Chapter 5

Mereology

Mereology emerged in the beginning of XXth century due to independent efforts of S. Leśniewski and A. N. Whitehead. In the scheme of Leśniewski, the predicate of being a part was taken as the primitive notion whereas in the development of Whitehead's ideas the primitive notion was adopted as the predicate of being connected. Mereology presents an alternative, holistic, approach to concepts which is especially suited to reasoning about extensional objects, e.g., spacial ones as witnessed, e.g., by the Tarski axiomatization of geometry of solids, or recent applications to geometric information systems or analysis of statements about spatial objects and relations in natural language.

0.1 Mereology: The Theory of Leśniewski

Mereology due to Leśniewski arose from attempts at reconciling antinomies of naïve set theory, see Leśniewski [42], Sobociński [58]. Leśniewski [41] was the first presentation of foundations of this theory, see also Leśniewski [44], [45]; cf., Lejewski [38] and Sobociński [59].

The primitive notion of mereology due to Leśniewski is a notion of a *part*. Given some objects, a relation of a part is a binary relation π which is required to be

M1 *Irreflexive*: *For each* $x \in U$ *it is not true that* $\pi(x, x)$

M2 *Transitive*: *For each triple* x, y, z *of objects in* U, *if* $\pi(x, y)$ *and* $\pi(y, z)$, *then* $\pi(x, z)$

Remark. In the original scheme of Leśniewski, the relation of parts is applied to *individual objects* as defined in Ontology of Leśniewski, see Leśniewski [43], Iwanuś [30], Słupecki [54]. Ontology is founded on the predicate ϵ to be read "is" (in Greek, *ei* "you are", cf., Plutarch [48]) which is required to satisfy the Ontology Axiom AO, formulated by Leśniewski as early as 1920, see Słupecki [54], Lejewski [39].

AO $x\epsilon\, y \Leftrightarrow \exists z.(z\epsilon\, x) \wedge \forall\, z.(z\epsilon\, x \Rightarrow z\epsilon\, y) \wedge \forall\, z, w.(z\epsilon\, x \wedge w\epsilon\, x \Rightarrow z\epsilon\, w)$

This axiom determines the meaning of the copula ϵ in the way adopted by Leśniewski: in spite of the copula occurring on either side of the equivalence, its meaning can be revealed by requiring the equivalence to be true. The three terms occurring on the right–hand side of the equivalence mean, respectively,

A $\exists z.(z\epsilon\ x)$ *means that the name* x *is not empty and that some object responds to that name*

B $\forall\ z, w.(z\epsilon\ x \wedge w\epsilon\ x \Rightarrow z\epsilon\ w)$: *letting* $z = w$ *if and only if* $z\epsilon\ w \wedge w\epsilon\ z$, *we infer that the term B means that* $z = w$ *for each pair* z, w *of objects responding to the name of* x. *Thus*, x *is a singular name*

C *The term* $\forall\ z.(z\epsilon\ x \Rightarrow\ z\epsilon\ y)$ *means that each object responding to the name of* x *responds as well to the name of* y: *in a sense* x *is contained in* y

From axiom AO, the following properties of the copula ϵ follow, cf., Słupecki [54]

1. $x\epsilon\ y \wedge z\epsilon x \Rightarrow z\epsilon\ y$;

2. $x\epsilon y \wedge z\epsilon x \Rightarrow x\epsilon\ z$;

3. $x\epsilon\ y \Rightarrow\ x\epsilon\ x$.

The phrase 'x is an object' is rendered as $x\epsilon\ V$ equivalent to $\exists y.x\epsilon\ y$. Then, the equivalence

4. $x\epsilon\ V \Leftrightarrow x\epsilon\ x$

does express the fact that x is an object.

In Mereology, the predicate *part* is applied to objects, called by singular names.

The relation of *part* induces the relation of an *ingredient*, *ingr*, defined as

$$ingr(x, y) \Leftrightarrow\ \pi(x, y)\ \vee x = y \tag{0.1}$$

Clearly,

Proposition 1. *The relation of ingredient is a partial order on objects, i.e.,*

1. $ingr(x, x)$*;*

2. $ingr(x, y) \wedge ingr(y, x) \Rightarrow (x = y)$*;*

3. $ingr(x, y) \wedge ingr(y, z) \Rightarrow ingr(x, z)$.

We formulate the third axiom with a help from the notion of an ingredient.

M3 (*Inference*) *For objects* x, y*, the property*

$I(x, y)$: *For each object* t*, if* $ingr(t, x)$*, then there exist objects* w, z *such that* $ingr(w, t), ingr(w, z), ingr(z, y)$

implies $ingr(x, y)$.

The predicate of *overlap*, Ov in symbols, is defined by means of

$$Ov(x, y) \Leftrightarrow \exists z.ingr(z, x) \wedge ingr(z, y) \tag{0.2}$$

Using the overlap predicate, one can write $I(x, y)$ down in the form

$I_{Ov}(x, y)$: *For each* t *with* $ingr(t, x)$*, there exists* z *such that* $ingr(z, y)$ *and* $Ov(t, z)$

The notion of a mereological class follows; for a non–vacuous property Φ of objects, the *class of* Φ, denoted $Cls\Phi$ is defined by the conditions

C1 *If* $\Phi(x)$*, then* $ingr(x, Cls\Phi)$

C2 *If* $ingr(x, Cls\Phi)$*, then there exists* z *such that* $\Phi(z)$ *and* $I_{Ov}(x, z)$

In plain language, the class of Φ collects in an individual object all objects satisfying the property Φ.

The existence of classes is guaranteed by an axiom.

M4 *For each non–vacuous property* Φ *there exists a class* $Cls\Phi$

The uniqueness of the class follows.

Proposition 2. *For each non–vacuous property* Φ*, the class* $Cls\Phi$ *is unique.*

Proof. Assuming that for some Φ there exist two distinct classes Y_1, Y_2, consider $ingr(t, Y_1)$. Then, by C2, and (0.2), there exists z such that $Ov(t, z)$ and $ingr(z, Y_2)$. It follows by M3 that $ingr(Y_1, Y_2)$. By symmetry, $ingr(Y_2, Y_1)$ holds and Proposition 1(2) implies that $Y_1 = Y_2$ □

Proposition 3. *For the non–vacuous property* Φ*, if for each object* z *such that* $\Phi(z)$ *it holds that* $ingr(z, x)$*, then* $ingr(Cls\Phi, x)$.

Proof. It follows directly from M3 □

The notion of an overlap allows for a succinct characterization of a class.

Proposition 4. *For each non–vacuous property Φ and each object x, it happens that $ingr(x, Cls\Phi)$ if and only if for each ingredient w of x, there exists an object z such that $Ov(w,z)$ and $\Phi(z)$.*

Remark. Proposition 2 along with existence of a class is an axiom in the Leśniewski [42] scheme, from which M3 is derived. Similarly, it is an axiom in the Tarski [60], [62] scheme.

Example 1. 1. The strict inclusion $\subset$ on sets is a part relation. The corresponding ingredient relation is the inclusion $\subseteq$. The overlap relation is the non–empty intersection. For a non–vacuous family F of sets, the class $ClsF$ is the union $\bigcup F$;

2. For reals in the interval $[0,1]$, the strict order $<$ is a part relation and the corresponding ingredient relation is the weak order $\leq$. Any two reals overlap; for a set $F \subseteq [0,1]$, the class of F is $supF$.

The notion of an element, Leśniewski [42], par. 6, Def. IV, $el(x,y)$ in symbols, is defined as follows

$$el(x,y) \Leftrightarrow \exists\Phi . y = Cls\Phi \ \wedge \ \Phi(x) \tag{0.3}$$

In plain words, $el(x,y)$ means that y is a class of some property and x responds to that property.To establish some properties of the notion of an element, we begin with

Proposition 5. *For each object x, and the property $INGR(x) = \{y : ingr(y,x)\}$, the identity $x = ClsINGR(x)$ holds.*

Proof. By Proposition 1(1), $INGR(x)$ is non–vacuous and

$$ingr(x, ClsINGR(x))$$

That $ingr(ClsINGR(x), x)$ follows by M3 $\square$

Proposition 6. *For each pair x,y of objects, $el(x,y)$ holds if and only if $ingr(x,y)$ holds. Hence, $el = ingr$.*

Proof. By (0.3) and Proposition 5, if $ingr(x,y)$ then $el(x,y)$. The converse follows from the definition of an element (0.3) and class requirement C1 $\square$

Corollary 1 *If $\pi(x,y)$, then $el(x,y)$.*

Corollary 2 *$el(x,x)$ holds for each object x.*

By Corollary 2, every object considered in mereology is non–empty in the sense of the element relation.

Corollary 3 *The property of objects of not being its own element is vacuous.*

Corollary 3 is one of means of expressing the impossibility of the Russell paradox within the mereology, cf., Leśniewski [42], Thms. XXVI, XXVII, see also Sobociński [58].

Extensionality of the overlap relation can be inferred from Proposition 5.

Proposition 7. *For each pair x, y of objects, $x = y$ if and only if for each object z, the equivalence $Ov(z, x) \Leftrightarrow Ov(z, y)$ holds.*

Proof. Assume the equivalence $Ov(z, x) \Leftrightarrow Ov(z, y)$ to hold for each z. By Proposition 5, if $ingr(t, x)$ then $Ov(t, x)$ and $Ov(t, y)$ hence by axiom M3 $ingr(t, y)$ and with $t = x$ we get $ingr(x, y)$. By symmetry, $ingr(y, x)$ and by Proposition 1(2), $x = y$ □

Concerning the class properties, we mention

Proposition 8. *For each pair of non–vacuous properties Φ, Ψ, from $\Phi \Rightarrow \Psi$ it follows that $ingr(Cls\Phi, Cls\Psi)$.*

Proof. The proposition is a direct consequence of the class definition C1, C2 and of M3 □

A corollary follows.

