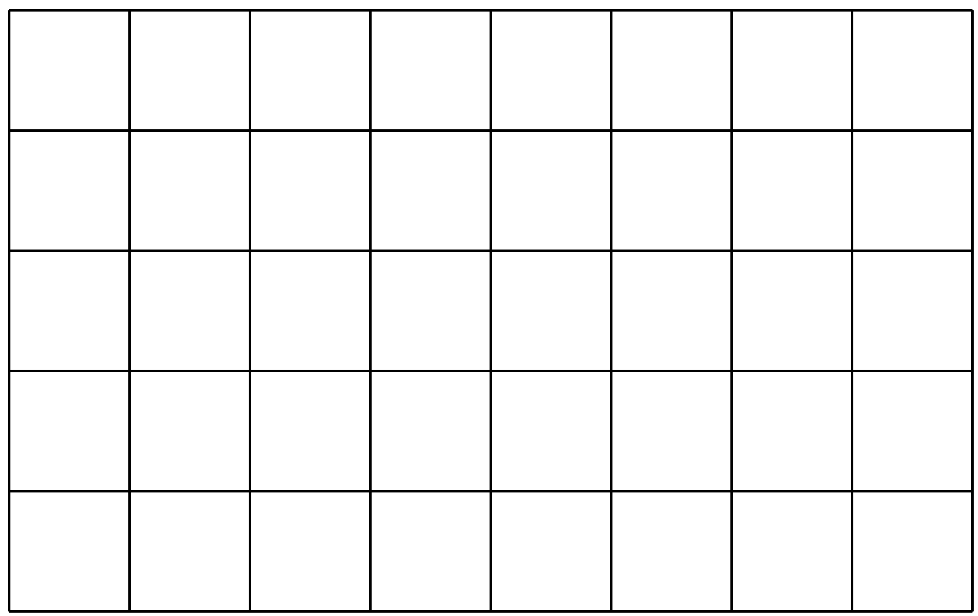
Lemma. *Given a T-indistinguishability operator E on a set X and an element $x \in X$, the column $\mu_x = E(x, \cdot)$ of x is extensional.*

Proposition. *Let E be a T-indistinguishability operator on a set X. The following properties are satisfied for all $\mu \in H_E$, $(\mu_i)_{i\in I}$ a family of extensional fuzzy subsets and $\alpha \in [0,1]$.*

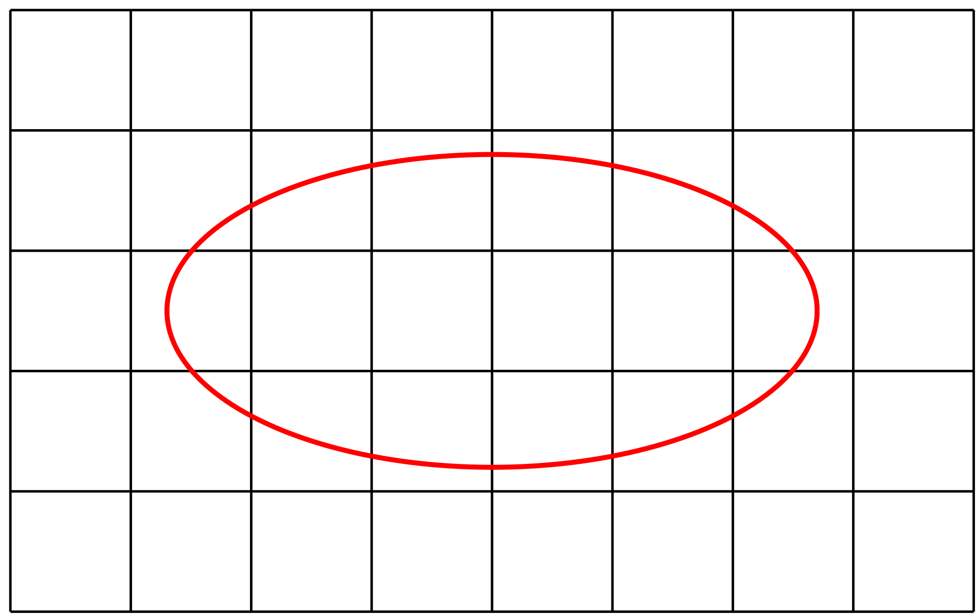
1. $\bigvee_{i\in I} \mu_i \in H_E$.
2. $\bigwedge_{i\in I} \mu_i \in H_E$.
3. $T(\alpha, \mu) \in H_E$.
4. $\overrightarrow{T}(\mu|\alpha) \in H_E$.
5. $\overrightarrow{T}(\alpha|\mu) \in H_E$.

