Chapter 6

Rough Mereology

A scheme of mereology, introduced into a collection of objects, see Ch. 5, sets an exact hierarchy of objects of which some are (exact) parts of others; to ascertain whether an object is an exact part of some other object is in practical cases often difficult if possible at all, e.g., a robot sensing the environment by means of a camera or a laser range sensor, cannot exactly perceive obstacles or navigation beacons. Such evaluation can be done approximately only and one can discuss such situations up to a degree of certainty only. Thus, one departs from the exact reasoning scheme given by decomposition into parts to a scheme which approximates the exact scheme but does not observe it exactly.

Such a scheme, albeit its conclusions are expressed in an approximate language, can be more reliable, as its users are aware of uncertainty of its statements and can take appropriate measures to fend off possible consequences.

Imagine two robots using the language of connection mereology for describing mutual relations; when endowed with touch sensors, they can ascertain the moment when they are connected; when a robot has as a goal to enter a certain area, it can ascertain that it connected to the area or overlapped with it, or it is a part of the area, and it has no means to describe its position more precisely.

Introducing some measures of overlapping, in other words, the extent to which one object is a part to the other, would allow for a more precise description of relative position, and would add an expressional power to the language of mereology. Rough mereology answers these demands by introducing the notion of a *part to a degree* with the degree expressed as a real number in the interval $[0, 1]$. Any notion of a part by necessity relates to the general idea of *containment*, and thus the notion of a part to a degree is related to the idea of *partial containment* and it should preserve the essential intuitive postulates about the latter.

The predicate of a part to a degree stems ideologically from and has as one of motivations the predicate of an element to a degree introduced by L. A. Zadeh as a basis for fuzzy set theory [24]; in this sense, rough mereology is to mereology as the fuzzy set theory is to the naive set theory. To the rough set theory, owes rough mereology the interest in concepts as objects of analysis.

The primitive notion of rough mereology is the notion of a *rough inclusion* which is a ternary predicate $\mu(x, y, r)$ where x, y are *objects* and $r \in [0, 1]$, read '*the object x is a part to degree at least of r to the object y*'. Any rough inclusion is associated with a mereological scheme based on the notion of a part by postulating that $\mu(x, y, 1)$ is equivalent to $ingr(x, y)$, where the ingredient relation is defined by the adopted mereological scheme, see Ch. 5, sect. 1. Other postulates about rough inclusions stem from intuitions about the nature of partial containment; these intuitions can be manifold, a fortiori, postulates about rough inclusions may vary. In our scheme for rough mereology, we begin with some basic postulates which would provide a most general framework. When needed, other postulates, narrowing the variety of possible models, can be introduced.

0.1 Rough inclusions

We have already stated that a rough inclusion is a ternary predicate $\mu(x, y, r)$. We assume that a collection of objects is given, on which a part relation π is introduced with the associated ingredient relation *ingr*. We thus apply inference schemes of mereology due to Leśniewski, presented in Ch. 5, sect. 1.

Predicates $\mu(x, y, r)$ were introduced in Polkowski and Skowron [18], [19]; they satisfy the following postulates, relative to a given part relation π and the induced by π relation *ingr* of an ingredient, on a set U of entities

RINC1 $\mu(x, y, 1) \Leftrightarrow ingr(x, y)$

This postulate asserts that parts to degree of 1 are ingredients.

RINC2 $\mu(x, y, 1) \Rightarrow \forall z[\mu(z, x, r) \Rightarrow \mu(z, y, r)]$

This postulate does express a feature of partial containment that a 'bigger' object contains a given object 'more' than a 'smaller' object. It can be called a *monotonicity condition* for rough inclusions.

RINC3 $\mu(x, y, r) \wedge s < r \Rightarrow \mu(x, y, s)$

This postulate specifies the meaning of the phrase 'a part to a degree at least of r'.

From postulates RINC1–RINC3, and known properties of the ingredient predicate, see Ch. 5, sect. 1, some consequences follow.

Proposition 1. *The immediate consequences of postulates RINC1–RINC3 are*

1. $\mu(x, x, 1)$*;*

2. $\mu(x, y, 1) \wedge \mu(y, z, 1) \Rightarrow \mu(x, z, 1)$;

3. $\mu(x, y, 1) \wedge \mu(y, x, 1) \Leftrightarrow x = y$;

4. $x \neq y \Rightarrow \neg\mu(x, y, 1) \vee \neg\mu(y, x, 1)$;

5. $\forall z \forall r[\mu(z, x, r) \Leftrightarrow \mu(z, y, r)] \Rightarrow x = y$.

Proof. Property 1 follows by by RINC1 and Property 1 of Proposition 1, Ch. 5, Property 2 is implied by transitivity of ingredient, Property 2 of Proposition 1, Ch. 5, and RINC1. Property 3 follows by RINC1 and Property 3 of Proposition 1, Ch. 5, Property 4 holds by Property 3 of Proposition 1, Ch. 5. Lastly, Property 5 is true by Property 3 of Proposition 1, Ch. 5 □

Property 5 above may be regarded as an *extensionality postulate* in rough mereology.

By a *model* for rough mereology, we mean a quadruple

$$M = (V_M, \pi_M, ingr_M, \mu_M)$$

where V_M is a set with a part relation $\pi_M \subseteq V_M \times V_M$, the associated ingredient relation $ingr_M \subseteq V_M \times V_M$, and a relation $\mu_M \subseteq V_M \times V_M \times [0, 1]$ which satisfies RINC1–RINC3.

We now describe some models for rough mereology which at the same time give us methods by which we can define rough inclusions, see Polkowski [10]–[14].

0.2 Rough inclusions: Residual models

We begin with continuous t–norms on the unit interval [0,1], see Ch. 4, sect. 7.

We recall that it follows from results in Mostert and Shields [7] and Faucett [2], see Ch. 4, sect. 9, cf., Hájek [3], that the structure of a continuous t–norm T depends on the set $F(T)$ of idempotents of T, i.e, values x such that $T(x, x) = x$; we denote with O_T the countable family of open intervals $A_i \subseteq [0, 1]$ with the property that $\bigcup_i A_i = [0, 1] \setminus F(T)$.

Then, see Ch. 4, Proposition 18,

Proposition 2. *$T(x, y)$ is an isomorph to either $L(x, y)$ or $P(x, y)$ when $x, y \in A_i$ for some i, and $T(x, y) = min\{x, y\}$, otherwise.*

We recall, see Ch. 4, sect. 10, that, for a continuous t–norm $T(x, y)$, the *residual implication, residuum*, $x \Rightarrow_T y$ is defined by the condition

$$x \Rightarrow_T y \geq z \Leftrightarrow T(x, z) \leq y \tag{0.1}$$

It follows that $x \Rightarrow_T y = 1$ if and only if $x \leq y$, as $T(x, x) \leq x$ for each continuous t–norm T.

For a continuous t–norm T, we define a relation $\mu_T \subseteq [0, 1]^3$ by means of

$$RIT\ \mu_T(x, y, r) \Leftrightarrow x \Rightarrow_T y \geq r \tag{0.2}$$

Proposition 3. *The quadruple* $M(T) = ([0, 1], <, \leq, \mu_T)$ *is a model for rough mereology induced by the residuum of the t–norm* T.

Proof. First, let us make positive that μ_T satisfies RINC1–RINC3. For RINC1, $\mu_T(x, y, 1)$ means that $x \Rightarrow_T y = 1$, hence, $x \leq y$, i.e., $ingr_M(x, y)$. For RINC2, assume that $\mu_T(x, y, 1)$ and $\mu_T(z, x, r)$, hence (i) $x \leq y$ (ii) $z \Rightarrow_T x \geq r$, i.e., by (1), (iii) $T(z, r) \leq x$. By (i), (iii), $T(z, r) \leq y$, hence, by (1), $z \Rightarrow_T y \geq r$. RINC3 follows by (0.2) □

Clearly, the underlying part relation in the above proposition is the strict ordering $<$ and the ingredient relation is $\leq$.

In particular important cases, of t–norms L, P, M, one obtains the specific models M_L, M_P, M_M. In each model $M(T)$, $\mu(x, y, 1) \Leftrightarrow x \leq y$, hence, we recall below only the case when $x > y$, see Ch. 4. sect. 10.

In case of the Lukasiewicz t–norm L, we have, $x \Rightarrow_L y = min\{1, 1-x+y\}$; accordingly,

$$\mu_L(x, y, r) \Leftrightarrow min\{1, 1 - x + y\} \geq r \tag{0.3}$$

equivalently for $x > y$

$$\mu_L(x, y, r) \Leftrightarrow x - y \leq 1 - r \tag{0.4}$$

From (0.4), we can extract a transitivity rule

Proposition 4. *From* $\mu_L(x, y, r), \mu_L(y, z, s)$ *it follows that* $\mu_L(x, z, L(r, s))$.