Proposition 9. *For each pair of non–vacuous properties Φ, Ψ, from $\Phi \Leftrightarrow \Psi$ it follows that $Cls\Phi = Cls\Psi$.*

The notion of a subset, $sub(x, y)$ is introduced in mereology, cf., Leśniewski [42], par. 10, Def. V, via the requirement

$$sub(x, y) \Leftrightarrow \forall z.[ingr(z, x) \Rightarrow ingr(z, y)] \tag{0.4}$$

It follows immediately that

Proposition 10. *For each pair x, y of objects, $sub(x, y)$ holds if and only if $el(x, y)$ holds if and only if $ingr(x, y)$ holds.*

For the property $Ind(x) \Leftrightarrow x \epsilon x$, one calls the class $ClsInd$, *the universe*, in symbols V, Leśniewski [42], par. 12, Def. VII.

It follows that

Proposition 11. *The following are properties of the universe V*

1. The universe is unique;

2. $ingr(x, V)$ holds for each object x;

3. For each non–vacuous property Φ, it holds true that $ingr(Cls\Phi, V)$.

The notion of an exterior object x to the object y, $extr(x, y)$ in symbols, Leśniewski [42], par. 13, Def. VIII, is the following

$$extr(x, y) \Leftrightarrow \text{ it is not true that } Ov(x, y) \tag{0.5}$$

In plain words, x is exterior to y when no object is an ingredient both to x and y.

Clearly,

Proposition 12. *The operator of being exterior has properties*

1. No object is exterior to itself;

2. $extr(x, y)$ implies $extr(y, x)$;

3. If for a non–vacuous property Φ, an object x is exterior to every object z such that $\Phi(z)$ holds, then $extr(x, Cls\Phi)$.

The notion of a complement to an object, with respect to another object, is rendered as a ternary predicate $comp(x, y, z)$, Leśniewski [42], par. 14, Def. IX, to be read:'x is the complement to y with respect to z', and it is defined by means of the following requirements

1. $x = ClsEXTR(y, z)$;

2. $ingr(y, z)$,

where $EXTR(y, z)$ is the property which holds for an object t if and only if $ingr(t, z)$ and $extr(t, y)$ hold.

This definition implies that the notion of a complement is valid only when there exists an ingredient of z exterior to y.

Proposition 13. *For each triple x, y, z such that $comp(x, y, z)$,*

1. $extr(x, y)$;

2. $\pi(x, z)$;

3. $comp(x, y, z)$ implies $comp(y, x, z)$.

Proof. Property 1 follows from Proposition 12, 3. Property 2 follows from Proposition 3 which implies $ingr(x, z)$ and by Property 1, x is exterior to y hence distinct from z. To prove Property 3, we need to verify that $y = ClsEXTR(x, z)$. Consider t such that $ingr(t, z)$ and $extr(t, x)$; was $extr(t, y)$, it would be that $ingr(t, x)$, a contradiction. Hence, $Ov(t, y)$ and it follows by arbitrariness of t that $y = ClsEXTR(x, z)$ □

We let for an object x, $-x = ClsEXTR(x, V)$. It follows from Proposition 13 that

Proposition 14. *The operation $-x$ has properties*

1. $-(-x) = x$ for each object x*;*

2. $-V$ does not exist*.*

The operator $-x$ can be a candidate for the boolean complement in a structure of a Boolean algebra within mereology, constructed in Tarski [61], and anticipated in Tarski [60]; cf., in this respect Clay [13].

This algebra will be obviously rid of the null element, as the empty object is not allowed in mereology of Leśniewski, and the meet of two objects will be possible only when these objects overlap. Under this caveat, the construction of Boolean operators of join and meet proceeds on the following lines.

We let $Booladd(x, y)$ to denote a property of objects defined by

$$Booladd(x, y)(t) \Leftrightarrow ingr(t, x) \vee ingr(t, y) \tag{0.6}$$

We denote the class $ClsBooladd(x, y)$ with the symbol $x + y$. As properties $ingr(t, x) \vee ingr(t, y)$ and $ingr(t, y) \vee ingr(t, x)$ are equivalent, we have by Proposition 9 that

Proposition 15. *$x + y = y + x$.*

Similarly, properties $(ingr(t, x) \vee ingr(t, y)) \vee ingr(t, z)$ and $ingr(t, x) \vee (ingr(t, y)) \vee ingr(t, z))$ are equivalent, hence

Proposition 16. *$(x + y) + z = x + (y + z)$.*

Consider now $x+(-x)$: the defining property is $\Phi(t) : ingr(t, x) \vee ingr(t, -x)$. For an arbitrary object z, either exists w such that $Ov(z, w) \wedge ingr(w, x)$ or exists v such that $Ov(z, v) \wedge ingr(v, -x)$, otherwise an easily perceived contradiction takes place, hence, $ingr(z, x+(-x))$ and a fortiori $x+(-x) = V$. We arrive at

Proposition 17. *$x + (-x) = V$.*

The object $x + y$ has the property

Proposition 18. *For each object z, the equivalence holds: $Ov(z, x + y) \Leftrightarrow Ov(z, x) \vee Ov(z, y)$.*

Proof. $Ov(z, x + y)$ means the existence of w such that $ingr(w, z)$ and $ingr(w, x + y)$ hence for some t such that $Ov(w, t)$ one has $ingr(t, x) \vee ingr(t, y)$ and thus for some u with $ingr(u, t), ingr(u, w)$ it follows that $ingr(u, z)$ and $ingr(u, x) \vee ingr(u, y)$ so finally $Ov(z, x) \vee Ov(z, y)$. The converse implication follows easily □

For each pair x, y of overlapping objects, we introduce a property $Boolprod(x, y)$ defined as

$$Boolprod(x, y)(t) \Leftrightarrow ingr(t, x) \wedge ingr(t, y) \tag{0.7}$$

We denote the class $ClsBoolprod(x, y)$ with $x \cdot y$.

As with + operation, one can prove properties of $\cdot$.

Proposition 19. *Under the assumption that* $Ov(x, y), Ov(y, z), Ov(x \cdot y, z)$ *the following properties hold*

1. $x \cdot y = y \cdot x$*;*

2. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$*;*

3. $x \cdot (-x)$ *is not defined.*

The property counterparting 18 reads as follows

Proposition 20. *Assume the existence of non–empty complement. Then, for each object* z *it holds that* $ingr(z, x \cdot y) \Leftrightarrow ingr(z, x) \wedge ingr(z, y)$.

Proof. Assume $ingr(z, x \cdot y)$ so for each v with $ingr(v, z)$, there is w such that $Ov(v, w)$ and $ingr(w, x), ingr(w, y)$; hence there is u such that $ingr(u, v)$, $ingr(u, w)$, $ingr(u, x)$, $ingr(u, y)$ and $ingr(u, x \cdot y)$. However, was, e.g., $\neg\, ingr(z, x)$ we would have t such that $ingr(t, x)$ and $extr(t, x)$, contradicting the case $v = t$ considered in the preceding sentence. Thus, $ingr(z, x)$, $ingr(z, y)$ must hold and the proposition is proved as the converse implication follows directly by the class definition □

Boolean relations hold between $x + y$ and $x \cdot y$, viz.,

Proposition 21. *The following relations between* + *and* $\cdot$ *are valid*

1. $-(x + y) = (-x) \cdot (-y)$*;*

2. $-(x \cdot y) = (-x) + (-y)$*.*

Proof. Defining properties, respectively, for $-(x+y)$ and $(-x)\cdot(-y)$ are $\neg[(ingr(t,x)\vee ingr(t,y)]$ and $(\neg ingr(t,x))\wedge(\neg ingr(t,y))$, respectively, and are equivalent, hence, classes of them are identical. Similarly, the second identity follows □

For each object x, we construct a *set* $\sigma(x)$ by letting

$$\sigma(x)=\{y: ingr(y,x)\} \tag{0.8}$$

The meaning of $\sigma(x)$ is that this set consists of objects which are ingredients of x and it does correspond to class of ingredients of x.

Proposition 22. *Assume the existence of non–vacuous complement for all pairs a,b with $\pi(a,b)$. Objects satisfying $Booladd(x,y)$ make the set $\sigma(x)\cup\sigma(y)$. Then, $x+y$ corresponds to $\sigma(x)\cup\sigma(y)$.*

Proof. Assume $ingr(t,x+y)$; then, for each w, if $ingr(w,t)$, then $Ov(w,z)$, $ingr(z,x)\vee ingr(z,y)$ for some z. Assume that neither $ingr(t,x)$ nor $ingr(t,y)$. By complement assumption, there exist u,v such that $ingr(u,t)$, $ingr(v,t)$, $extr(u,x)$, $extr(v,y)$. The exteriority of u to x and v to y contradicts the preceding sentence □

Objects satisfying $Boolprod(x,y)$ constitute the set $\sigma(x)\cap\sigma(y)$ by Proposition 20. The class $x\cdot y$ corresponds to the set $\sigma(x)\cap\sigma(y)$.

Finally, we consider the set $V\setminus\sigma(x)$ corresponding to the class $Cls\neg Ov(x)$ of the property $\neg Ov(x)$ which we denote $-x$. Then

Proposition 23. *The correspondence $\iota: x\to\sigma(x)$ is an isomorphism satisfying $ingr(u,v)\Leftrightarrow\sigma(u)\subseteq\sigma(v)$. At this correspondence, $x+y$ maps onto $\sigma(x)\cup\sigma(y)$, $x\cdot y$ maps onto $\sigma(x)\cap\sigma(y)$, and $-x$ maps onto $U\setminus\sigma(x)$. The Cls operator acts as the least upper bound operator, mapping a class of a property Φ onto the union $\bigcup\{\sigma(x):\Phi(x)\}$.*

0.2 A modern structural analysis of mereology

An ex post analysis of the structure of Leśniewski's mereological theory has resulted in some stratification of the theory into more or less stronger sub–theories, e.g., in the way proposed in Casati–Varzi [10].