Theorem. *Let H be a subset of $[0,1]^X$ satisfying the properties of the last proposition. Then there exists a T-indistinguishability operator E on X such that $H = H_E$. E is uniquely determined and it is generated (using the Representation Theorem) by the family of elements of H.*

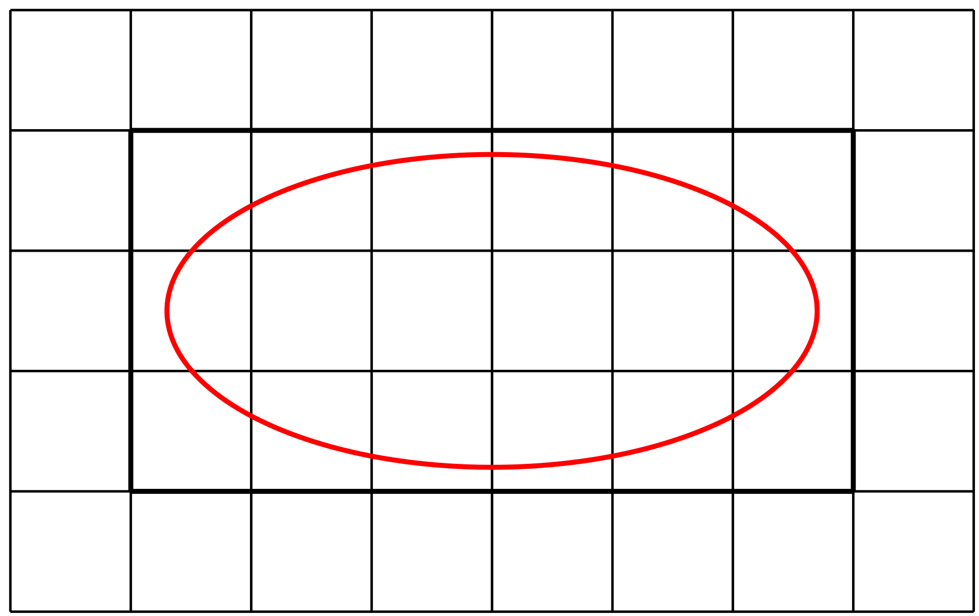
Upper and Lower Approximations



Upper and Lower Approximations



Upper and Lower Approximations



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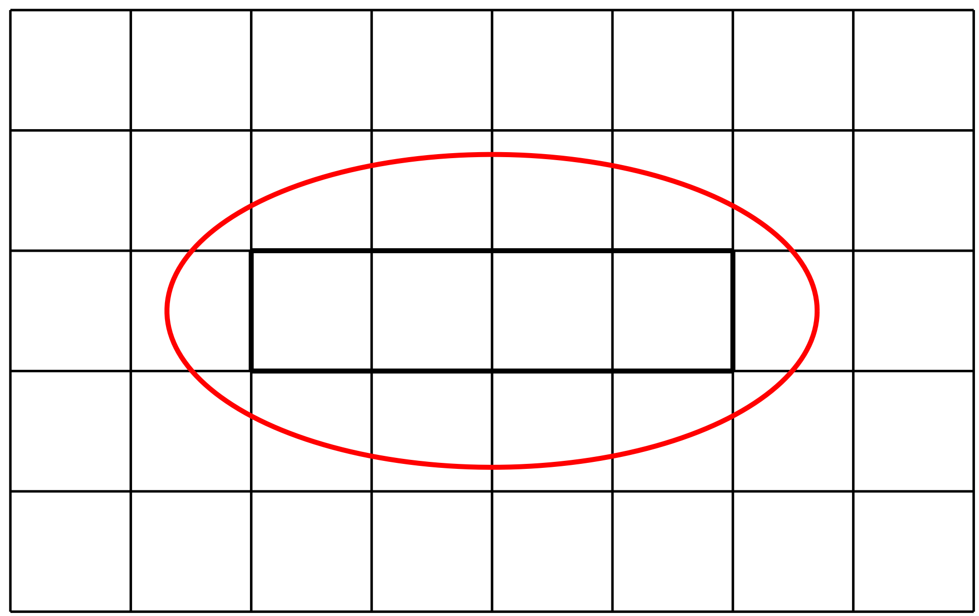