Proof. It suffices to consider the case when $x > y$ and $y > z$, by (0.4), we have $x - y \leq 1 - r$ and $y - z \leq 1 - s$, hence, $x - z \leq 1 - (r + s - 1)$, i.e., $\mu_L(x, y, L(r, s))$ □

The proposition does encompass as well cases when $x \leq y$ or $y \leq z$.

For the product t–norm P, if $x > y$ then we have $x \Rightarrow_P y = \frac{y}{x}$; hence

$$\mu_P(x, y, r) \Leftrightarrow \frac{y}{x} \geq r \tag{0.5}$$

The transitivity rule follows.

Proposition 5. *$\mu_P(x, y, r), \mu_P(y, z, s)$ imply $\mu_P(x, zP(r, s)$.*

Proof. The conclusion follows by (0.5) from $\frac{y}{x} \geq r$, $\frac{z}{y} \geq s$, which imply $\frac{z}{x} \geq r{\cdot}s$. Cases when either $x \leq y$ or $y \leq z$, or $x \leq z$ are discussed analogously □

Finally, we consider the minimum t–norm M with $x \Rightarrow_P y = y$, hence

$$\mu_M(x, y, r) \Leftrightarrow y \geq r \qquad (0.6)$$

Proposition 6. *If $\mu_M(x, y, r), \mu_M(y, z, s)$, then $\mu_M(x, z, min\{r, s\})$.*

Proof. From $\mu_M(x, y, r), \mu_M(y, z, s)$ it follows that $min\{x, r\} \leq y$, $min\{y, s\} \leq z$, hence, $min\{x, min\{r, s\}\} \leq z$, i.e., $\mu_M(x, z, M(r, s))$ □

The partial results in Propositions 0.3, 0.5, 0.6, can be generalized to,

Proposition 7. *For each continuous t–norm T, the transitivity rule is obeyed by the rough inclusion μ_T: if $\mu_T(x, y, r), \mu_T(y, z, s)$, then $\mu_T(x, z, T(r, s))$.*

Proof. $\mu_T(x, y, r)$ is equivalent to $T(x, r) \leq y$, and, $\mu_T(y, z, s)$ is equivalent to $T(y, s) \leq z$. By coordinate–wise monotonicity of T, it follows that $T(T(x, r), s) \leq z$, and, by associativity of T, one obtains $T(x, T(r, s)) \leq z$, hence, $\mu_T(x, z) \geq T(r, s)$ □

Let us also put for the record the observation

Proposition 8. *For each r, and every continuous t–norm T, the set $D_T(r) = \{(x, y) : \mu_T(x, y, r)\}$ is a closed subset of the unit square $[0, 1]^2$; sets $D_L(r)$, $D_M(r)$ are moreover convex.*

Proof. $\mu(x, y, r)$ is equivalent to $T(x, r) \leq y$ and the result follows by continuity of T □

To carry further a topological analysis of rough inclusions of the form μ_T, we consider for a given $r \in [0, 1]$, the set $D_T(r)$. It is equal, by RINC3, to the set $D_T(r^+) = \{(x, y) : \exists s \geq r \mu(x, y, s)\}$. The structure of $D_T(r^+)$ can be revealed by RINC2.

Proposition 9. *If $(x, y) \in D_T(r^+)$, then the segment $\{x\} \times [0, y] \subseteq D_T(r^+)$.*

Proof. By RINC2, if $y' \leq y$, and $\mu(x, y', r)$ then $\mu(x, y, r)$ □

We know, by Proposition 7, that each μ_T is transitive. As such it satisfies

Proposition 10. *If $\mu_T(x, y, r)$ and $ingr_M(z, x)$, then $\mu_T(z, y, r)$.*

Proof. As $ingr_M(z, x)$, it follows that $\mu_T(z, x, 1)$ by RINC1, hence, by transitivity, $\mu_T(z, y, T(1, r))$, i.e., $\mu_T(z, y, r)$ □

The structure theorem for μ_T can be strengthened,

Proposition 11. *If* $(x, y) \in D_T(r^+)$*, then the Cartesian product of segments* $[0, x] \times [0, y] \subseteq D_T(r^+)$*.*

As for each r, the set $D_T(r^+)$ is closed, the topological characterization of μ_T follows,

Proposition 12. μ_T *is upper–semicontinuous, in the sense that the set* $\{(x, y) : \mu(x, y, r)$ *is closed for each* $r \in [0, 1]$*.*

We may observe that μ_T is in fact a ternary relation, hence, the continuity property of μ_T can be expressed – more accurately even – when μ_T is regarded as a many–valued mapping, i.e., given (x, y), the value of $\mu_T(x, y, r)$, by RINC3, is an interval $[0, r_{max}(x, y)]$. Given $s \in [0, 1]$, the following holds

Proposition 13. *The set* $E_T(s) = \{(x, y) \in [0, 1]^2 : r_{max}(x, y) < s\}$ *is open for each* s*, i.e.,* μ_T *is upper–semicontinuous as a many–valued mapping.*

Proof. $E_T(s)$ is open as the complement to the closed set $D_T(s^+)$ □

Proposition 17 in Ch. 4, implies the converse to the above Proposition 13, see Ch. 2, sect. 12, for the notion of semi–continuity for multi–valued mappings,

Proposition 14. *Each rough inclusion* $\mu(x, y, r)$*, with* $x, y, r \in [0, 1]$*, which is non–increasing in the first coordinate, non–decreasing in the second coordinate and upper semi–continuous as a multi–valued mapping is of the form* $\mu_T(x, y, r)$ *for some t–norm* T*.*

Proof. It follows from assumptions that the function $f_\mu(x, y) = r_{max}$ satisfies assumptions of Proposition 17 in Ch. 4, hence, $f_\mu(x, y) = x \Rightarrow_T y$ for some t–norm T, and our thesis follows □

We now turn to Archimedean t–norms of Ch. 4 in search for new rough inclusions.

0.3 Rough inclusions: Archimedean models

We recall that a continuous t–norm T is *Archimedean*, see Ch. 4, sect. 8, when $T(x, x) < x$ for each $x \in (0, 1)$. Thus, the only idempotents of T are $0, 1$.

We also recall, see Ch. 4, sect. 9, that Archimedean t–norms admit a functional characterization, a very special case of the general Kolmogorov [5] theorem, viz., for any Archimedean t–norm T, the following functional equation holds

$$T(x, y) = g_T(f_T(x) + f_T(y)) \tag{0.7}$$

where the function $f_T : [0,1] \to R$ is continuous decreasing with $f_T(1) = 0$, and $g_T : R \to [0,1]$ is the pseudo–inverse to f_T, i.e., $g \circ f = id$, see Ling [6].

We consider two Archimedean t–norms: L and P. Their representations are

$$f_L(x) = 1 - x;\ g_L(y) = 1 - y \tag{0.8}$$

and

$$f_P(x) = exp(-x);\ g_P(y) = -ln\ y \tag{0.9}$$

For an Archimedean t–norm T, we define the rough inclusion μ^T on the interval $[0,1]$ by means of

$$ARI\ \mu^T(x,y,r) \Leftrightarrow g_T(|x-y|) \geq r \tag{0.10}$$

equivalently,

$$\mu^T(x,y,r) \Leftrightarrow |x-y| \leq f_T(r) \tag{0.11}$$

It follows from (0.11), that

Proposition 15. *The relation μ^T satisfies conditions RINC1–RINC3 with ingr as identity =.*

Proof. For RINC1: $\mu^T(x,y,1)$ if and only if $|x-y| \leq f_T(1) = 0$, hence, if and only if $x = y$. This implies RINC2. In case $s < r$, and $|x-y| \leq f_T(r)$, one has $f_T(r) \leq f_T(s)$ and $|x-y| \leq f_T(s)$ □

Specific recipes are: for μ^L

$$\mu^L(x,y,r) \Leftrightarrow |x-y| \leq 1 - r \tag{0.12}$$

and for μ^P,

$$\mu^P(x,y,r) \Leftrightarrow |x-y| \leq -ln\ r \tag{0.13}$$

The counterpart of Proposition 7 obeys for Archimedean rough inclusions,

Proposition 16. *For each Archimedean t–norm T, if $\mu^T(x,y,r)$ and $\mu^T(y,z,s)$, then $\mu^T(x,z,T(r,s))$.*

Proof. Assume $\mu^T(x,y,r)$ and $\mu^T(y,z,s)$, i.e., $|x-y| \leq f_T(r)$ and $|y-z| \leq f_T(s)$. Hence, $|x-z| \leq |x-y| + |y-z| \leq f_T(r) + f_T(s)$, hence, $g_T(|x-z|) \geq g_T(f_T(r) + f_T(s)) = T(r,s)$, i.e., $\mu^T(x,z,T(r,s))$ □

It may be worth–while to relate the residual and archimedean approaches to rough inclusions. For an Archimedean t–norm $T(x,y) = g(f(x) + f(y))$, one can easily calculate that in case $x > y$, the residual implication $x \Rightarrow_T y$ can be expressed as $g(f(y) - f(x))$, which, e.g., for the t–norm L comes down to $g(y - x)$, i.e., a formula (0.10), albeit with different mereological context.