The basic notion is that of ingredient, subject to postulates

P1 $ingr(x,x)$

P2 $ingr(x,y)\wedge ingr(y,x)\Rightarrow(x=y)$

P3 $ingr(x, y) \wedge ingr(y, z) \Rightarrow ingr(x, z)$

The theory supporting P1–P3 is called *Ground Mereology* and denoted M. It does encompass a theory of ingredients and parts as partial and strict orders on a universe of objects. Clearly, parts are defined by means of

$$\pi(x, y) \Leftrightarrow ingr(x, y) \wedge \neg ingr(y, x) \tag{0.9}$$

The relation Overlap is defined as above and the dual predicate of *underlap* is defined by means of

$$U(x, y) \Leftrightarrow \exists z.ingr(x, z) \wedge ingr(y, z) \tag{0.10}$$

Combinations of these predicates for expressing various possible positions of each object relative to the other are possible, like *proper over–crossing* PO

$$PO(x, y) \Leftrightarrow Ov(x, y) \wedge \neg ingr(x, y) \wedge \neg ingr(y, x) \tag{0.11}$$

Further extensions are proposed by means of postulates which would secure the existence of complementary parts within objects , cf., the notion of a complement above; complementation is expressed by means of two postulates, see Casati–Varzi, op. cit., and Simons [52], a throughout discussion in Simons [53], viz., the *Postulate of Weak Supplementation*

P4 $\pi(x, y) \Rightarrow \exists z.ingr(z, y) \wedge \neg Ov(x, z)$

and the *Postulate of Strong Supplementation*

P5 $\neg ingr(y, x) \Rightarrow \exists z.ingr(z, y) \wedge \neg Ov(x, z)$

As from $\pi(x, y)$ it follows that $\neg ingr(y, x)$, P4 follows from P5. Since $\neg ingr(y, x)$ does not imply $\pi(x, y)$, the converse does not hold. The theory M+P5 denoted EM and called the *extensional mereology* is stronger than the theory M+P4 denoted MM and called the *minimal mereology.* In the latter, extensionality with respect to parts holds: If objects x, y have parts, then $\forall z.\pi(z, x) \Leftrightarrow \pi(z, y) \Rightarrow (x = y)$, see Casati–Varzi [10], p. 40.

Further postulates bring closer to fusion properties – the existence of sums (classes in the sense of Leśniewski) and intersections (called also products); a finitary version requires that objects in underlap relation should be encompassed in the smallest object which contains them

P6 $U(x, y) \Rightarrow \exists z.\forall w.[Ov(w, z) \Leftrightarrow Ov(w, x) \vee Ov(w, y)]$

and, dually, overlapping objects contain the greatest object contained in either of them

P7 $Ov(x,y) \Rightarrow \exists z.\forall w.[ingr(w,z) \Leftrightarrow ingr(w,x) \wedge ingr(w,y)]$

The theory M + P6, P7 is called the *closure mereology* denoted CM, MM+ P6,P7 is called the *closure minimal mereology* denoted CMM, and EM +P6, P7 is the *closure extensional mereology* denoted CEM. As M $\subseteq$ MM $\subseteq$ EM, it follows that M $\subseteq$ CM, MM $\subseteq$ CMM, EM $\subseteq$ CEM, CMM $\subseteq$ CEM. Actually, as pointed to in Casati–Varzi [10], p. 44, P4 $\Rightarrow$(P7 $\Rightarrow$ P5), hence CEM=CMM. Finally, GEM, the *general extensional mereology* is the mereology exposed above in sect. 5.1., i.e., the mereology as envisioned by Leśniewski, with full power of fusion, secured by the general notion of a class.

0.3 Mereotopology

Topological structures which arise in mereology due to Leśniewski, can be induced from overlap relations. As the first approximation to topology, let us define for each object x, its *closure* $c(x)$ by means of

$$c(x) = ClsOv(x) \tag{0.12}$$

where the property $Ov(x)$ is defined by $Ov(x)(y) \Leftrightarrow Ov(x,y)$, i.e., we build the closure $c(x)$ as the class of objects which overlap with x.

We have

Proposition 24. *The closure operator* $c(.)$ *has the following properties*

Cl1 $ingr(x, c(x)$

Cl2 If $ingr(x,y)$*, then* $ingr(c(x), c(y))$

Cl3 $ingr(c(x \cdot y), c(x) \cdot c(y))$

Cl4 $c(x+y) = c(x) + c(y)$

Proof. Cl1 and Cl2 follow from definition of the overlap relation and the class definition. For Cl3, if $ingr(t, c(x \cdot y))$, then there is z such that $Ov(t,z)$ and $Ov(z, x \cdot y)$ thus for some w one has $Ov(z,w)$ and $ingr(w,x)$, $ingr(w,y)$ which imply that $ingr(t, c(x))$, $ingr(t, c(y))$ and finally $ingr(t, c(x) \cdot c(y))$. By M3, $ingr(c(x \cdot y), c(x) \cdot c(y))$.

For Cl4, it suffices to observe that $Ov(z, x+y) \Leftrightarrow Ov(z,x) \vee Ov(z,y)$ □

Another possibility for a topology is in iteration of the operator c, viz, we let

$$Ov^{n+1}(x,y) \Leftrightarrow \exists z.Ov(x,z) \wedge Ov^n(z,y); Ov^1(x,y) \Leftrightarrow Ov(x,y) \tag{0.13}$$

and we define

$$OVLP(x)(y) \Leftrightarrow \exists n.Ov^n(x,y) \tag{0.14}$$

The closure $Cl(x)$ is defined as the class of the property $OVLP(x)$, i.e.,

$$Cl(x) = ClsOVLP(x) \tag{0.15}$$

The operator $Cl(x)$ has the following properties

Proposition 25. *Properties of $Cl(x)$ are*

CL1 $Cl(Cl(x)) = Cl(x)$

CL2 $ingr(x, Cl(x))$

CL3 $ingr(x,y)$ *implies* $ingr(Cl(x), Cl(y))$

CL4 $Cl(x+y) = Cl(x) + Cl(y)$

Proof. CL2, CL3 follow straightforwardly from definitions. For CL1, observe that $ingr(t, Cl(x))$ if and only if $OVLP(x)(t)$. Thus, $ingr(t, Cl(Cl(x)))$ if and only if $OVLP(Cl(x))(t)$ if and only if $OVLP(x)(t)$ if and only if $ingr(t, Cl(x))$.

For CL4, assume first that $ingr(t, Cl(x+y))$ hence $OVLP(t, x+y)$ and thus $OVLP(t,x) \vee OVLP(t,y)$, i.e., $ingr(t, Cl(x)) \vee ingr(t, Cl(y))$ and thus $ingr(t, Cl(x) + Cl(y))$. Assume now that $ingr(t, Cl(x) + Cl(y))$, i.e., $OVLP(t,x) \vee OVLP(t,y)$, so there exists m such that $Ov^m(t,x) \vee Ov^m(t,y)$, i.e., $Ov^m(t, x+y)$, hence, $ingr(t, Cl(x+y))$ □

It follows that the operator Cl is a genuine closure operator; its properties are weak, as it in fact delineates components of objects with respect to the overlap property: it is not even a T_0–closure operator (a Kolmogoroff operator) which would mean that distinct objects should have distinct closures, as distinct objects in the same component have the same closure.

A definition of a *boundary* can be attempted on the lines of topological boundary concept. For an object x, let a property $\Upsilon(x)$ be defined as follows

$$\Upsilon(x)(t) \Leftrightarrow ingr(t,x) \wedge \forall z.[Ov(z,x) \wedge Ov(z,-x) \Rightarrow Ov(z,t)] \tag{0.16}$$

We may define the *boundary of x*, $Fr(x)$, by letting

$$Fr(x) = Cls\Upsilon(x) \tag{0.17}$$

Proposition 26. *Properties of $Fr(x)$ following directly from definitions above are*

1. $ingr(Fr(x), x)$*;*

2. $\forall z.Ov(z,x) \wedge Ov(z,-x) \Rightarrow Ov(z,Fr(x))$.

The above notion of a boundary has a topological flavor; however, the notion of a boundary has a much wider scope. It can also support the idea of a *separator* between two objects within a third, which does encompass either, like a river flowing through a town separates parts on opposite banks. To implement this idea, for objects x, y, such that $extr(x,y)$, we define the property

$$\Omega(x,y)(t) \Leftrightarrow extr(t,x) \wedge extr(t,y) \tag{0.18}$$

and we let

$$Bd(x,y) = Cls\Omega(x,y) \tag{0.19}$$

Then

Proposition 27. *The boundary operation Bd has properties*

1. $Bd(x,y) = Bd(y,x)$.
2. $Bd(x+y,z) = Bd(x,z) \cdot Bd(y,z)$.

Proof. Property 1 is obvious. Property 2 follows from the equivalence $extr(x+y,z) \Leftrightarrow extr(x,z) \wedge extr(y,z)$ □

A relative variant can be defined; assuming that $ingr(x,z)$, $ingr(y,z)$ and $extr(x,y)$, a *boundary relative to z between x* and y, $Bd_z(x,y)$, is the class of objects t such that $ingr(t,z)$, $extr(t,x), extr(t,y)$ provided this property is non–vacuous.

0.4 Timed mereology

Timed component of mereology was introduced in Tarski [62], and presented in a systematic way in Woodger [68], [69]. The time component is introduced into the framework of mereology with a set of notions and postulates (axioms) concerning aspects of time like *m*omentariness, coincidence in time, time slices.

Objects are considered as spatial only and their relevance to time is expressed as momentary or as spatial and extended in time and then the predicate of part is understood as a global descriptor covering spatio–temporal extent of objects whereas the temporal extension is described by the predicate Temp (T) with the intended meaning that $T(u,v)$ means that the object *u precedes* in time the object v (in terminology of Leśniewski, Tarski and Woodger: *u wholly precedes v*) meaning that, e.g., when u and v have some temporal extent, then u ends before or at the precise moment when v begins.