The Map ϕ_E

Definition. *Let* E *be a T-indistinguishability operator on a set* X*. The map* $\phi_E : [0,1]^X \to [0,1]^X$ *is defined for all* $x \in X$ *by*

$$\phi_E(\mu)(x) = \sup_{y \in X} T\left(E(x,y), \mu(y)\right).$$

Proposition. *For all* $\mu, \mu' \in [0,1]^X$,

1. *If* $\mu \leq \mu'$ *then* $\phi_E(\mu) \leq \phi_E(\mu')$.
2. $\mu \leq \phi_E(\mu)$.
3. $\phi_E(\bigvee_{i\in I} \mu_i) = \bigvee_{i\in I} \phi_E(\mu_i)$.
4. $\phi_E(\phi_E(\mu)) = \phi_E(\mu)$.
5. $\phi_E(\{x\})(y) = \phi_E(\{y\})(x)$
6. $\phi_E(T(\alpha,\mu)) = T(\alpha, \phi_E(\mu))$.

Theorem. *Let* $\phi : [0,1]^X \to [0,1]^X$ *be a map satisfying the properties of the last proposition. The fuzzy relation* E_ϕ *on* X *defined for all* $x, y \in X$ *by*

$$E_\phi(x,y) = \phi(\{x\})(y)$$

is a T*-indistinguishability operator on* X.

Proposition. *$\mu \in H_E$ if and only if $\phi_E(\mu) = \mu$.*

Hence, H_E is characterized as the set of fixed points of ϕ_E.

Proposition. Im$(\phi_E) = H_E$.

Proposition. *For any $\mu \in [0,1]^X$, $\phi_E(\mu) = \inf_{\mu' \in H_E}\{\mu \leq \mu'\}$.*

So, $\phi_E(\mu)$ is the most specific extensional set that contains μ (i.e. $\mu \leq \phi_E(\mu)$) and in this sense it is the optimal upper approximation of μ in H_E.

The Map ψ_E

Definition. *Let* E *be a T-indistinguishability operator on a set* X. *The map* $\psi_E : [0,1]^X \to [0,1]^X$ *is defined by*

$$\psi_E(\mu)(x) = \inf_{y \in X} \overrightarrow{T}\left(E(x,y) | \mu(y)\right) \ \forall x \in X.$$

Proposition. *For all $\mu, \mu' \in [0,1]^X$, we have:*

1. $\mu \leq \mu' \Rightarrow \psi_E(\mu) \leq \psi_E(\mu')$.
2. $\psi_E(\mu) \leq \mu$.
3. $\psi_E(\bigwedge_{i\in I} \mu_i) = \bigwedge_{i \in I} \psi_E(\mu_i)$.
4. $\psi_E(\psi_E(\mu)) = \psi_E(\mu)$.
5. $\psi_E(\overrightarrow{T}(\{x\}|\alpha))(y) = \psi_E(\overrightarrow{T}(\{y\}|\alpha))(x)$.
6. $\psi_E(\overrightarrow{T}(\alpha|\mu)) = \overrightarrow{T}(\alpha|\psi_E(\mu))$.

Theorem. *Let $\psi : [0,1]^X \to [0,1]^X$ be a map satisfying the properties of the last proposition. The fuzzy relation E_ψ on X defined for all $x, y \in X$ by*

$$E_\psi(x,y) = \inf_{\alpha \in [0,1]} \overrightarrow{T}(\psi(\overrightarrow{T}(\{x\}|\alpha)(y)|\alpha)).$$

is a T-indistinguishability operator on X.

Proposition. *$\mu \in H_E$ if and only if $\psi_E(\mu) = \mu$.*

Hence, H_E is also characterized as the set of fixed points of ψ_E.

Proposition. Im$(\psi_E) = H_E$.