From Proposition 2, a general result on structure of μ_T can be inferred

Proposition 17. *For a continuous t–norm T, in notation of Proposition 2, a rough inclusion μ_T can be defined as follows,*

$$\mu_T(x,y,r) \text{ } if \text{ } and \text{ } only \text{ } if \begin{cases} \mu_T(x,y,r) \text{ } for \text{ } x,y \in A_i, T = L \text{ } or \text{ } T = P \\ \mu^M(x,y,r) \text{ } otherwise \end{cases} \tag{0.14}$$

For future applications, we need rough inclusions on sets.

0.4 Rough inclusions: Set models

Consider now a finite set X along with the family 2^X of its subsets. We define a rough inclusion $\mu^S \subseteq 2^X \times 2^X \times [0,1]$, by letting

$$\mu^S(A,B,r) \Leftrightarrow \frac{|A \cap B|}{|A|} \geq r \tag{0.15}$$

where $|X|$ denotes the cardinality of X.

Then, we observe

Proposition 18. *The relation μ^S is a rough inclusion with the associated ingredient relation of containment $\subseteq$ and the part relation being the strict containment $\subset$.*

Proof. Clearly, $\mu^S(A,B,1)$ if and only if $A \subseteq B$; in that case, for every $Z \subseteq X$, $Z \cap A \subseteq Z \cap B$, hence, $\mu^S(Z,A,r)$ implies $\mu^S(Z,B,R)$ for every r. RINC3 is obviously satisfied □

For containment on sets, there is no transitivity rule.

0.5 Rough inclusions: Geometric models

The set model can be modified in a geometric context; consider, e.g., a Euclidean space of a finite dimension E with objects as compact convex regions; this environment is usually applied, e.g., in modeling problems of intelligent mobile robotics, where compact convex regions model robots as well as obstacles.

For regions A, B, we define a rough inclusion μ^G by means of

$$\mu^G(A,B,r) \Leftrightarrow \frac{||A \cap B||}{||A||} \geq r \tag{0.16}$$

where $||A||$ denotes the area (the Lebesgue measure) of the region A.

We have

Proposition 19. μ^G *is a rough inclusion with the containment* $\subseteq$ *as the associated ingredient relation.*

Proof. Clearly, if $||A \cap B|| = ||A||$, then $A \subseteq B$, as $A \setminus B \neq \emptyset$ implies $||A \setminus B|| \neq 0$ for compact convex regions A, B. The rest is already standard □

Again, as with μ^S, no transitivity rule can be given here.

0.6 Rough inclusions:Information models

An important domain where rough inclusions will play a dominant role in our analysis of reasoning by means of parts is the realm of information systems, see Ch. 4., sect. 2. We will define information rough inclusions denoted with a generic symbol μ^I. We assume that *indiscernibility = identity*, i.e., each indiscernibility class, see Ch. 4., sect. 2, is represented by a unique object.

In order to define μ^I, for each pair $u, v \in U$, we define the set

$$DIS(u, v) = \{a \in A : a(u) \neq a(v)\} \tag{0.17}$$

We begin with an Archimedean rough inclusion T, and we define a rough inclusion μ^I_T by means of

$$\mu^I_T(u, v, r) \Leftrightarrow g_T(\frac{|DIS(u, v)|}{|A|}) \geq r \tag{0.18}$$

Then, it is true that

Proposition 20. μ^I_T *is a rough inclusion with the associated ingredient relation of identity and the part relation empty.*

Proof. We have $g_L(y) = 1 - y$ and $g_P(y) = exp(-y)$, i.e, $g^{-1}(1) = 0$ in either case, so $\mu^I_T(u, v, 1)$ implies $DIS(u, v) = \emptyset$, hence,$IND_A(u, v)$ which, by our assumption, is the identity =. This verifies the condition RINC1, the rest follows along standard lines □

Rough inclusions defined by means of (0.18) will be called *archimedean information rough inclusions* (shortened to *airi's*).

In specific cases, for the Łukasiewicz t–norm, L, the *airi* μ^I_L is given by means of the formula

$$\mu^I_L(u, v, r) \Leftrightarrow 1 - \frac{|DIS(u, v)|}{|A|} \geq r \tag{0.19}$$

We introduce the set $IND(u, v) = A \setminus DIS(u, v)$. With its help, we obtain a new form of (0.19)

$$\mu_L^I(u,v,r) \Leftrightarrow \frac{|IND(u,v)|}{|A|} \geq r \tag{0.20}$$

The formula (0.20) witnesses that the reasoning based on the rough inclusion μ_L^I is the probabilistic one. At the same time, we have given a logical proof for formulas like (0.20) that are very frequently applied in Data Mining and Knowledge Discovery, also in rough set methods in those areas, see, e.g., Kloesgen and Zytkow [4]. It also witness that μ_L^I is a generalization of indiscernibility relation to the relation of partial indiscernibility.

In case of the product t–norm P, the formula (0.18) specifies to

$$\mu_P^I(u,v,r) \Leftrightarrow exp(-\frac{|DIS(u,v)|}{|A|}) \geq r \tag{0.21}$$

We can prove, for any *airi* μ_T^I, the transitivity property in the form, see Polkowski [11]

$$If\ \mu_T^I(u,v,r)\ and\ \mu_T^I(v,w,s),\ then\ \mu_T^I(u,w,T(r,s)) \tag{0.22}$$

Proof. We begin with the observation that

$$DIS(u,w) \subseteq DIS(u,v) \cup DIS(v,w) \tag{0.23}$$

hence

$$\frac{|DIS(u,w)|}{|A|} \leq \frac{|DIS(u,v)|}{|A|} + \frac{|DIS(v,w)|}{|A|} \tag{0.24}$$

We let

$$\begin{cases} g_T(\frac{|DIS(u,v)|}{|A|}) = r \\ g_T(\frac{|DIS(v,w)|}{|A|}) = s \\ g_T(\frac{|DIS(u,w)|}{|A|}) = t \end{cases} \tag{0.25}$$

Then

$$\begin{cases} \frac{|DIS(u,v)|}{|A|} = f_T(r) \\ \frac{|DIS(v,w)|}{|A|} = f_T(s) \\ \frac{|DIS(u,w)|}{|A|} = f_T(t) \end{cases} \tag{0.26}$$

Finally, by (0.24)

$$f_T(t) \leq f_T(r) + f_T(s) \tag{0.27}$$

hence

$$t = g_T(f_T(t)) \geq g_T(f_T(r) + f_T(s)) = T(r,s) \tag{0.28}$$

witnessing $\mu_T(u,w,T(r,s))$. This concludes the proof □

We would like as well to exploit, in the context of information systems, residual implications that served us well in case of the unit interval. The formalism of descriptor logic, see Ch. 4, sect. 2, however, gives us not many possibilities for characterization of objects, save sets DIS and IND. To come as close as possible to requirements for a rough inclusion, we select an object $s \in U$ referred to as a *standard object, or, a pattern.* From application point of view, s may be, e.g., the best classified case, or, the pattern set as ideal.