We follow in our exposition the presentation in Woodger [69], to whom all results belong. Proofs are mostly supplied by this author to make the exposition more accessible.

The property (predicate) Mom meaning *momentary being* is introduced to denote objects having only spatial aspect. This predicate is introduced by means of the following postulate

$$MOM\ Mom(x) \Leftrightarrow T(x,x) \tag{0.20}$$

Thus, x begins and ends at the same time, so its time aspect is like a spike in time; it renders the phrase 'to exist in a moment of time'.

The predicate T is required to satisfy postulates

TM1 $T(x,y) \wedge T(y,z) \Rightarrow T(x,z)$

TM2 $Mom(x) \wedge Mom(y) \Rightarrow T(x,y) \vee T(y,x)$

TM3 $T(x,y) \Leftrightarrow \forall u,v.ingr(u,x) \wedge ingr(v,y) \Rightarrow T(u,v)$

Postulate TM1 states that T is transitive, Postulate TM2 does state that of two momentary things, one precedes the other and Postulate TM3 relates T to the class operator, i.e., x precedes y if and only if each ingredient of x precedes each ingredient of y. Postulate TM3 provides a link between the part based mereology and the timed mereology, bonding spatial and temporal properties of objects.

As a consequence to postulates TM1–TM3, one obtains

Proposition 28. *The following properties result from postulates TM1–TM3*

1. $Mom(x) \Leftrightarrow \exists y.T(x,y) \wedge T(y,x)$;

2. $ingr(u,x) \wedge T(x,y) \Rightarrow T(u,y)$;

3. $T(Cls\Phi,x) \Leftrightarrow \forall y.\Phi(y) \Rightarrow T(y,x)$;

4. $T(x,Cls\Phi) \Leftrightarrow \forall y.\Phi(y) \Rightarrow T(x,y)$;

5. $Mom(Cls\Phi) \Leftrightarrow \forall x,y.\Phi(x) \wedge \Phi(y) \Rightarrow T(x,y)$.

Proof. Property 1 follows by Postulate (0.20) and Postulate TM2: $T(x,y) \wedge T(y,x)$ implies $T(x,x)$ by Postulate TM2, hence, $Mom(x)$ by (0.20). Property 2 follows by Postulate TM3 as $ingr(u,x), ingr(y,y), T(x,y)$ imply $T(u,y)$. Property 3 follows from Postulate TM3: from $ingr(x,x), ingr(u,x)$ it follows that for some z, it happens that $Ov(u,z) \wedge \Phi(z)$, hence $ingr(z,Cls\Phi)$, hence, by Postulate TM3, $T(z,x)$ and by the class definition, $T(u,x)$.

Property 4 is dual to Property 3. Property 5 follows by (0.20) and either Property 3, or Property 4 applied to the pair $Cls\Phi, Cls\Phi$ □

Remark. Concisely, one can write down Property 2 in the form: $ingr \circ T = T$, hence, $T = T \circ ingr^{-1}$, where *ingr* denotes the ingredient relation.

The notion of a *coincidence in time*, CT in symbols, needs a defining postulate

$$CT(x, y) \Leftrightarrow T(x, y) \wedge T(y, x) \tag{0.21}$$

Hence, the relation of coincidence in time is symmetric and transitive.

Properties of coincidence in time can be summed up as follows

Proposition 29. *Operation CT has properties*

1. $Mom(x) \Leftrightarrow \forall y. ingr(y, x) \Rightarrow CT(y, x)$;

2. $Mom(x) \wedge Mom(y) \wedge Ov(x, y) \Rightarrow CT(x, y)$;

3. $Mom(Cls\Phi) \Leftrightarrow \exists x.[\Phi(x) \wedge \forall y.\Phi(y) \Rightarrow CT(y, x)]$.

Proof. Property 1 follows by (0.20) implying $T(x, x)$ from which by Postulate TM3 it follows that if $ingr(y, x)$ then $T(y, x)$ and by Proposition 28(1) by which $T(x, y)$, hence, $CT(x, y)$. This proves the implication from left to right. The converse implication follows by definition (0.21). Property 2 follows by Property 1 and transitivity of CT. Property 3 follows from Property 1 with $Cls\Phi$ in place of x and transitivity of CT □

The notion of being *wholly before in time* is rendered as a predicate Z defined as follows

$$Z(x, y) \Leftrightarrow Mom(x) \wedge Mom(y) \wedge \neg T(y, x) \tag{0.22}$$

Hence, $Z(x, y)$ means that the momentary object x is wholly before the momentary object y in time.

By Postulate TM2, $Z(x, y)$ implies $T(x, y)$. Hence,

Proposition 30. *For each pair of momentary objects x, y the equivalence*

$$T(x, y) \Leftrightarrow CT(x, y) \vee Z(x, y)$$

holds. Meaning that in case $T(x, y)$ holds, either x, y coincide in time or x is wholly before y in time.

Proof. By definitions of CT and Z, if $T(x, y)$ then either $T(y, x)$ and hence CT or $\neg T(y, x)$, hence, $Z(x, y)$ □

Proposition 31. *If x, y are momentary objects and $Z(x, y)$, then neither $ingr(x, y)$ nor $ingr(y, x)$.*

Proof. By Proposition 29, if either $ingr(x, y)$ or $ingr(y, x)$, then $CT(x, y)$ which excludes $Z(x, y)$ by virtue of respective definitions □

Proposition 31 sets another important link between spatial and temporal aspects of objects. Actually in this proposition the condition that neither of objects is an ingredient of the other can be replaced by demanding that the objects do not overlap.

Finally, the notion of a *time–slice* is introduced, as a predicate $Slc(x, y)$ by means of

$$Slc(x, y) \Leftrightarrow Mom(x) \wedge ingr(x, y) \wedge \forall z.[ingr(z, y) \wedge C(z, x) \Rightarrow ingr(z, x)] \tag{0.23}$$

and thus a time–slice of an object y is an ingredient of y which is spatially so arranged that any ingredient of y coinciding with it in time is also its ingredient.Time slices are unique up to coincidence in time.

Proposition 32. *If x, y are time–slices of z, then x, y coincide in time if and only if $x = y$.*

Proof. By definition (0.23), one has $ingr(x, y), ingr(y, x)$, hence, $x = y$ □

Each object is the class of its time slices. To prove this statement, first we note that

Proposition 33. *Given a momentary ingredient x of an object y, the class $CT(y, x)$ of all ingredients of y which coincide in time with x is a time–slice of y coincident in time with x.*

Proof. Clearly, $CT(y, x)$ is an ingredient of y. By Postulate TM3, the class $CT(y, x)$ coincides in time with x. By transitivity of T, any ingredient of y which coincides in time with $CT(y, x)$ also coincides in time with x, hence, is an ingredient of $CT(y, x)$. It follows that $C(y, x)$ is a time–slice of y □

Proposition 33 suggests the way of embedding any ingredient of an object into a time–slice of that object. As any object is the class of its ingredients, it follows that

Proposition 34. *Each object x is the class of its ingredients which are its time–slices.*

Among time–slices of an object, one can distinguish the first (beginning) and the last (ending) time–slices.

A time–slice x of an object y is a *first time–slice* of y if and only if each time–slice z of y such that $T(z, x)$ is identical with x; similarly, a time–slice w of y is an *ending time–slice* of y if each time–slice z of y such that $T(w, z)$ is identical with w.

Proposition 35. *The important properties of first and last time–slices are*

1. *Each first time–slice x of an object y satisfies $T(x, y)$;*

2. *Each last time–slice w of an object y satisfies $T(y, w)$.*

Proof. As Statement 2 is dual to Statement 1, it suffices to verify the latter. Consider thus the first time–slice x of y. For any other time–slice $z \neq x$ of y, we have by Proposition 32 and by Postulate TM2 that $T(x, z)$, hence, by Proposition 34 and Postulate TM3 it follows that $T(x, y)$ □

One can define a *temporal interior of an object y* as the separator $Fr(x, w, y)$ between a first time–slice x and the last time–slice w of y relative to y.

0.5 Spatio–temporal reasoning: cells

A theory of cells, motivated by phenomenology of cell biology, but in fact an abstract timed mereology of time–evolving spatial structures, was developed in detail in Woodger [69], Ch. 2, see also Tarski [62].

By a *cell* an object y is understood for which an ingredient x exists with $T(x, y)$ and an ingredient w exists such that $T(y, w)$. Hence, a cell is time–bounded by some time–preceding object and some time–following object.