Proposition. *For any $\mu \in [0,1]^X$, $\psi_E(\mu) = \sup_{\mu' \in H_E}\{\mu' \leq \mu\}$.*

So, $\psi_E(\mu)$ is the greatest extensional set contained in μ (i.e. $\mu \geq \phi_E(\mu)$) and in this sense it is the optimal lower approximation of μ in H_E.

Fuzzy Points

Definition. *Let E be a T-indistinguishability operator on a set X. $\mu \in H_E$ is a fuzzy point of X with respect to E if and only if*

$$T\left(\mu(x_1), \mu(x_2)\right) \leq E(x_1, x_2)\ \forall x_1, x_2 \in X.$$

P_X *will denote the set of fuzzy points of X with respect to E.*

The Map Λ_E

Definition. *Let E be a T-indistinguishability operator on a set X. The map $\Lambda_E : [0,1]^X \to [0,1]^X$ is defined by*

$$\Lambda_E(\mu)(x) = \inf_{y \in X} \overrightarrow{T}\,(\mu(y)|E(y,x)) \;\; \forall x \in X.$$

Proposition. *Let μ be a normal fuzzy subset of X (i.e. $\exists x_0 \in X$ such that $\mu(x_0) = 1$) $\Lambda_E(\mu) = \mu$ if and only if μ is a column μ_x of E.*

Proposition. *Let E be a T-indistinguishability operator on X and $\mu \in H_E$. $\Lambda_E(\mu) \geq \mu$ if and only if $\mu \in P_X$.*

Theorem. *Let E be a T-indistinguishability operator on X.* Fix(Λ_E) *is the set of all fuzzy points $\mu \in P_X$ which are maximal in P_X.*

Fuzzy Points and the Representation Theorem

Proposition. *Let $(\mu_i)_{i\in I}$ be a family of fuzzy subsets of X and E the T-indistinguishability operator generated by this family ($E(x,y) = \inf_{i\in I} E_{\mu_i}(x,y)$). Then E is the greatest T-indistinguishability operator for which all the fuzzy subsets of the family are extensional.*

Proposition. *Let $(\mu_i)_{i\in I}$ be a family of normal fuzzy subsets of X and $(x_i)_{i\in I}$ a family of elements of X such that $\mu_i(x_i) = 1$ for all $i \in I$. Then the following two properties are equivalent.*

a) *There exists a T-indistinguishability operator E on X such that*

$$\mu_i(x) = E(x, x_i)\ \forall i \in I\ \forall x \in X.$$

b) *For all $i, j \in I$,*

$$\sup_{x\in X} T(\mu_i(x), \mu_j(x)) \leq \inf_{y\in X} E_T(\mu_i(y), \mu_j(y)).$$

Proposition. *Let* $(\mu_i)_{i\in I}$ *be a family of normal fuzzy subsets of* X *and* $(x_i)_{i\in I}$ *a family of elements of* X *such that* $\mu_i(x_i)=1$ *for all* $i\in I$ *satisfying*

$$\sup_{x\in X} T(\mu_i(x),\mu_j(x)) \le \inf_{y\in X} E_T(\mu_i(y),\mu_j(y)).$$

for all $i,j\in I$*. Then the* T*-indistinguishability operator* $E=\sup_{i\in I} E^{\mu_i}$ *is the smallest* T*-indistinguishability operator on* X *satisfying*

$$\mu_i(x)=E(x,x_i)\ \forall i\in I,\ \forall x\in X.$$

Indistinguishability Operators Between Fuzzy Subsets.

Definition. *The natural T-indistinguishability operator on $[0,1]^X$ is defined for all $\mu, \nu \in [0,1]^X$ by*

$$E_T(\mu, \nu) = \inf_{x \in X} E_T(\mu(x), \nu(x)).$$

$\min$-indistinguishability Operators

Definition. *A map* $m : X \times X \to \mathbb{R}$ *is a pseudo ultrametric if and only if for all* $x, y, z \in X$

1. $m(x,x) = 0$.
2. $m(x,y) = m(y,x)$.
3. $\max(m(x,y), m(y,z)) \geq m(x,z)$.