For any object $x \in U$, we let

$$IND(x, s) = \{a \in A : a(x) = a(s)\} \tag{0.29}$$

For a continuous t–norm T, we define a rough inclusion $\mu_T^{IND,s}$, under generic name of *indri*, by letting

$$INDRI\ \mu_T^{IND,s}(u, v, r) \Leftrightarrow \frac{|IND(u, s)|}{|A|} \Rightarrow_T \frac{|IND(v, s)|}{|A|} \geq r \tag{0.30}$$

We obtain a rough inclusion, indeed

Proposition 21. *The indri $\mu_T^{IND,s}$ satisfies conditions RINC1–RINC3 with ingredient relation $ingr_{IND}(u, v) \Leftrightarrow |IND(u, s)| \leq |IND(v, s)|$ and identity defined as $u =_{IND} v \ \Leftrightarrow |IND(u, s)| = |IND(v, s)|$.*

Proof. $\mu_T^{IND,s}(u, v, 1)$ is equivalent to $|IND(u, s)| \leq |IND(v, s)|$, i.e., to $ingr_{IND}(u, v)$. That is for RINC1. For RINC2, assume that $|IND(u, s)| \leq |IND(v, s)|$, and $\frac{|IND(w,s)|}{|A|} \Rightarrow_T \frac{|IND(u,s)|}{|A|} \geq r$. Then, by monotonicity of T, $\frac{|IND(w,s)|}{|A|} \Rightarrow \frac{|IND(v,s)|}{|A|} \geq r$ follows. RINC3 is obviously satisfied □

The rough inclusion $\mu_T^{IND,s}$ specifies to distinct formulas for three basic t–norms, L, P, M. The specific formulas are, for $r < 1$,

$$\mu_L^{IND,s}(u, v, r) \Leftrightarrow 1 - |IND(u, s)| + |IND(v, s)| \geq r \cdot |A| \tag{0.31}$$

$$\mu_P^{IND,s}(u, v, r) \Leftrightarrow |IND(v, s)| \geq r \cdot |IND(u, s)| \tag{0.32}$$

and,

$$\mu_M^{IND,s}(u, v, r) \Leftrightarrow |IND(v, s)| \geq r \cdot |A| \tag{0.33}$$

We may observe that the quotient $\frac{|IND(u,s)|}{|A|}$ is the value of the *(reduced modulo A) Hamming distance between u and s*, and $ingr_{IND}(u, v)$ means in this context that v is ‘closer’ to the standard s than u.

This, as well as property (0.4), suggests usage of metrics in definitions of rough inclusions; actually, metrics were used in Poincaré [9] to give an example of a tolerance relation, see Ch. 1, sect. 13. We will build on this idea.

0.7 Rough inclusions: Metric models

We consider a metric space (X, ρ) and we let

$$\mu_\rho(x, y, r) \Leftrightarrow \rho(x, y) \leq 1 - r \tag{0.34}$$

We check that μ_ρ is a rough inclusion.

Proposition 22. *The relation μ_ρ satisfies conditions RINC1–RINC3 with the ingredient relation of identity $=$ and the part relation empty.*

Proof. $\mu_\rho(x, y, 1)$ means that $\rho(x, y) \leq 0$, i.e., $x = y$. This proves RINC1 and RINC2, RINC3 follow □

The rough inclusion μ_ρ obeys a transitivity law

Proposition 23. *If $\mu_\rho(x, y, r)$ and $\mu_\rho(y, z, s)$, then $\mu_\rho(x, z, L(r, s))$.*

Proof. From $\rho(x, y) \leq 1-r$ and $\rho(y, z) \leq 1-s$, by the triangle inequality for ρ, it follows that $\rho(x, z) \leq (1-r)+(1-s)$, i.e., $\rho(x, z) \leq 1-(1-r+s) = 1-L(r, s)$, hence, $\mu_\rho(x, z, L(r, s))$ □

In particular, we may consider the *discrete metric* $D(x, y)$ defined as

$$D(x, y) = \begin{cases} 1 \text{ } in \text{ } case \text{ } x \neq y \\ 0 \text{ } in \text{ } case \text{ } x = y \end{cases} \tag{0.35}$$

The rough inclusion μ_D satisfies the following

Proposition 24. *1. $\mu_D(x, y, 1) \Leftrightarrow x = y$.*
2. For $r < 1$, $\mu_D(x, y, r) \Leftrightarrow \mu_D(x, y, 0) \Leftrightarrow x \neq y$.

We have produced a *two–valued* rough inclusion, which we may justly call the *discrete rough inclusion.*

We conclude our review of known to us types of rough inclusions with a 3–valued rough inclusion.

0.8 Rough inclusions: A 3–valued rough inclusion on finite sets

We define a 3–valued rough inclusion μ_3 by formulas

$$\mu_3(A, B, 1) \Leftrightarrow A \subseteq B \tag{0.36}$$

$$\mu_3(A, B, 1/2) \Leftrightarrow A \triangle B \neq \emptyset, \tag{0.37}$$

and

$$\mu_3(A, B, 0) \Leftrightarrow A \cap B = \emptyset \tag{0.38}$$

Symmetric rough inclusions offer technical advantage over non–symmetric ones; the possibility of inverting roles of objects allows for better control of mutual relationships between pairs of objects and from technical point of view offers a considerable advantage. Symmetric by definitions are rough inclusions of sects. 4, 6, 8.

0.9 Symmetrization of rough inclusions

Assume μ a transitive non–symmetric rough inclusion; by μ^{sym}, we denote the symmetrized version of μ defined by letting

$$\mu^{sym}(x, y, r) \Leftrightarrow \mu(x, y, r) \wedge \mu(y, x, r) \tag{0.39}$$

Proposition 25. *Properties of μ^{sym} are summed in*

1. μ^{sym} is a rough inclusion with the ingredient relation of identity =;

2. μ^{sym} is a transitive symmetric rough inclusion.

Proof. $\mu^{sym}(x, y, 1)$ means $\mu(x, y, 1)$ and $\mu(y, x, 1)$, hence, $ingr(x, y)$ and $ingr(y, x)$ thus $x = y$. RINC2, RINC3 follow easily. Symmetry is obvious by definition and transitivity follows easily □

Here our discussion of basics of rough inclusions ends and we proceed to the analysis of structures which can be defined in rough mereological universes.

0.10 Mereogeometry

Elementary geometry was defined by Alfred Tarski in His Warsaw University lectures in the years 1926–27 as a part of Euclidean geometry which can be described by means of 1st order logic. Alfred Tarski proposed an axiomatization of elementary geometry, and many others including David Hilbert, Moritz Pasch, Eugenio Beltrami proposed also some axiomatizations of geometry.

There are two main aspects in formalization of geometry: one is metric aspect dealing with the distance underlying the space of points which carries geometry and the other is affine aspect taking into account linear relations.

In Tarski axiomatization, Tarski [23], the metric aspect is expressed as a relation of equidistance (congruence) and the affine aspect is expressed by means of the betweenness relation. The only logical predicate required is the identity $=$.

We recall here the Tarski formalism, although it is not our goal to discuss elementary geometry; we include this for completeness' sake as well as to be used in the sequel.

Equidistance relation denoted $Eq(x, y, u, z)$ (or, as a congruence: $xy \equiv uz$) means that the distance from x to y is equal to the distance from u to z (pairs x, y and u, z are equidistant) is subject to requirements

1. *Eq–reflexivity*: $Eq(x, y, y, x)$;

2. *Eq–identity*: *If* $Eq(x, y, z, z)$, *then* $x = y$;

3. *Eq–transitivity*: *If* $Eq(x, y, u, z)$ *and* $Eq(x, y, v, w)$, *then* $Eq(u, z, v, w)$.

Betweenness relation, denoted $B(x, y, z)$, (y is between x and z) is required to satisfy the requirements

1. *B–identity*: *If* $B(x, y, x)$, *then* $x = y$;

2. *B–Pasch axiom*: *If* $B(x, u, z)$ *and* $B(y, v, z)$, *then there is some* a *such that* $B(u, a, y)$ *and* $B(v, a, x)$;

3. *B–continuity*: *Let* $\phi(x)$ *and* $\psi(y)$ *be first–order formulas in which objects* a, b *do not occur as free, and,* x *is not free in* $\psi(y)$ *and* y *is not free in* $\phi(x)$. *If there is* a *such that for each pair* x, y *[from* $\phi(x)$ *and* $\psi(y)$ *it follows that* $B(a, x, y)$*], then there is* b *such that for each pair* x, y *[from* $\phi(x)$ *and* $\psi(y)$ *it follows that* $B(b, x, y)$*];*

4. *B–lower dimension*: *For some triple* a, b, c *[not* $B(a, b, c)$ *or not* $B(b, c, a)$ *or not* $B(c, a, b)$*].*

Requirements concerning mutual relations between the two predicates are following

1. *B, Eq–upper dimension*: *If* $Eq(x, u, x, v)$ *and* $Eq(y, u, y, v)$ *and* $Eq(z, u, z, v)$ *and* $u \neq v$, *then* $B(x, y, z)$ *or* $B(y, z, x)$, *or* $B(z, x, y)$;

2. *B, Eq–parallel postulate*: *If* $B(x, y, w)$ *and* $Eq(x, y, y, w)$, $B(x, u, v)$, $Eq(x, u, u, v)$, $B(y, u, z)$ *and* $Eq(y, u, z, u)$, *then* $Eq(y, z, v, w)$;

3. *B, Eq–five segment postulate*: *If* $(x \neq y)$ *and* $B(x, y, z)$, $B(x', y', z')$, $Eq(x, y, x', y')$, $Eq(y, z, y', z')$, $Eq(x, u, x', u')$, $Eq(y, u, y', u')$, *then* $Eq(z, u, z', u')$;

4. *B, Eq–segment extension postulate*: *There exists* a, *such that* $B(w, x, a)$ *and* $Eq(x, a, y, z)$.