We denote the fact that y is a cell with the symbol $cell(y)$. Formally

TM4 $cell(y) \Leftrightarrow \exists x, w.[ingr(x, y) \wedge ingr(w, y) \wedge T(x, y) \wedge T(y, w)]$

The next postulate asserts that no cell is a momentary object,

TM5 $cell(y) \Rightarrow \neg Mom(y)$

We recall that the part relation is denoted with the symbol π. A first time–slice x of a cell y, respectively, a last time–slice w of y, will be denoted by $Beg(y)$, respectively $End(y)$. The last postulate about cells follows

TM6 $cell(y_1) \wedge cell(y_2) \wedge y_1 \neq y_2 \wedge Ov(y_1, y_2) \Rightarrow \pi(Beg(y_1), End(y_2)) \vee \pi(End(y_1), Beg(y_2))$

Properties of cells are collected in

Proposition 36. *The following are properties of cells*

1. *If $cell(y)$, then the class C of ingredients x of y for which $T(x, y)$ holds is a momentary ingredient of y such that $T(C, y)$ holds;*

2. The class C of Statement 1 is a first time–slice of y;

3. If $cell(y)$, then the class D of ingredients w of y for which $T(y, w)$ holds is a momentary ingredient of y such that $T(y, D)$ holds;

4. The class D of Statement 2 is a last time–slice of y.

Proof. As Statements 3 and 4 are dual to, respectively, Statements 1 and 2, it suffices to verify the latter. Concerning Statement 1, by Remark 1, for ingredients z, y of x such that $T(z, x), T(y, x)$ it follows (in view of $T \circ ingr^{-1} = T$) that $T(z, y)$ which by Postulate TM3 implies that $T(y, z)$ hence by Proposition 28, $T(x, x)$, hence, $Mom(x)$ follows by Postulate TM1. Clearly, $T(C, x)$ by Proposition 28, again. Concerning Statement 2, clearly, C is a first time–slice by its definition □

Proposition 37. *Properties of first and last time–slices are*

1. For any cell x, the first time–slice $Beg(x)$ is not the last time–slice $End(x)$;

2. For any cell x, $T(Beg(x), End(x))$;

3. For any cell x, $Z(Beg(x), End(x))$.

Proof. By transitivity of T, in case 1, was it otherwise, we would have $T(x, x)$ contradicting Postulate TM5 in view of Postulate TM1. For Statement 2, it follows by transitivity of T from Proposition 36 1, 3. Statement 3 follows by definition (0.22) of the predicate Z and by Postulate TM4 about cells, and transitivity of T □

Postulate TM6 implies that

Proposition 38. *For two distinct cells x, y, if*

$$C(Beg(x), Beg(y)) \text{ or } C(End(x), End(y))$$

then neither $ingr(Beg(x), End(y))$ nor $ingr(End(x), Beg(y))$.

Proof. Was, e.g., $C(Beg(x), Beg(y))$ and $ingr(Beg(x), End(y))$, one would have $Z(Beg(y), Beg(x))$ contradicting $C(Beg(x), Beg(y))$. Similarly for other combinations of terms in the premise and in the conclusion □

The following is true in consequence

Proposition 39. *For cells x, y with the property that $C(Beg(x), Beg(y))$ or $C(End(x), End(y))$, if $Ov(x, y)$, then $x = y$.*

Proof. Premises imply by Proposition 38 that neither $ingr(Beg(x), End(y))$ nor $ingr(End(x), Beg(y))$ contradicting Postulate TM6 in case $x \neq y$ □

A corollary specializes Proposition 39

Corollary 1. *For cells x, y, the following are equivalent*

1. $Beg(x) = Beg(y)$;

2. $End(x) = End(y)$;

3. $x = y$.

Proof. Assumptions of Proposition 39 are satisfied □

Operations on cells include *division* and *fusion*.

A cell x *arises by division from a cell* y if and only if $\pi(Beg(x), End(y))$. We write the fact down as $Div(x, y)$.

Then, by already proven statements,

Proposition 40. *If $Div(x, y)$, then $Z(Beg(x), Beg(y)$, $T(y, x)$, and $x \neq y$.*

A dual operation of *fusion* is defined as follows.

A cell x *arises by fusion of a cell* y *with some other cell* if and only if $\pi(End(y), Beg(x))$. We write down the fact as $Fus(x, y)$.

A dual proposition to 40 is stated as follows

Proposition 41. *If $Fus(x, y)$, then $Z(Beg(y), Beg(x))$, $T(y, x)$, and $x \neq y$.*

It follows from definitions of division and fusion that

Proposition 42. *There exists no cell x which can divide and fuse with the same other cell y. There can be no cell which could arise by division and fusion from the same other cell.*

0.6 Mereology based on connection

A dual approach to parts, was initiated in Whitehead [64], [65], [66] in a form of propositions of axioms for the notion of 'x extends over y', dual to that of a part, and of vague proposals of desired properties of the notion. Th. de Laguna [37] published a variant of the Whitehead scheme, which led Whitehead [67] to another version of his approach, based on the notion of 'x is extensionally connected to y'. Connection Calculus based on the notion of a 'connection' was proposed in Clarke [12], and this version is presented here, see also Calculus of Individuals of Leonard and Goodman [40].

The relation/predicate of connection C is subject to basic requirements, see Clarke [12], A0.1, A0.2

CN1 $C(x,x)$ *for each object* x

CN2 *If* $C(x,y)$, *then* $C(y,x)$ *for each pair* x,y *of objects*

It follows that connection is reflexive and symmetric. This theory is sometimes called *Ground Topology* T, cf., Casati–Varzi [10]. Adding the extensionality requirement

CN3 *If* $\forall z.[C(z,x) \Leftrightarrow C(z,y)]$, *then* $x = y$

produces the *Extensional Ground Topology* ET., loc. cit.

Let us observe that within the mereology M, the predicate C can be realized by taking $C = Ov$; clearly, CN1–CN3 are all satisfied with Ov. We call this model of connection mereology, the *Overlap model*, denoted OVM.

In the universe endowed with C, satisfying CN1, CN2, one can define the notion of an ingredient $ingr_C$ by letting

$$IC\ ingr_C(x,y) \Leftrightarrow \forall z.[C(z,x) \Rightarrow C(z,y)] \tag{0.24}$$

Then,

Proposition 43. *(Clarke [12]) The following properties of* $ingr_C$ *hold*

1. $ingr_C(x,x)$;

2. $ingr_C(x,y) \wedge ingr_C(y,z) \Rightarrow ingr_C(x,z)$;

3. *In presence of CN3,* $ingr_C(x,y) \wedge ingr_C(y,x) \Rightarrow x = y$;

4. $ingr_C(x,y) \Leftrightarrow \forall z.[ingr(z,x) \Rightarrow ingr(z,y)]$;

5. $ingr_C(x,y) \wedge C(z,x) \Rightarrow C(z,y)$;

6. $ingr_C(x,y) \Rightarrow C(x,y)$.

The notion of a part π_C can be introduced as

$$PC\ \pi_C(x,y) \Leftrightarrow ingr_C(x,y) \wedge x \neq y \tag{0.25}$$

By definition (0.25), π_C satisfies requirements of mereology for the notion of a part

Proposition 44. *(Clarke [12]) Properties of* part *are*

1. $\neg\pi_C(x,x)$;

2. $\pi_C(x,y) \wedge \pi_C(y,z) \Rightarrow \pi_C(x,z)$;

3. $\pi_C(x,y) \Rightarrow ingr_C(x,y)$;

4. $\pi_C(x,y) \Rightarrow \neg\pi_C(y,x)$.

The predicate of *overlapping*, $Ov_C(x,y)$ is defined by means of

$$OC\ Ov_C(x,y) \Leftrightarrow \exists z.[ingr_C(z,x) \wedge ingr_C(z,y)] \tag{0.26}$$

Basic properties of overlapping follow.

Proposition 45. *(Clarke [12]) Properties of overlapping are*

1. $Ov_C(x,x)$;

2. $Ov_C(x,y) \Leftrightarrow Ov_C(y,x)$;

3. $Ov_C(x,y) \Rightarrow C(x,y)$;

4. $ingr_C(x,y) \wedge Ov_C(z,x) \Rightarrow Ov_C(z,y)$;

5. $ingr_C(x,y) \Rightarrow Ov_C(x,y)$.

The counterpart of the notion of an *exterior* object, $extr_C$ is defined by means of

$$EC\ extr_C(x,y) \Leftrightarrow \neg Ov_C(x,y) \tag{0.27}$$

Proposition 46. *(Clarke [12]) The property*

$$[ingr_C(x,y) \wedge extr_C(z,y)] \Rightarrow extr_C(z,x).$$

holds by Property 4 in Proposition 45.

A new notion due to connectedness is the notion of *external connectedness*, EC in symbols, defined as follows

$$EC\ EC(x,y) \Leftrightarrow C(x,y) \wedge extr(x,y) \tag{0.28}$$

It is easy to see that in the model OVM, EC is a vacuous notion. Clearly, by definition (0.28),

Proposition 47. *(Clarke [12]) The following properties of external connectedness hold*

1. $\neg EC(x,x)$;

2. $EC(x,y) \Leftrightarrow EC(y,x)$;

3. $C(x,y) \Leftrightarrow EC(x,y) \vee Ov_C(x,y)$;

4. $Ov_C(x,y) \Leftrightarrow C(x,y) \wedge \neg EC(x,y)$;

5. $\neg EC(x,y) \Leftrightarrow [Ov_C(x,y) \Leftrightarrow C(x,y)]$: *This is a logical rendering of our remark that in OVM, no pair of objects is in EC, hence, $\neg EC(x,y) = TRUE$ for each pair of objects;*

6. $\neg\exists z.EC(z,x) \Rightarrow \{ingr_C(x,y) \Leftrightarrow [\forall w.Ov_C(w,x) \Rightarrow Ov_C(w,y)]\}$.

Proof. A comment in the way of proof. The implication

$$ingr_C(x,y) \Rightarrow [\forall w.Ov_C(w,x) \Rightarrow Ov_C(w,y)]$$

is always true. Thus, it remains to assume that

(i) $\neg\exists z.EC(z,x)$

and to prove that

$$(*)\ [\forall w.Ov_C(w,x) \Rightarrow Ov_C(w,y)] \Rightarrow ingr_C(x,y)$$

Assumption (i), can be written down as

(ii) $\forall z.\neg C(z,x) \vee Ov_C(z,x)$

To prove that $ingr_C(x,y)$ it should be verified that

(iii) $\forall z.(C(z,x) \Rightarrow C(z,y))$.