If $m(x,y) = 0$ *implies* $x = y$*, then it is called an ultrametric.*

Balls of Ultrametrics

Proposition. *Let m be an ultrametric on X. Then*

1. *If $B(x,r)$ denotes the ball of centre x and radius r and $y \in B(x,r)$, then $B(x,r) = B(y,r)$. (All elements of a ball are its centre).*
2. *If two balls have non-empty intersection, then one of them is contained in the other one.*

Proposition. *Let E be a fuzzy relation on a set X. E is a $\min$-indistinguishability operator on X if and only if $m = 1 - E$ is a pseudo ultrametric.*

Corollary. *The cardinality of Im$(E) = \{E(x,y)\}$ is smaller than or equal to the cardinality of X. In particular, if X is finite of cardinality n and E is identified with a matrix, then the number of different entries of the matrix is less or equal than n.*

Definition. *Let E be a fuzzy relation on X and $\alpha \in [0,1]$, the α-cut of E is the set E_α of pairs $(x,y) \in X \times X$ such that $E(x,y) \geq \alpha$.*

Proposition. *Let E be a fuzzy relation on X. E is a $\min$-indistinguishability operator on X if and only if for each $\alpha \in [0,1]$, the α-cut of E is an equivalence relation on X.*

$\min$-indistinguishability Operators and Hierarchical Trees

Definition. *A hierarchical tree of a finite set X is a sequence of partitions $A_1, A_2, ..., A_k$ of X such that every partition refines the preceding one.*

A hierarchical tree is indexed if every partition A_i has associated a non-negative number λ_i and $\lambda_i < \lambda_{i+1}$ for all $i = 1, 2, ..., k-1$.

Proposition. *Every $\min$-indistinguishability operator E on a finite set X generates an indexed hierarchical tree on X.*

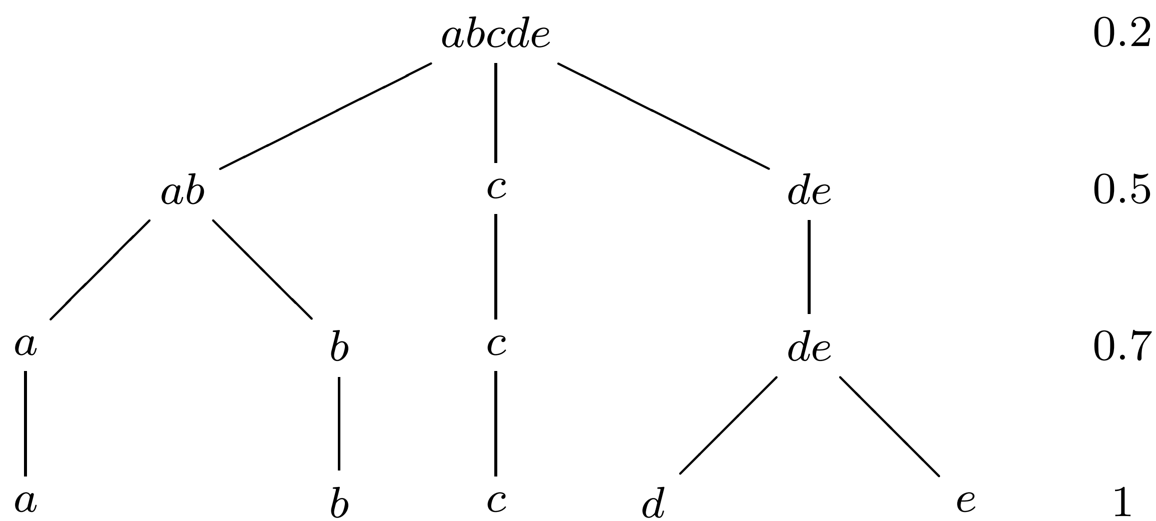
Reciprocally,

Proposition. *Every indexed hierarchical tree $A_1, A_2, ..., A_k$ of a finite set X with $\lambda_k = 1$ generates a $\min$-indistinguishability operator E on X.*

Example. *Let* $X = \{a, b, c, d, e\}$ *and* E *the* $\min$*-indistinguishability operator with matrix*

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 0.5 & 0.2 & 0.2 & 0.2 \\ 0.5 & 1 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 1 & 0.7 \\ 0.2 & 0.2 & 0.2 & 0.7 & 1 \end{pmatrix} \end{array}.$$

The corresponding tree is



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