Van Benthem [1] took up the subject proposing a version of betweenness predicate based on the nearness predicate which was a departing point for rough mereological geometry, see Polkowski and Skowron [20], and has served us in our definition of robot formations, see Ch. 8.

We are interested in introducing into the mereological world defined by μ of a geometry in whose terms it will be possible to express spatial relations among objects; a usage for this geometry has been found in navigation and control tasks of mobile robotics, see Polkowski and Osmialowski [16], [17], Osmiałowski [8], Polkowski and Szmigielski [21], Szmigielski [22].

We first introduce a notion of a distance κ, induced by a rough inclusion μ, see Polkowski and Skowron [20], and applications in Polkowski and Osmialowski [16], [17], Osmiałowski [8], Polkowski and Szmigielski [21], Szmigielski [22]

$$\kappa(X, Y) = min\{max\ r, max\ s :\ \mu(X, Y, r), \mu(Y, X, s)\} \tag{0.40}$$

Observe that the mereological distance differs essentially from the standard distance: the closer are objects, the greater is the value of κ: $\kappa(X, Y) = 1$ means $X = Y$ whereas $\kappa(X, Y) = 0$ means that X, Y are either externally connected or disjoint, no matter what is the Euclidean distance between them.

The notion of *betweenness in the Tarski sense* $T(Z, X, Y)$ (read: Z is between X and Y), due to Tarski [23], is

$$T(Z, X, Y) \Leftrightarrow \text{for each region W, } \kappa(Z, W) \in [\kappa(X, W), \kappa(Y, W)] \tag{0.41}$$

Here, $[a, b]$ means the non–oriented interval with endpoints a, b.

Proposition 26. *The relation* T *satisfies the basic properties resulting from axioms of elementary geometry of Tarski [23] for the notion of betweenness*

1. *TB1 $T(Z,X,X)$ if and only if $Z = X$ (identity);*

2. *TB2 $T(V,U,W)$ and $T(Z,V,W)$ imply $T(V,U,Z)$ (transitivity);*

3. *TB3 $T(V,U,Z)$, $T(V,U,W)$ and $U \neq V$ imply $T(Z,U,W)$ or $T(W,U,Z)$ (connectivity).*

Proof. Indeed, by means of κ, the properties of betweenness in our context are translated into properties of betweenness in the real line which hold by the Tarski theorem, Tarski [23], Thm. 1 $\square$

We apply κ to define in our context the relation N of *nearness* proposed in Van Benthem [1]

$$N(X,U,V) \Leftrightarrow \kappa(X,U) > \kappa(V,U) \tag{0.42}$$

Here, $N(X,U,V)$ means that X *is closer to* U *than* V *is to* U.
Then, N does satisfy all axioms for nearness in Van Benthem [1]

Proposition 27. *The relation N does satisfy the following postulates*

NB1 $N(Z,U,V)$ and $N(V,U,W)$ imply $N(Z,U,W)$ (transitivity)

NB2 $N(Z,U,V)$ and $N(U,V,Z)$ imply $N(U,Z,V)$ (triangle inequality)

NB3 $N(Z,U,Z)$ is false (irreflexivity)

NB4 $Z = U$ or $N(Z,Z,U)$ (selfishness)

NB5 $N(Z,U,V)$ implies $N(Z,U,W)$ or $N(W,U,V)$ (connectedness)

Proof. For NB1, assumptions are $\kappa(Z,U) > \kappa(V,U)$ and $\kappa(V,U) > \kappa(W,U)$; it follows that $\kappa(Z,U) > \kappa(W,U)$ i.e. the conclusion $N(Z,U,W)$ follows.
For NB2, assumptions $\kappa(Z,U) > \kappa(V,U)$, $\kappa(V,U) > \kappa(Z,V)$ imply $\kappa(Z,U) > \kappa(Z,V)$, i.e., $N(U,Z,V)$.
For NB3, it cannot be true that $\kappa(Z,U) > \kappa(Z,U)$.
For NB4, $Z \neq U$ implies in our world that $\kappa(Z,Z) = 1 > \kappa(Z,U) \neq 1$.
For NB5, assuming that neither $N(Z,U,W)$ nor $N(W,U,V)$, we have $\kappa(Z,U) \leq \kappa(W,U)$ and $\kappa(W,U) \leq \kappa(V,U)$ hence $\kappa(Z,U) \leq \kappa(V,U)$, i.e., $N(Z,U,V)$ does not hold $\square$

In addition to betweenness T, we make use of a *betweenness* relation in the sense of Van Benthem T_B introduced in Van Benthem [1]

$$T_B(Z,U,V) \Leftrightarrow [\text{for each } W \ (Z = W) \text{ or } N(Z,U,W) \text{ or } N(Z,V,W)] \tag{0.43}$$

The principal example bearing, e.g., on our approach to robot control deals with rectangles in 2D space regularly positioned, i.e., having edges parallel to coordinate axes. We model robots (which are represented in the plane as discs of the same radii in 2D space) by means of their safety regions about robots; those regions are modeled as rectangles circumscribed on robots. One of advantages of this representation is that safety regions can be always implemented as regularly positioned rectangles.
Given two robots a, b as discs of the same radii, and their safety regions as circumscribed regularly positioned rectangles A, B, we search for a proper choice of a region X containing A, and B with the property that a robot C contained in X can be said to be between A and B. In this search we avail ourselves with the notion of betweenness relation T_B.
Taking the rough inclusion μ^G defined in (0.16), sect. 6, for two disjoint rectangles A, B, we define the *extent*, $ext(A, B)$ of A and B as the smallest rectangle containing the union $A \cup B$. Then we have the claim, obviously true by definition of T_B.

Proposition 28. *We consider a context in which objects are rectangles positioned regularly, i.e., having edges parallel to axes in R^2. The measure μ is μ^G. In this setting, given two disjoint rectangles C, D, the only object between C and D in the sense of the predicate T_B is the* extent *$ext(C, D)$ of C, D, , i.e., the minimal rectangle containing the union $C \cup D$.*

Proof. As linear stretching or contracting along an axis does not change the area relations, it is sufficient to consider two unit squares A, B of which A has (0,0) as one of vertices whereas B has (a,b) with $a, b > 1$ as the lower left vertex (both squares are regularly positioned). Then the distance κ between the extent $ext(A, B)$ and either of A, B is $\frac{1}{(a+)(b+1)}$.

For a rectangle $R : [0, x] \times [0, y]$ with $x \in (a, a + 1), y \in (b, b + 1)$, we have that

$$\kappa(R, A) = \frac{(x - a)(y - b)}{xy} = \kappa(R, B) \tag{0.44}$$

For $\phi(x, y) = \frac{(x-a)(y-b)}{xy}$, we find that

$$\frac{\partial \phi}{\partial x} = \frac{a}{x^2} \cdot (1 - \frac{b}{y}) > 0 \tag{0.45}$$

and, similarly, $\frac{\partial \phi}{\partial y} > 0$, i.e., ϕ is increasing in x, y reaching the maximum when R becomes the extent of A, B.

An analogous reasoning takes care of the case when R has some (c,d) with $c, d > 0$ as the lower left vertex $\square$

Further usage of the betweenness predicate is suggested by the Tarski axiom of *B,Eq*–upper dimension, which implies collinearity of x, y, z. Thus,a line segment may be defined via the auxiliary notion of a pattern; we introduce this notion as a relation Pt.

We let $Pt(u, v, z)$ if and only if $T_B(z, u, v)$ or $T_B(u, z, v)$ or $T_B(v, u, z)$.

We will say that a finite sequence $u_1, u_2, ..., u_n$ of objects *belong in a line segment* whenever $Pt(u_i, u_{i+1}, u_{i+2})$ for $i = 1, ..., n-2$; formally, we introduce the functor *Line* of finite arity defined by means of

$$Line(u_1, u_2, ..., u_n) \ if \ and \ only \ if \ Pt(u_i, u_{i+1}, u_{i+2}) \ for \ i < n - 1$$

For instance, any two disjoint rectangles A, B and their extent $ext(A, B)$ form a line segment.