Consider an arbitrary object z'; either $\neg C(z', x)$ in which case implication in (iii) is satisfied with z', or, $Ov_C(z', x)$, hence, $Ov_C(z', y)$ by the assumed premise in (*), which implies that $C(z', y)$. The implication (iii) is proved and this concludes the proof □

The richer structure of connection based calculus allows for some notions of a topological nature; the first is the notion of a *tangential ingredient*, in symbols: $Tingr_C(x, y)$ defined by means of

$$TI\ Tingr_C(x, y) \Leftrightarrow ingr_C(x, y) \wedge \exists z.EC(z, x) \wedge EC(z, y) \tag{0.29}$$

Basic properties of tangential parts follow by definition TI

Proposition 48. *The following are basic properties of the predicate* $Tingr_C$

1. $\exists z.EC(z, x) \Rightarrow Tingr_C(x, x)$;

2. $\neg\exists z.EC(z, y) \Rightarrow \neg existsx.Tingr_C(x, y)$;

3. $Tingr_C(z, x) \wedge ingr_C(z, y) \wedge ingr_C(y, x) \Rightarrow Tingr_C(y, x)$.

Proof. For Property 3, some argument may be in order; consider w such that $EC(w, x), EC(w, z)$ existing by $Tingr_C(z, x)$. hence, $C(w, y)$. As $\neg Ov_C(w, x)$, it follows that $\neg Ov_C(w, y)$, hence, $EC(w, y)$, and $Tingr_C(y, x)$ □

These properties witness the fact that if there is something externally connected to x, then x is its tangential ingredient. This fact shows that the notion of a tangential ingredient falls short of the idea of a boundary. Dually, in absence of objects externally connected to y, no ingredient of y can be a tangential ingredient.

An object y is a *non–tangential ingredient* of an object x, $NTingr_C(y, x)$ in symbols, in case, it is an ingredient but not any tangential ingredient of x

$$NTI\ NTingr_C(y, x) \Leftrightarrow \neg Tingr_C(y, x) \wedge ingr_C(y, x) \tag{0.30}$$

Proposition 49. *Basic properties of the operator NTI are*

1. $NTingr_C(y, x) \Rightarrow \forall z.\neg EC(z, y) \vee \neg EC(z, x)$;

2. $\neg\exists z.EC(z, x) \Rightarrow NTingr_C(x, x)$.

In absence of externally connected objects, each object is a non–tangential ingredient of itself.

Hence, in the model OVM, each object is its non–tangential ingredient and it has no tangential ingredients.

To produce models in which EC, $NTingr_C, Tingr_C$ will be exhibited, we may resort to topology; we recall, see Ch.2, sect. 6, that *a regular open set* in a topological space X, is a set $A \subseteq X$ such that $A = IntClB$ for some set $B \subseteq X$.

We define in the space $RO(X)$ of regular open sets in a regular space X (recall that a topological space is regular if for each element x, each neighborhood of x contains closure of another neighborhood of x, see Ch. 2, sect. 6) the connection C by demanding that $C(A, B) \Leftrightarrow ClA \cap ClB \neq \emptyset$. For simplicity sake, we assume that the regular space X is connected, so no set in it is clopen, equivalently, the boundary of each set is non–empty.

When investigating properties of a model ROM, we refer to Ch. 2, sect. 6.

First, we investigate what $ingr_C$ means in ROM. By definition IC in (0.24), for $A, B \in RO(X)$,

$$ingr_C(A, B) \Leftrightarrow \forall Z \in RO(X).ClZ \cap ClA \neq \emptyset \Rightarrow ClZ \cap ClB \neq \emptyset$$

This excludes the case when $A \setminus ClB \neq \emptyset$ as then we could find a $Z \in RO(X)$ with

$$Z \cap A \neq \emptyset = ClZ \cap ClB$$

(as our space X is regular). It remains that $A \subseteq ClB$, hence, $A \subseteq IntClB = B$.

It follows finally that

Proposition 50. *In model ROM, $ingr_C(A, B) \Leftrightarrow A \subseteq B$.*

It follows that in ROM ingredient means containment with connection C as intersection of closures.

Now, we can interpret overlapping in ROM. For $A, B \in RO(X)$, $Ov_C(A, B)$ means that there exists $Z \in RO(X)$ such that $Z \subseteq A$ and $Z \subseteq B$ hence$Z \subseteq A \cap B$, hence

$$A \cap B \neq \emptyset$$

This condition is also sufficient by regularity of X. We obtain

Proposition 51. *In ROM, $Ov_C(A, B) \Leftrightarrow A \cap B \neq \emptyset$.*

The status of EC in ROM is then

Proposition 52. *In ROM,*

$$EC(A, B) \Leftrightarrow ClA \cap ClB \neq \emptyset \wedge A \cap B = \emptyset$$

This means that closed sets ClA, ClB do intersect only at their boundary points.

We can address the notion of a tangential ingredient: $Tingr_C(A, B)$ means the existence of $Z \in RO(X)$ such that

$$ClZ \cap ClA \neq \emptyset \neq ClZ \cap ClB$$

and

$$Z \cap A = \emptyset = Z \cap B$$

along with $A \subseteq B$.

In case

$$ClA \cap (ClB \setminus B) \neq \emptyset$$

letting $Z = X \setminus ClB$ we have

$$ClZ = Cl(X \setminus ClB)$$

and

$$BdZ = ClZ \setminus Z = Cl(X \setminus ClB) \setminus (X \setminus ClB)$$

which in turn is equal to

$$Cl(X \setminus ClB) \cap ClB = Cl(X \setminus B) \cap ClB = BdB$$

Hence, $ClB \setminus B \subseteq ClZ$, and $ClZ \cap ClA \neq \emptyset$; a fortiori, $ClB \cap ClZ \neq \emptyset$. As $Z \cap B = \emptyset$, a fortiori $Z \cap A = \emptyset$ follows.

We know, then, that if

$$ClA \cap (ClB \setminus B) \neq \emptyset \Rightarrow Tingr_C(A, B)$$

Was to the contrary, $ClA \subseteq B$, from $Z \cap ClA \neq \emptyset$ it would follow that $Z \cap B \neq \emptyset$, negating $EC(A, B)$.

It follows finally that

Proposition 53. *In the model ROM, $Tingr_C(A, B)$ if and only if $A \subseteq B$ and $ClA \cap (ClB \setminus B) \neq \emptyset$. From this analysis we obtain also that $NTingr_C(A, B)$ if and only if $ClA \subseteq B$.*

Further properties of the predicate $NTingr_C$ are collected in

Proposition 54. *(Clarke [12]) Properties of $NTingr_C$ are*

1. $NTingr_C(y, x) \wedge C(z, y) \Rightarrow C(z, x)$;

2. $NTingr_C(y, x) \wedge Ov_C(z, y) \Rightarrow Ov_C(z, x)$;

3. $NTingr_C(y, x) \wedge C(z, y) \Rightarrow Ov_C(z, x)$;

4. $ingr_C(y, x) \wedge NTingr_C(x, z) \Rightarrow NTingr_C(y, z)$;

5. $ingr_C(y,z) \wedge NTingr_C(x,y) \Rightarrow NTingr_C(x,z)$*;*

6. $NTingr_C(y,z) \wedge NTingr_C(z,x) \Rightarrow NTingr_C(y,x)$*.*

Proof. For Property 3, from already known $\forall z.\neg EC(z,y) \vee \neg EC(z,x)$, it follows

$$(i)\ \forall w.\neg C(w,x) \vee Ov_C(w,x) \vee \neg C(w,y) \vee Ov_C(w,y)$$

As $C(z,y)$, one obtains $C(z,x)$. Thus, by (i), $Ov_C(z,y) \vee Ov_C(z,x)$ and $Ov_C(z,x)$.

For Property 4, assuming that $ingr_C(y,x), ingr_C(x,z)$ and hence, $ingr_C(y,z)$ (else, there is nothing to prove), consider $\neg NTingr_C(y,z)$, i.e., for some w: $EC(w,z), EC(w,y)$. Thus, $C(w,z)$, $\neg Ov_C(w,z)$, $C(w,y)$, $\neg Ov_C(w,y)$.

Then, $C(w,x)$ and $\neg Ov_C(w,x)$, hence, $EC(w,x)$ and $\neg NTingr_C(x,z)$, a contradiction. Similarly, one justifies Properties P5 and P6 □

0.7 Classes in connection mereology

Definition of a class $Cls\Phi$ of a non–vacuous property Φ, is given in Clarke's version of connection mereology in terms of the primitive notion of the predicate C to satisfy the desired property

$$C(z, Cls_C\Phi) \Leftrightarrow \exists x.\Phi(x) \wedge C(z,x) \tag{0.31}$$

The existence of a class is secured by an axiom

CN4 *For each non–vacuous property* Φ *of objects, there exists an object* $Cls_C\Phi$ *(a C–class of* Φ*) which satisfies* (0.31)

Proposition 55. *(Clarke [12])*

1. *By CN3 and CN4, there is a unique object* $Cls_C\Phi$ *for each non–vacuous property* Φ*;*

2. *If* $\Phi(z)$*, then* $ingr_C(z, Cls_C\Phi)$*;*

3. $ingr_C(z, Cls_C\Phi) \Rightarrow \exists w.\Phi(w) \wedge C(z,w)$*;*

4. $C(z, Cls\Phi) \wedge \neg\exists w.EC(z,w) \Rightarrow \exists w.\Phi(w) \wedge Ov_C(z,w)$*;*

Properties 3, 4 witness that

5. In the model OVM*, the class defined in connection mereology by* Cls_C *satisfies Le'sniewski's postulates for a class;*

6. $Ov_C(z, Cls_C\Phi) \Rightarrow \exists w.\Phi(w) \wedge C(z,w)$*;*

Indeed, $Ov_C(z, Cls_C\Phi)$ *implies* $\exists t.[ingr_C(t,z) \wedge ingr_C(t, Cls_C\Phi)]$*, hence,* $\exists w.[\Phi(w) \wedge C(t,w)]$*. Thus,* $C(z,w)$*.*

7. $\neg existsw.EC(z,w) \Rightarrow [Ov_C(z, Cls_C\Phi) \Rightarrow \exists t.\Phi(t) \wedge Ov_C(z,t)]$*;*

8. $\exists w.\Phi(w) \wedge C(z,w) \Rightarrow Ov_C(z, Cls_C\Phi)$*.*

As with the Leśniewski notion of a class, connection based classes have properties

