# Aspects of the Development of Many-Valued and Fuzzy Logic

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# Some
# (Pre-) History

The prehistory of MVL may be traced back to **Aristotle** (-310 to about -230).

In **De Interpretatione**, chapt. 9, he discussed the problem of future contingents (**contingentia futura**).

This is the problem to determine “today” the truth value of a proposition which asserts some future event.

This problem is closely tied with the philosophical problems of determinism and the understanding of modalities.

That a future event is (actually) “possible” or “undetermined” is read as a third “truth value” besides $\top$ and $\bot$.

This reading is not the necessary one, but it may be an interesting one.

The ancient philosophical school of **Epicureans** (ca. -300 to about 200) tended toward indeterminism and rejected the principle of bivalence.

The ancient philosophical school of the **Stoics** (ca. -300 to about 200) did accepted the principle of bivalence – and strongly advocated determinism.

Actually, the topic of the relationship between many-valuedness and determinism is discussed only occasionally.

The problem of **contingentia futura** was also the source for several extended discussions during the Middle Ages without getting resolved.

During the second half of the 19th century the idea of neglecting – at least partly – the principle of bivalence appeared (partly without clear mentioning of this fact) to H. McColl (1837–1909) and Ch.S. Peirce (1839–1914), but also to A. Meinong (1853–1920), whose work even was influential on J. Łukasiewicz.

Besides Łukasiewicz, these authors did not develop systems of MVL. They often mentioned only the idea that propositions may have other values besides "true" and "false". And sometimes their systems are also **not truth functional**.

Sometimes also the "non-aristotelian" systems which the Russian logician N.A. Vasiliev (1880–1940) introduced around 1910 have been considered as forerunners of systems of MVL.

It seems, however, that they are systems of paraconsistent logics.

P. Bernays used 1926 more than the usual two truth values of classical logic to study independence problems for systems of axioms for systems of classical propositional calculus. But in his case these multiple values were only formal tools for his unprovability results.

MVL really started in the 1920s, and the main force of development was the Polish school of logic under J. Łukasiewicz.

Łukasiewicz intended 1920 a modal reading to his MVL.

He claimed that only the three-valued and the infinite valued case (with the set of all rationals between 0 and 1 as truth degree set) are really of interest for applications.

Later on also all finitely many-valued propositional systems are discussed, always based on a negation and an implication connective as primitive ones.

In the beginning 1940s Łukasiewicz again argued for a modal reading to his MVL, now for the 4-valued case.

Basic theoretical results for systems of many-valued logics which followed this initial phase of "Polish" many-valued logic have been:

- An axiomatization of $L_3$ by M. Wajsberg 1931.
- The extension of $L_3$ to a functionally complete system and its axiomatization by J. Słupecki 1936.
- The work of K. Gödel 1932 and S. Jaśkowski 1936 which clarified the mutual relations of intuitionistic and many-valued logic: intuitionistic (propositional) logic is not a MVL with finitely many truth degrees.
- An application of MVL to the problems of logical antinomies by D.A. Bočvar 1938/43 with the third truth value read as "senseless".
- An application of three-valued logic to problems of partially defined functions by S.C. Kleene 1938 with the third truth value read as "undefined".

During the 1940s basic approaches were generalized, and essential results proved by J.B. Rosser and A.R. Turquette.

## Preliminaries:

$$\mathcal{W}_m = \{0, \frac{1}{m-1}, \ldots, \frac{m-2}{m-1}, 1\}\,, \qquad \mathcal{W}_\infty = [0,1]\,, \qquad \mathcal{W}_0 = [0,1] \cap \mathbb{Q}\,,$$

and with respect to some particular $m$ also

$$\tau_k = \frac{m-k}{m-1}\,.$$

Particularly important truth degree function are

$$\mathsf{seq}_{\,L}(x,y) = \min\{1, 1-x+y\}\,, \qquad \mathsf{seq}_{\,G}(x,y) = \begin{cases} 1\,, & \text{if } x \le y \\ y & \text{otherwise.} \end{cases}$$

$$\mathsf{non}_l(x) = 1-x\,, \qquad \mathsf{non}_G(x) = \begin{cases} 1\,, & \text{if } x = 0 \\ 0 & \text{otherwise,} \end{cases} \qquad \mathsf{j}_u(x) = \begin{cases} 1\,, & \text{if } x = u \\ 0 & \text{otherwise.} \end{cases}$$

## Conditions for Connectives / Truth Degree Functions:

**Standard condition** means: Behaves w.r.t. designated/undesignated truth degrees like the corresponding classical connective.

**Normal condition** means: The restriction to $\{0, 1\}$ behaves like a classical connective, i.e. has values in $\{0, 1\}$ again.

# The
# Łukasiewicz Systems

The Łukasiewicz systems $\mathsf{L}_m, \mathsf{L}_0, \mathsf{L}_\infty$ have as truth degree sets $\mathcal{W}_m$ for $m \geq 2$, $\mathcal{W}_0$ and $\mathcal{W}_\infty$, respectively.

Both infinite-valued systems have the same sets of tautologies because all (basic) truth degree functions are continuous ones. (A simple continuity argument.)

If the particular set of truth degrees is not of importance, simply write $\mathsf{L}_\nu$ or even only $\mathsf{L}$.