Proposition 29. *The relation T_B does satisfy the Tarski properties TB1–TB2, in the case of regular rectangles. The condition TB3 is easily falsified with very simple examples.*

Proof. For the case of rectangles, TB1 follows by the fact that $ext(X, X) = X$ (we interpret $Z \subseteq X$ as $Z = X$). As for TB2, assume that $T_B(V, U, W)$, $T_B(Z, V, W)$ hold, i.e., $V = ext(U, W)$, and $Z = ext(V, W)$ hence $Z = V$ and $V = ext(U, Z)$, i.e. $T_B(V, U, Z)$ holds.

In the case of robots, assume robots are modeled as squares of side length 1. Then TB1 follows as for rectangles, and for TB2, assume that $T_B(a, b, c)$, $T_B(d, a, c)$ hold. We describe this situation by assuming that in the extent $ext(b, c)$, b is at the left upper vertex of $ext(b, c)$, and c is in the right lower corner of $ext(b, c)$.

We will say: a is down and right to b and a is up and left to c. Similarly for $T_B(d, a, c)$: d is down and right to a and d is up and left to c. Hence, a is down and right to b, and a is up and left to d, meaning that $T_B(a, b, d)$ holds
□

We now return to the theme of mereotopology discussed in Ch. 5 for mereology.

0.11 Rough mereotopology

We analyze now topological structures in rough mereological framework. We consider separately some cases depending on types of rough inclusions.

0.11.1 The case of transitive and symmetric rough inclusions

Here we have rough inclusions of the form μ^T induced by Archimedean t–norms in sect. 0.3, rough inclusions of the form of *airi's*, see sect. 0.6 (18), and rough inclusions of the form $\mu_\rho$induced by a metric ρ, see sect. 0.7 . We will use the general notation of $\mu_{s,t}$ to denote either of these forms. We thus assume that a rough inclusion $\mu_{s,t}$ is given, on a collection of objects, which obeys the transitivity law

$$\mu_{s,t}(x,y,r), \mu_{s,t}(y,z,s) \Rightarrow \mu_{s,t}(x,z,T(r,s)) \tag{0.46}$$

and is symmetric, i.e.

$$\mu_{s,t}(x,y,r) \Leftrightarrow \mu_{s,t}(y,x,r) \tag{0.47}$$

For each object x, we define an object $O_r(x)$ as the class of property $M(x,r)$, where

$$M(r,x)(y) \Leftrightarrow \mu_{s,t}(y,x,r), \tag{0.48}$$

and,

$$O_r(x) = ClsM(x,r) \tag{0.49}$$

Hence,

Proposition 30. *$ingr(z, O_r(x))$ if and only if $\mu_{s,t}(z,x,r)$.*

Proof. By Ch. 5, Proposition 4, $ingr(z, O_r(x))$ if and only if there exists t such that $Ov(z,t)$ and $\mu_{s,t}(t,x,r)$, hence, there exists w such that $ingr(w,z)$, $ingr(w,t)$, hence $w = z = t$, and finally $\mu_{s,t}(z,x,r)$ □

We regard the object $O_r(x)$ as an analogue of the notion of the 'closed ball about x of the radius r'.

To define the analogue of an open ball, we consider the property

$$M_r^+(x)(y) \Leftrightarrow \exists q > r.\mu_{s,t}(y,x,q) \tag{0.50}$$

The class of the property $M_r^+(x)$ will serve as the open ball analogue

$$Int(O_r(x)) = ClsM_r^+(x) \tag{0.51}$$

The counterpart of Proposition 30 for the new property is

Proposition 31. *$ingr(z, Int(O_r(x)))$ if and only if $\exists q > r.\mu_{s,t}(z,x,q)$.*

Proof. We follow the lines of the preceding proof. It is true that

$$ingr(z, Int(O_r(x)))$$

if and only if there exists t such that $Ov(z,t)$ and there exists $q > r$ for which $\mu_T(t,x,q)$ holds, hence, there exists w such that $ingr(w,z)$, $ingr(w,t)$, which implies that $w = z = t$, and finally $\mu_T(z,x,q)$ □

From Propositions 30 and 31, we infer that

Proposition 32. *$ingr(Int(O_r(x)), O_r(x)$.*

Proposition 33. *If $s < r$, then $ingr(O_r(x), O_s(x))$, $ingr(Int(O_r(x)),$ $Int(O_s(x)))$.*

Consider z with $ingr(z, Int(O_r(x)))$. By Proposition 31, $\mu_T(z, x, s)$ holds with some $s > r$. We can choose $\alpha \in [0, 1]$ with the property that $T(\alpha, s) > r$. For any object w with $ingr(w, O_\alpha(z))$, we can find an object u such that $\mu_T(u, z, \alpha)$ and $Ov(w, u)$.

For an object t such that $ingr(t, u)$ and $ingr(t, w)$, we have $\mu_T(t, w, 1)$, $\mu_T(t, u, 1)$, hence, $\mu_T(t, x, T(\alpha, s))$, i.e, $ingr(t, Int(O_r(x)))$. As $t = w$, we find that $ingr(w, Int(O_r(x)))$. We have verified

Proposition 34. *For each object z with $ingr(z, Int(O_r(x)))$, there exists $\alpha \in [0, 1]$ such that $ingr(O_\alpha(z), Int(O_r(x)))$.*

For any object z, what happens when $ingr(z, Int(O_r(x))$ and $ingr(z, O_s(y))$? We would like to find some $O_q(z)$ which would be an ingredient in either of $O_r(x)$, $O_s(y)$).

Proposition 34 answers this question positively: we can find $\alpha, \beta \in [0, 1]$ such that $ingr(O_\alpha(z), Int(O_r(x)))$ and $ingr(O_{beta}(z), Int(O_s(y)))$. By Proposition 33, for $q = max\{\alpha, \beta\}$, $ingr(O_q(z), Int(O_r(x)))$, $ingr(O_q(z), Int(O_s(y)))$.

We can sum up the last few facts

Proposition 35. *The collection $\{Int(O_r(x)) : x$ an object, $r \in [0, 1]\}$ is an open basis for a topology on the collection of objects.*

We will call an object x *open*, $Open(x)$ in symbols, in case it is a class of some property of objects of the form $Int(O_r(x))$.

$$Open(x) \Leftrightarrow \exists \Psi,\ a\ non--vacuous\ property\ of\ basic\ open\ sets.\ x = Cls\Psi\} \tag{0.52}$$

Hence,

Proposition 36. *In consequence of (0.52)*

1. *If Φ is any non–vacuous property of objects of the form $Open(x)$, then $Open(Cls\Phi)$;*

2. *If $Ov(Open(x), Open(y))$, then $Open(Open(x) \cdot Open(y))$.*

We define closures of objects, and to this end, we introduce a property $\Phi(x)$ for each object x

$$\Phi(x)(y) \Leftrightarrow \forall s < 1. Ov(O_s(y), x) \tag{0.53}$$

Closures of objects are defined by means of

$$Cl(x) = Cls\Phi(x) \tag{0.54}$$

We find what it does mean to be an ingredient of the object $Cl(x)$.

Proposition 37. *$ingr(z, Cl(x))$ if and only if $Ov(O_r(z), x)$ for every $r < 1$.*

Proof. By definition, there exists w such that $Ov(w, z)$ and $\Phi(w)$. For t which is an ingredient of z, w, hence, $z = t = w$, we have $\Phi(z)$, i.e., $Ov(O_r(z), x)$ for all $r < 1$ □

In particular

Proposition 38. *$ingr(z, Cl(O_w(x)))$ if and only if $Ov(O_r(z), O_w(x))$ for every $r < 1$.*

We follow this line of analysis. $Ov(O_r(z), O_w(x))$ means that we find q such that $ingr(q, O_r(z))$, $ingr(q, O_w(x))$, hence, by Proposition 30, $\mu_{s,t}(q, z, r)$ and $\mu_{s,t}(q, x, w)$, hence, by symmetry of $\mu_{s,t}$, we have $\mu_{s,t}(z, x, T(r, w))$, and, by continuity of T, with $r \to 1$, we obtain $\mu_{s,t}(z, x, w)$, i.e., by Proposition 30, $ingr(z, O_w(x))$. We have proved

Proposition 39. *$ingr(z, Cl(O_w(x)))$ if and only if $ingr(z, O_w(x))$.*

A corollary follows

Proposition 40. *$Cl(O_w(x)) = O_w(x)$.*

We can also address the notion of the *interior of an object*. We define $Int(x)$, the *interior of* x as

$$ingr(z, Int(x)) \Leftrightarrow \exists w.[Ov(z, w) \wedge \exists r < 1.ingr(O_r(w), x)] \tag{0.55}$$

A standard by now reasoning shows

Proposition 41. *$ingr(z, Int(x))$ if and only if there exists $r < 1$ such that $ingr(O_r(z), x)$.*

This implies that the notion of an interior is valid for objects of the form $O_r(x)$, where $r < 1$ and x is an object.