9. If $\Phi \Rightarrow \Psi$*, then* $ingr_C(Cls_C\Phi, Cls_C\Psi)$*;*

10. If $\Phi \Leftrightarrow \Psi$*, then* $Cls_C\Phi = Cls_C\Psi$*;*

11. $x{=}Cls_C\ INGR(x)$*.*

0.8 C–Quasi–Boolean algebra

By Tarski [62], [61], as with the Leśniewski mereology, in connection mereology a quasi–Boolean algebra arises. By Postulate CN4, there exists for each pair x, y of objects, an object denoted $x{+}y$ defined as the class of the property $+(x,y)$,

$$+(x,y)(t) \Leftrightarrow ingr_C(t,x) \vee ingr_C(t,y) \tag{0.32}$$

Then,

$$Cls+\ \ x+y = Cls+(x,y) \tag{0.33}$$

Proposition 56. *(Clarke [12]) The operation + has properties*

1. $ingr_C(x, x+y)$ *by 9, Proposition 55;*

2. $C(z, x+y) \Leftrightarrow \exists w.[(ingr_C(w,x) \vee ingr_C(w,y)] \wedge C(z,w)$ *by* Cls_C*;*

3. $C(z, x+y) \Leftrightarrow C(z,x) \vee C(z,y)$ *by Property 2;*

From Property 10 of Proposition 55, we obtain

4. $x + y = y + x$*;*

5. $x + (y + z) = (x + y) + z$*;*

6. $x + x = x$*.*

The dual property is defined as

$$\times(x, y)(t) \Leftrightarrow ingr_C(t, x) \wedge ingr_C(t, y) \quad (0.34)$$

Clearly, the definition (0.34) makes sense only if $Ov_C(x, y)$; in the sequel, we will tacitly assume this condition any time the following operation $x \cdot y$ is mentioned.

The Boolean product $x \cdot y$ is defined as

$$Cls\times \;\; x \cdot y = Cls \times (x, y) \quad (0.35)$$

It follows immediately by (0.35)

Proposition 57. *(Clarke [12]) The following are properties of the Boolean product*

1. $C(z, x \cdot y) \Leftrightarrow \exists w.[(ingr_C(w,x) \wedge ingr_C(w,y)] \wedge C(z,w)$*;*

2. $C(z, x \cdot y) \Rightarrow C(z,x) \wedge C(z,y)$ *by Property 1;*

3. $ingr_C(x \cdot y, x)$*;* $ingr_C(x \cdot y, y)$ *by Property 2;*

4. $ingr_C(z, x \cdot y) \Leftrightarrow ingr_C(z,x) \wedge ingr_C(z,y)$ *by Property 2 and Proposition 55;*

By Property 10 of Proposition 55, the following are true

5. $x \cdot y = y \cdot x$;

6. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;

7. $x \cdot (y + z) = x \cdot y + x \cdot z$.

The universe V is defined as the class of the property Con

$$Con(x) \Leftrightarrow C(x, x) \tag{0.36}$$

$$V = ClsCon \tag{0.37}$$

By (0.37)

Proposition 58. *For each object* z*, one has* $ingr_C(z, V)$.

The complement $-x$ is defined as the class of property

$$N(x)(t) \Leftrightarrow \neg C(t, x) \tag{0.38}$$

by means of

$$-x = ClsN(x) \tag{0.39}$$

Then, by Property 10 of Proposition 55

Proposition 59. *(Clarke [12]) The properties of complement are*

1. $-(-x) = x$;

2. $C(z, -x) \Leftrightarrow \exists w.\neg C(x, w) \wedge C(z, w)$;

3. $C(z, -x) \Leftrightarrow \neg ingr_C(z, x)$ *by Property 2;*

4. $ingr_C(x, y) \Leftrightarrow ingr_C(-y, -x)$;

Complement operator $-$ *flip–flops between boolean addition and multiplication*

5. $-(x \cdot y) = (-x) + (-y)$*;*

6. $(-x) \cdot (-y) = -(x + y)$

7. $\neg EC(x, -y) \Rightarrow [-x + y = V \Leftrightarrow ingr_C(x, y)]$*.*

Proof. As Property 5 obtains from Property 4 by means of Property 1, it suffices to prove Property 4. We have $C(t, -x \cdot y) \Leftrightarrow \neg ingr_C(t, x \cdot y)$ by 3, and by Property 3 of Proposition 57,

$$\neg ingr_C(t, x \cdot y) \Leftrightarrow \neg ingr_C(t, x) \vee \neg ingr_C(t, y)$$

which by the Property 3 is equivalent to $C(t, -x) \vee C(t, -y)$ and it is by Property 3 of Proposition 56, equivalent to $C(t, (-x) + (-y))$. By CN3, $-(x \cdot y) = (-x) + (-y)$.

For Property 6, premise $\neg EC(x, -y)$ is equivalent to $\neg C(x, -y) \vee Ov_C(x, -y)$ and in turn, by Property 3, to $ingr_C(x, y) \vee Ov_C(x, -y)$. The implication $ingr_C(x, y) \Rightarrow -x + y = V$ being obvious, we may observe that conditions $-x+y = V$ and $Ov_C(x, -y)$ are contradictory: hence, the premise $-x+y = V$ implies $ingr_C(x, y)$ □

In particular, in any model OVM, $-x + y = V \Leftrightarrow ingr_C(x, y)$.

0.9 C–Mereotopology

Topological operators are constructed in connection mereology under same caveat as quasi–Boolean operators: absence of the null object causes to make reservations concerning existence of some objects necessary for topological constructions. We will make this reservations not trying to add new axioms which would guarantee existence of some auxiliary objects.

The *C–interior* $Int_C(x)$ of an object x is defined as the class of non–tangential ingredients of x.

We define the property $NTP(x)$

$$NTP(x)(z) \Leftrightarrow NTP(z, x) \tag{0.40}$$

The interior $Int_C(x)$ is defined by means of

$$INT_C \; Int_C(x) = ClsNTP(x) \tag{0.41}$$

hence, properties follow

Proposition 60. *(Clarke [12]) properties of the operator* Int_C *are*

1. $C(z, Int_C(x)) \Leftrightarrow \exists w.NTingr_C(w,x) \wedge C(z,w)$ *by the class definition;*

2. $\neg \exists z.EC(z,x) \Rightarrow (Int_C(x) = x)$. *In particular, in the model* OVM, $Int_C(x) = x$ *for each object* x*;*

3. $ingr_C(Int_C(x), x)$ *as* $C(z, Int_C(x)) \Rightarrow C(z,x)$*;*

4. $C(z, Int_C(x)) \Rightarrow Ov_C(z,x)$*;*

5. $EC(z,x) \Rightarrow \neg C(z, Int_C(x))$*;*

6. $ingr_C(z, Int_C(x)) \Leftrightarrow NTingr_C(z,x)$*;*

7. $ingr_C(z,x) \Rightarrow ingr_C(Int_C(z), Int_C(x))$*;*

8. $Int_C(x) = x \Leftrightarrow C(z,x) \Rightarrow Ov_C(z,x)$*;*

9. $Int_C(x) = x \Leftrightarrow NTingr_C(x,x)$.

An *open* object is x such that $Int_C(x) = x$.

Under additional axiomatic postulate that the boolean product of any two open sets is open, see Clarke [12], A2.1, one can prove that $Int_C(x \cdot y) = Int_C(x) \cdot Int_C(y)$.

The notion of a topological *closure* $Cl_C(x)$ of x, can be introduced by means of the standard duality

$$Cl_C \; Cl_C(x) = -int(-x) \tag{0.42}$$

By properties of the interior and by duality (0.42), one obtains dual properties of closure

Proposition 61. *(Clarke [12]) Properties of the operator* Cl_C *are*

1. $ingr_C(x, Cl_C(x))$*;*

2. $Cl_C(Cl_C(x)) = Cl_C(x)$*;*

3. $ingr_C(x, y) \Rightarrow ingr_C(Cl_C(x), Cl_C(y))$;

4. $Int_C(x \cdot y) = Int_C(x) \cdot Int_C(y) \Leftrightarrow Cl_C(x + y) = Cl_C(x) + Cl_C(y)$;

5. $C(z, Cl_C(x)) \Leftrightarrow \exists w.NTingr_C(w, -x) \wedge C(z, w)$.

The notion of a *boundary* can be introduced along standard topological lines

$$Bd_C \ Bd_C(x) = -(int(x) + int(-x)) \tag{0.43}$$

We collect basic properties of the boundary in

Proposition 62. *The operator Bd_C has properties*

1. *Under Property 4 of Proposition 61, $Bd_C(x) = Cl_C(x) \cdot -Int_C(x)$, i.e., it can be expressed as the difference between the closure and the interior of the object;*

2. *$Bd(x) = Bd(-x)$ by Property 1 of Proposition 59;*

3. *$ingr_C(Bd_C(x), Cl_C(x))$.*

We now include a section on applications of reasoning schemes based on mereology in spatial reasoning.

0.10 Spatial Reasoning: Mereological calculi

Spatial reasoning belongs to the oldest known to humankind forms of reasoning: the space around man and orientation in it were doubtless the earliest experiences they had had. The strict relations among various objects, e.g., segments of various length, especially those forming figures in the plane were among the first scientific observations of man: for many historians of civilization, see Bronowski [9], the Pythagorean Theorem is the most important mathematical fact known to man and one can argue plausibly the same about the Tales Theorem most probably known to Imhotep, and other pyramid builders, see Shaw [51].

It comes thus as no surprise that axiomatic system of geometry due to Euclid, see Joyce [31], is one of the two earliest formal systems besides the Syllogistic of Aristotle, see Ch. 1. Formal questions posed by the Euclidean system, like the famous Vth Postulate have led to many fundamental discoveries like non–Euclidean geometries. In XX century, axiomatization of geometry was pursued among others by Tarski [63].

With Riemann, geometry entered the realm of abstract spaces and work of Cantor alongside early anticipations by Riemann brought forth topology – a theory of invariants of continuous mappings, cf., Ch. 2. Topology created a more flexible and adaptable to various contexts reasoning framework which have grown to a many–faceted theory permeating the whole of mathematical insight into reality, mostly due to the universal and simple notion of a neighborhood.

Formal discussion of topology have been carried out within the framework of set theory, initiated by Cantor, whose elementary notion is the relation of being an element, in symbols, $\in$.

The philosophical assumption concerning the nature of a set was therefore that sets are built of elements and identical sets possess of the same elements. This assumption brought forth some paradoxes like the well–known Russell paradox whether there exists a set of all sets. The emergence of paradoxes demonstrated the difficulty with the notion of a set; mathematicians and philosophers of the turn of XX century found seemingly the way out of difficulty by imposing on the notion of a set some strict requirements which allowed the existence of some sets and prohibited the construction of some too abstract sets, see Kanamori [32]. Yet, some questions have turned out to be very difficult to answer and even undecidable, so currently there exist many models of set theory to choose from, cf., Kanamori, op. cit.

Qualitative Spatial Reasoning developed from the above mentioned ideas of Leśniewski and Whitehead and it has become a basic ingredient in a variety of problems, e.g., in mobile robotics, see, e.g., Kuipers [34], Kuipers and Byun [35], Kuipers and Levitt [36]. Spatial reasoning which deals with objects like solids, regions etc., by necessity refers to and relies on mereological theories of concepts based on the opposition part–whole, see Gotts et al. [27].

For expressing relations among entities, mathematics proposes two basic languages: the language of set theory, based on the opposition element–set, where distributive classes of entities are considered as sets consisting of (discrete) atomic entities, and languages of mereology, for discussing entities continuous in their nature, based on the opposition part–whole. Due to continuous nature of spatial real objects, Spatial Reasoning relies to great extent on mereological theories of part, cf., Asher and Vieu [5], Asher et al. [4], Aurnague and Vieu [6], Cohn and Gotts [15], Gotts and Cohn [26], Cohn et al. [16], Galton [24], Smith [57], [56], Masolo and Vieu [47].

Qualitative Reasoning aims at studying concepts and calculi on them that arise often at early stages of problem analysis when one is refraining from qualitative or metric details, cf., Cohn [14]; as such it has close relations with design, cf., Booch [8] as well as planning stages, cf., Glasgow [25] of the model synthesis process.

Classical formal approaches to spatial reasoning, i.e., to representing spatial entities (points, surfaces, solids) and their features (dimensionality, shape, connectedness degree) rely on Geometry or Topology, i.e., on formal theories whose models are spaces (universes) constructed as sets of points; contrary to this approach, qualitative reasoning about space often exploits pieces of space (regions, boundaries, walls, membranes) and argues in terms of relations abstracted from a commonsense perception (like *connected*, *discrete from*, *adjacent*, *intersecting*). In this approach, points appear as ideal objects, e.g., ultrafilters of regions/solids as in Tarski [60].

Qualitative Spatial Reasoning has a wide variety of applications, among them, to mention only a few, representation of knowledge, cognitive maps and navigation tasks in robotics, see Kuipers, Kuipers and Byun, Kuipers and Levitt op. cit., op. cit., op. cit., as well as AISB97 [1], Arkin [3], Dorigo and Colombetti [18], Kortenkamp [33], Freksa [23], Geographical Information Systems and spatial databases including *Naive Geography*, see Frank and Campari [21], Frank and Kuhn [22], Hirtle and Frank [29], Egenhofer and Golledge [20], Mark [46], and in studies in semantics of orientational lexemes and in semantics of movement in Asher et al. [4] and Aurnague and Vieu [6].

Spatial Reasoning establishes a link between Computer Science and Cognitive Sciences, cf., Freksa [23] and it has close and deep relationships with philosophical and logical theories of space and time, cf., Reichenbach [50] , vanBenthem [7], Allen [2].

Any formal approach to Spatial Reasoning requires Ontology, which differs from the above exposed Ontology due to Leśniewski, and is based on established hierarchies of concepts, cf., Guarino [28], Smith [55], Casati et al. [11]. In reasoning with spatial objects, of primary importance is to develop an ontology of spatial objects, taking into account complexity of these objects.

The scheme for Connection calculus, presented above, sect. 5, has inspired many authors toward creation of a calculus on specified geometric objects which would implement connection predicate, although in a modified according to a given context form. A good example of such approach is the calculus of regions RCC (Region Connection Calculus), see Randell et al. [49]. Primitive geometric objects considered here are regions; this standpoint distinguishes RCC from the original setting by Clarke in [12] where points were primitives and connection C was interpreted as having a point in common. In RCC authors tend to interpret connection as the property of having a common point in closures of regions. Thus, RCC is essentially a calculus of regularly open regions, i.e., regions R such that $R = IntClR$: each region is the interior of its closure, see sect. 5. Hence, the framework adopted by authors of RCC is that of the model ROM.

0.10.1 On Region Connection Calculus

RCC adopts requirements CN1–CN3 of Connection calculus in Clarke [12], and defines other predicates in a similar way; we recall them here preserving the notation of RCC.

DISCONNECTED FROM(x)(y): $DC(x, y)$ *if and only if not* $C(x, y)$

IMPROPER PART OF(x)(y): $P(x, y)$ *if and only if for each* z, $C(z, y) \rightarrow C(z, x)$ *(*y *is an improper part of* x*)*

PROPER PART OF(x)(y): $PP(x, y)$ *if and only if* $P(x, y)$ *and not* $P(y, x)$ *(*y *is a proper part of* x*)*

EQUAL(x)(y): $EQ(x, y)$ *if and only if* $P(x, y)$ *and*$P(y, x)$ *(*y *is identical to* x*)*

OVERLAP(x)(y): $Ov(x, y)$ *if and only if there exists* z *such that* $P(x, z)$ *and* $P(y, z)$ *(*y *overlaps* x*)*

DISCRETE FROM(x)(y): $DR(x, y)$ *if and only if not* $Ov(x, y)$ *(*y *is discrete from* x*)*

PARTIAL OVERLAP(x)(y): $POv(x, y)$ *if and only if* $Ov(x, y)$ *and not* $P(x, y)$*, and not* $P(y, x)$ *(*y *partially overlaps* x*)*

EXTERNAL CONNECTED(x)(y): $EC(x, y)$ *if and only if* $C(x, y)$ *and not* $Ov(x, y)$ *(*y *is externally connected to* x*)*

TANGENTIAL PART OF(x)(y): $TPP(x, y)$ *if and only if* $PP(x, y)$ *and there exists* z *such that* $EC(x, z)$ *and* $EC(y, z)$ *(*y *is a tangential proper part of* x*)*

NON–TANGENTIAL PART OF(x)(y): $NTPP(x, y)$ *if and only if* $PP(x, y)$ *and not* $TPP(x, y)$ *(*y *is a non–tangential proper part of* x*)*

Each non–symmetric predicate X among the above is also accompanied by its inverse Xi (e.g., $TPP(x, y)$ by $TPPi(y, x)$).

Of these predicates, the eight: DC, EC, PO, EQ, TPP, NTPP, TPPi, NTPPi are shown to have the JEPD property (Jointly Exclusive and Pairwise Disjoint) and they form the fragment of RCC called the RCC8 calculus.

Due to topological assumptions, RCC has some stronger properties than Clarke's calculus of C. Witness, the two properties, not mentioned by us when discussing the model ROM.

The first is the extensionality of the Overlap predicate

If for each z $Ov(x, z) \leftrightarrow Ov(y, z)$*, then* $x = y$

Indeed, for regular open sets A, B, the condition that $Ov(x, z) \Leftrightarrow Ov(y, z)$

means that $ClA = ClB$, hence, $A = IntClA = IntClB = B$.

The second property concerns complementarity

If $PP(A,B)$, *then there exists* C *such that* $P(C,B)$ *and* $DR(C,A)$

Indeed, was $B \subseteq ClA$, we would have $ClB = ClA$, a fortiori, $B = A$, contrary to the assumptions. Hence, $B \setminus ClA \neq \emptyset$, and there exists, by regularity of X, a $C \in RO(X)$ such that $ClC \subseteq B \setminus ClA \neq \emptyset$.
One more consequence of topological assumptions is that the region and its complement are externally connected, as their closures do intersect,

$EC(x, -x)$ *for each region* x

For practical reasons, RCC8 is presented in the form of the *transition table*: a table in which for entries $R_1(x,y)$ and $R_2(y,z)$ a result $R_3(x,z)$ is given, see Egenhofer [19]. The transition table for RCC8 is shown in the table of Fig. 0.1.

Fig. 0.1 Transition table for RCC8 calculus

-	DC	EC	PO	TPP	NTPP	TPPi	NTPPi
DC	-	DR,PO,PP	DR,PO,PP	DR,PO,PP	DC	DC	
EC	DR,PO,PPi	DR,PO,TPP,TPi	DR,PO,P	EC,PO,PP	PO,PP	DR	DC
PO	DR,PO,PPi	DR,PO,PPi	-	PO,PP	PO,P	DR,PO,PPi	DR,PO,PPi
TPP	DC	DR	DR,PO,PP	PP	NTPP	DR,PO,PP	-
NTPP	DC	DC	DR,O,PP	NTPP	NTPP	DR,PO,PP	-
TPPi	DR,PO,PPi	EC,PO,PPi	PO,PPi	PO,TPP,TPi	PO,PP	PPi	NTPPi
NTPPi	DR,PO,PPi	PO,PPi	PO,PPi	PO, PPi	0	NTPPi	NTPPi

RCC8 allows for additional predicates characterizing shape, connectivity, see Gotts et al. [27] and regions with vague boundaries ("the egg–yolk" approach), see Cohn and Gotts [15].

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