We can now address the problem of a *boundary* of any object of the form $O_r(x)$. We define the boundary $Bd(O_r(x))$ as

$$Bd(O_r(x)) = O_r(x) \cdot -Int(O_r(x)) \tag{0.56}$$

We have a characterization of boundary ingredients

Proposition 42. *$ingr(z, Bd(O_r(x)))$ if and only if*

$$\mu_{s,t}(z, x, r) \wedge \neg\exists q > r.\mu_{s,t}(z, x, q)$$

Proof. $ingr(z, Bd(O_r(x)))$ if and only if

$$ingr(z, O_r(x)), \neg ingr(z, Int(O_r(x)))$$

hence, $\mu_{s,t}(z, x, r)$ and $\mu_{s,t}(z, x, q)$ for no $q > r$ □

We introduce a symbol $\overline{\mu}_{s,t}(z,x,r) = sup\{q : \mu_T s, t(z,x,q)\}$. With its help, we write the last result down as

$$ingr(z, Bd(O_r(x))) \Leftrightarrow \mu_T(z,x,r) \wedge \overline{\mu(z,x)} = r \qquad (0.57)$$

Summing up our discussion, we may state that in case of transitive symmetric rough inclusions, objects to whom notions of closure as well as interior can be assigned are 'collective' objects of the form $O_r(x)$ but not object of the form x; in one sense we obtain a 'pointless' topology, on the other hand, this topology is like the orthodox topology as open objects are 'neighborhoods' of the form $O_r(x)$.

0.11.2 The case of transitive non–symmetric rough inclusions

This case is more difficult as lack of symmetry of μ prohibits some inferences; on the other hand, it is more interesting, as, e.g., no longer $ingr(x, y)$ implies $x = y$.We use a symbol μ to denote a rough inclusion induced by a continuous t–norm T; this time, μ is transitive but not symmetric, e.g., it is a residual rough inclusion of the form $x \Rightarrow_T y \geq r$, see sect. 0.2.

We preserve definitions of objects $O_r(x)$, $Int(O_r(x))$ and we analyze these notions for a transitive, non–symmetric rough inclusion μ. When transitivity issue comes into play, we assume that it is due to a continuous t–norm T.

We begin with $O_r(x)$, and we prove

Proposition 43. *$ingr(z, O_r(x))$ if and only if there exists t such that $ingr(t, z)$ and $\mu(t, x, r)$.*

Proof. $ingr(z, O_r(x))$ means that $Ov(z, w)$ and $\mu(w, x, r)$. For t with $ingr(t, w)$ and $ingr(t, z)$, we have $\mu(t, w, 1)$ and $\mu(t, z, 1)$, hence $\mu(t, x, T(r, 1)$, i.e., $\mu(t, x, r)$ $\square$

We will call t in the conclusion of Proposition 43, an 'O–*witness*' for the property $ingr(z, O_r(x))$ of z.

Similarly, we characterize interiors of O–objects

Proposition 44. *$ingr(z, Int(O_r(x)))$ if and only if there is t such that $ingr(t, z)$ and $\mu(t, x, s)$ for some $s > r$.*

By analogy, we call t in the conclusion of Proposition 44, an 'O^+–*witness*' for the property $ingr(z, Int(O_r(x)))$ of z.

The witness has good neighborhood properties.

Proposition 45. *For an O^+–witness t for $ingr(z, Int(O_r(x)))$, there exists $s < 1$ such that $ingr(O_s(t), Int(O_r(x)))$ holds.*

Proof. We have $\mu(t,x,s)$ with $s > r$, and, $ingr(t,z)$. We choose $q < 1$ such that $T(q,s) > r$. Consider y with $ingr(y, O_q(t))$, so there is w with properties $ingr(w,y)$, $\mu(w,t,q)$.

Hence, $\mu(w,x,T(s,q)$ and $ingr(w, Int(O_r(x)))$. By mereology axiom M3, Ch. 5, sect. 1, $ingr(O_q(t), Int(O_r(x)))$ □

As an upshot to Proposition 45, we will call an object x *open*, $Open(x)$ in symbols, when the condition is satisfied

$$Open(x) \Leftrightarrow [ingr(z,x) \Leftrightarrow \exists t, 0 < r < 1.ingr(O_r(t),x) \wedge ingr(t,z)] \quad (0.58)$$

A corollary follows

$$Open(Int(O_r(x)) \quad (0.59)$$

for each object x and each $r < 1$.

Following this line of reasoning, we define the *interior* of an object x, $Int(x)$ in symbols, letting

$$ingr(z, Int(x)) \Leftrightarrow \exists t.[ingr(t,z) \wedge \exists r < 1.ingr(O_r(t),x)] \quad (0.60)$$

Clearly

Proposition 46. *Properties of the operator Int are*

1. $ingr(Int(x),x)$;

2. $ingr(Int(Int(x)), Int(x))$;

3. $ingr(x,y) \Rightarrow ingr(Int(x), Int(y))$.

Dually, we define the *closure* of x, $Cl(x)$ in symbols, by means of

$$ingr(z, Cl(x)) \Leftrightarrow \exists t.[Ov(t,z) \wedge \forall r < 1.Ov(O_r(t),x)] \quad (0.61)$$

This condition can be disentangled; there exists w with $ingr(w,t), ingr(w,z)$ and there exists u_r such that $ingr(q_r,x), \mu(q_r,z,r)$ for each $r < 1$.

This intricate requirement can be simplified in *discrete* case when the number of objects is *finite*. Then, for r sufficiently close to 1, the only q_r satisfying the condition $\mu(q_r,z,r)$ is z itself, hence, $ingr(z,x)$ and finally, $Ov(z,x)$. We obtain

Proposition 47. *In discrete case,* $ingr(z, Cl(x))$ *if and only if* $Ov(z,x)$.

Hence

Proposition 48. *In discrete case,*

1. $ingr(x, Cl(x))$*;*

2. $ingr(Cl(x), Cl(Cl(x)))$*;*

3. $ingr(x, y) \Rightarrow ingr(Cl(x), Cl(y))$*.*

We obtain a *quasi–Čech* closure operation. Finally, we take up the boundary question. Again, we define the boundary of x, $Bd(x)$ as $Cl(x) \cdot -Int(x)$. The specific condition can be revealed as

Proposition 49. *In discrete case, for each object* x*, the boundary* $Bd(x)$ *satisfies* $ingr(z, Bd(x))$ *if and only if* $Ov(z, x)$ *and* $\exists t.[ingr(t, z) \wedge \forall r < 1.\neg ingr(O_r(t), x)]$.

0.12 Connections from rough inclusions

We will explore now the possibility of inducing a connection in an environment endowed with a rough inclusion μ. As before, we consider cases when μ is symmetric or not. We still retain the symbol μ as generic for rough inclusions.

0.12.1 The case of transitive and symmetric rough inclusions

For a transitive and symmetric rough inclusion μ, we define a connection C_μ by letting

$$C_\mu(x, y) \Leftrightarrow \forall r < 1.Ov(O_r(x), O_r(y)) \tag{0.62}$$

We need to make sure that C_μ is a connection, see Ch. 5, 5

Proposition 50. C_μ *does satisfy conditions CN1–CN3 for connections.*

Proof. CN1, CN2 are satisfied obviously. For CN3, observe that when $x \neq y$ then, e.g., $\neg ingr(x, y)$, i.e., $\mu(x, y, r)$ implies $r < 1$, hence, there is $s < 1$ that $\mu(x, y, s)$ does not hold,i.e., it is not true that $Ov(O_q(x), O_q(y))$ holds for q large enough to satisfy $T(q, q) > s$ □

We can explore the form of other predicates induced from C_μ. We know that $Ov(x, y)$ means $x = y$, hence, external connectedness is expressed by means of

$$EC_{\mu}(x,y) \Leftrightarrow (x \neq y) \wedge C_{\mu}(x,y). \quad (0.63)$$

This implies the form of $Tingr_{C_{\mu}}$ and $NTingr_{C_{\mu}}$.

In discrete case, for r sufficiently close to 1, $O_r(x) = x$ for each x, hence, $C_{\mu}(x,y)$ means simply $x = y$, $EC_{\mu}(x,y)$ is not defined, hence, $Tingr_{C_{\mu}}$ is not defined and $NTingr_{C_{\mu}}(x) = x$ for each x. In continuous case, the situation depends on additional properties of μ: if μ is *continuous* in the sense

$$lim_{r \to s} O_r(x) = O_s(x) \quad (0.64)$$

then, again, $C_{\mu}(x,y) \Leftrightarrow (x = y)$, with $EC_{mu}, TingrC_{\mu}$, $NTingr_{C_{\mu}}$ as above.

An example of continuous μ is μ_{ρ}: as $\mu_{\rho}(x,y,r) \Leftrightarrow \rho(x,y) \leq 1 - r$, where ρ a metric, (0.64) is satisfied with μ_{ρ}.

0.12.2 The case of symmetric non–transitive rough inclusions and the general case

The general case is represented by rough inclusions of the type *indri*, see sect. 0.6 (30), and, as we know, $ingr(x,y)$ means in this case that $|IND(x,s)| \leq |IND(y,s)|$. Thus, if there exists an object z with $IND(z,s) = \emptyset$, then z is an ingredient of each object, hence, any two objects overlap, and the definition (0.62) is not plausible. The only definition that remains is to accept $C(x,y)$ if and only of $x = y$, i.e., when $|IND(X,s)| = |IND(y,s)|$. The case of symmetric non–transitive rough inclusions is represented, e.g., by set–theoretic rough inclusions $\mu^S(x,y,r) \Leftrightarrow \frac{|x \cap y|}{|x|} \geq r$, see sect. 0.4, and geometric rough inclusions $\mu^G(x,y,r) \Leftrightarrow \frac{||x \cap y||}{||x||} \geq r$, see sect. 0.5. We consider a family $\mathcal{F}$ of finite subsets of a certain universe U and a family $\mathcal{C}$ of convex compact regions in a certain space R^k which share the following properties

$$RINC4 \; extr(x,y) \Rightarrow \exists r < 1.extr(O_r(x), O_r(y)) \quad (0.65)$$

$$RINC5 \; \neg ingr(x,y) \Rightarrow \exists z.ingr(z,x) \wedge extr(z,y) \quad (0.66)$$

With respect to the property RINC4, one may say that objects in $\mathcal{F}$, respectively, in $\mathcal{C}$ are *well–separated*. We adopt the definition of a connection C in terms of overlapping

$$C(x,y) \Leftrightarrow \forall r < 1.Ov(O_r(x), O_r(y)) \quad (0.67)$$

Proposition 51. *Under RINC4,RINC5, C is a connection.*

Proof. Again, CN1, CN2, are obviously satisfied, and only CN3 needs to be verified. In case $x \neq y$, assume, e.g., that $\neq ingr(x,y)$. Under complementation RINC5, there exists z with $ingr(z, x)$ and $extr(z, y)$. By RNC4, there is $r < 1$ such that $extr(O_r(z), O_r(y))$, whereas, $Ov(O_r(z), O_r(x))$ for each $r < 1$ □

Under this definition, external connectedness can be defined as

Proposition 52. *$EC(x, y)$ if and only if $Ov(O_r(x), O_r(y))$ for each $r < 1$ but not $Ov(x, y)$.*

The notion of non–tangential part comes down to

Proposition 53. *$NTingr_C(x, y) \Leftrightarrow C(z, x) \Rightarrow Ov(z, y)$ for each z.*

We define the interior $Int_C(x)$ for each object x, as usual as the class of the property of being a non–tangential part

$$NTP(x)(y) \Leftrightarrow NTingr_C(y, x), \tag{0.68}$$

and

$$Int_C(x) = ClsNTP(x) \tag{0.69}$$

Hence,

$$ingr(z, Int_C(x)) \Leftrightarrow \exists w.Ov(z, w) \wedge NTingr_C(w, x) \tag{0.70}$$

Our claim is that

Proposition 54. *$ingr(z, Int_C(x))$ implies that $ingr(O_r(z), x)$ for some $r < 1$.*

Proof. Assuming to the contrary, that it is not true that $ingr(O_r(z), x)$ for $r < 1$, we obtain that the complement $-x$ is externally connected to x and z, a contradiction □

A corollary follows

Proposition 55. *$ingr(Int_C(x), Int(x))$, where $Int(x)$ is the interior of x defined in terms of μ.*

The converse also holds

Proposition 56. *$ingr(Int(x), Int_C(x))$ holds for each object x.*

Proof. For z with $ingr(z, Int(x)$, we have that $ingr(O_r(z), x)$ for some r; if $C(z, w)$ for any w, then $Ov(O_r(z), O_r(w)$, but this means that $Ov(x, w)$ by RINC4 □

A corollary is

Proposition 57. *$Int_C(x) = Int(x)$ for every x.*

Relations of rough mereology to the fuzzy set theory are highlighted by the fact that rough inclusions are higher–order fuzzy equivalences in the sense of Zadeh [25].

0.13 Rough inclusions as many–valued fuzzy equivalences

Rough inclusions are defined as relations of a part to a degree and this makes them affine to fuzzy constructs, though they stem form different inspirations and have a different logical structure. Nevertheless, one may try to relate both worlds. We have a result that states, see, e.g., Polkowski [10], [11] that any rough inclusion $\mu(x, y, r)$ does induce on its universe a fuzzy similarity relation in the sense of Zadeh [25], see Ch. 4, sect. 12. First, writing $\mu_y(x) = r$ instead of $\mu(x, y, r)$, we convert the relational notation into the fuzzy–style one. We observe that fuzzy sets of the form μ_y are higher–level sets: values of fuzzy membership degrees here are convex sub–intervals of the unit interval $[0, 1]$ of the form $[0, r]$, i.e., with the left–end point 0, hence the formula $\mu_y(x) = t$ is understood as the statement that the sub–interval $\mu_y(x)$ contains t. Under this proviso, the *fuzzy tolerance relation* $\tau^{\mu}_y(x)$ is defined by means of

$$\tau^{\mu}_y(x) = r \Leftrightarrow \mu_y(x) = r \text{ and } \mu_x(y) = r \tag{0.71}$$

and it does satisfy, clearly,

1. $\tau^{\mu}_x(x) = 1$;

2. $\tau^{\mu}_x(y) = \tau^{\mu}_y(x)$.

We will use now the notation $\tau_r(x, y)$ for $\tau_x(y) = r$, disregarding the rough inclusion μ. Following Zadeh [25], we define *similarity classes* $[x]_\tau$ as fuzzy sets satisfying the condition,

$$\chi_{[x]_\tau}(y) = r \Leftrightarrow \tau_r(x, y) \tag{0.72}$$

and in this interpretation, τ becomes a fuzzy equivalence in the sense of Zadeh [25], see Polkowski [12], i.e., the family $\{[x]_\tau : x \subseteq U\}$ does satisfy the requirements for a *T–fuzzy partition* in the sense of Zadeh [25], cf., Ch. 4, sect. 12

$$\forall x \exists y. \chi_{[x]_\tau}(y) = 1 \tag{0.73}$$

and

$$[x]_\tau \neq [z]_\tau \Rightarrow max_y\{min\{\chi_{[x]_\tau}(y), \chi_{[z]_\tau}(y)\}\} < 1, \tag{0.74}$$

and,

$$\bigcup_x [x]_\tau \times_T [x]_\tau = \tau \tag{0.75}$$

where $A \times_T B$ denotes the fuzzy set defined via,

$$\chi_{A \times_T B}(u, v) = T(\chi_A(u), \chi_B(v)) \tag{0.76}$$

and $\bigcup$ denotes the supremum operator.

We include an argument, cf., Polkowski [12], see Ch. 4, sect. 12, for properties FS1–FS3. For (0.73), it is satisfied with $x = y$. To justify (0.74), observe that the existence of y with $\tau(x,y) = 1 = \tau(z,y)$ would imply $\tau(x,z) = 1 = \tau(z,x)$, hence, $x = z$. For (0.75), we have for given x, y, z by FS3 that

$$T(\tau(x,y), \tau(x,z)) = T(\tau(y,x), \tau(x,z)) \leq \tau(y,z)$$

But, for $x = y$, we have by FS1 that

$$[x]_\tau \times_T [x]_\tau(y,z) = [y]_\tau \times_T [y]_\tau(y,z) = T(\tau(y,y), \tau(y,z)) = T(1, \tau(y,z)) = \tau(y,z)$$

so (0.75) follows.

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