## The Propositional Systems

These systems have negation and implication as basic connectives:

$$\neg \text{ (negation)}, \qquad \to_{\mathsf{L}} \text{ (implication)}$$

with the corresponding truth degree functions

$$\mathsf{ver}^{\mathsf{L}}_{\neg} = \mathsf{non}_L : x \mapsto 1 - x, \qquad \mathsf{ver}^{\mathsf{L}}_{\to_{\mathsf{L}}} = \mathsf{seq}_{\,L} : (x, y) \mapsto \min\{1 - x + y, 1\}$$

The truth degree 1 is the only designated one.

One has for any wffs $G, H$ of these systems and any valuation $\beta : \mathcal{V}_0 \to \mathcal{W}^{\mathsf{L}}$:

$$\begin{aligned} \mathsf{Val}^{\mathsf{L}}(\neg H, \beta) &= 1 - \mathsf{Val}^{\mathsf{L}}(H, \beta)\,, \\ \mathsf{Val}^{\mathsf{L}}(G \to_{\mathsf{L}} H, \beta) &= \min\{1, 1 - \mathsf{Val}^{\mathsf{L}}(G, \beta) + \mathsf{Val}^{\mathsf{L}}(H, \beta)\}\,. \end{aligned}$$

Obviously, $\mathsf{L}_2$ is the classical propositional logic formulated as a system in negation and implication.

Because both truth degree functions $\mathsf{non}_1, \mathsf{seq}_2$ satisfy the normal condition, the Łukasiewicz systems $\mathsf{L}_\nu$ with $\nu \neq 2$ are **not functionally complete**.

One extends the Łukasiewicz systems by further connectives:

$$\begin{aligned} H_1 \vee H_2 &=_{\mathrm{def}} (H_1 \to_{\mathsf{L}} H_2) \to_{\mathsf{L}} H_2\,, \\ H_1 \wedge H_2 &=_{\mathrm{def}} \neg(\neg H_1 \vee \neg H_2)\,, \\ H_1 \leftrightarrow_{\mathsf{L}} H_2 &=_{\mathrm{def}} (H_1 \to_{\mathsf{L}} H_2) \wedge (H_2 \to_{\mathsf{L}} H_1)\,. \\ H_1 \,\&\, H_2 &=_{\mathrm{def}} \neg(H_1 \to_{\mathsf{L}} \neg H_2)\,, \\ H_1 \veebar H_2 &=_{\mathrm{def}} \neg H_1 \to_{\mathsf{L}} H_2\,. \end{aligned}$$

These connectives have well known truth degree functions:

| connective | $\vee$ | $\wedge$ | $\&$ | $\veebar$ |
|---|---|---|---|---|
| truth degree function | $\max$ | $\min$ | $T_L$ | $S_L$ |

And the truth degree function $\mathsf{ver}^{\mathsf{L}}_{\leftrightarrow_{\mathsf{L}}}$ for the biimplication connective $\leftrightarrow_{\mathsf{L}}$ is characterized by the equation

$$\mathsf{ver}^{\mathsf{L}}_{\leftrightarrow_{\mathsf{L}}}(x, y) = 1 - |x - y| \, .$$

**Remark:** The truth degree function determined by $\neg(H_1 \leftrightarrow_{\mathsf{L}} H_2)$ is a metric (in the precise mathematical sense of this word), i.e. measures the distance of the truth degrees of $H_1$ and $H_2$.

Neither $\wedge, \&$ as generalized conjunctions nor $\vee, \veebar$ as generalized disjunctions are equivalent connectives.

The connectives $\&, \veebar$ are called **strong** conjunction and **strong** disjunction, and $\wedge, \vee$ are the **weak** conjunction and the **weak** disjunction.

The Łukasiewicz connectives can easily be used to express relationships between truth degrees of wffs:

$$\models_{\mathsf{L}} H \quad \text{iff} \quad \mathsf{Val}^{\mathsf{L}}(H, \beta) = 1 \text{ for all valuations } \beta : \mathcal{V}_0 \to \mathcal{W}^{\mathsf{L}}$$

hence

$$\models_{\mathsf{L}} (H_1 \to_{\mathsf{L}} H_2) \quad \text{iff} \quad \mathsf{Val}^{\mathsf{L}}(H_1, \beta) \leq \mathsf{Val}^{\mathsf{L}}(H_2, \beta) \text{ for each valuation } \beta\,,$$

and for each valuation $\beta : \mathcal{V}_0 \to \mathcal{W}^{\mathsf{L}}$ separately

$$\mathsf{Val}^{\mathsf{L}}(H_1 \to_{\mathsf{L}} H_2, \beta) = 1 \quad \text{iff} \quad \mathsf{Val}^{\mathsf{L}}(H_1, \beta) \leq \mathsf{Val}^{\mathsf{L}}(H_2, \beta)\,.$$

Similarly for each valuation $\beta : \mathcal{V}_0 \to \mathcal{W}^{\mathsf{L}}$

$$\mathsf{Val}^{\mathsf{L}}(H_1 \leftrightarrow_{\mathsf{L}} H_2, \beta) = 1 \quad \text{iff} \quad \mathsf{Val}^{\mathsf{L}}(H_1, \beta) = \mathsf{Val}^{\mathsf{L}}(H_2, \beta)\,,$$

and thus in general

$$\models_{\mathsf{L}} (H_1 \leftrightarrow_{\mathsf{L}} H_2) \quad \text{iff} \quad \mathsf{Val}^{\mathsf{L}}(H_1, \beta) \;=\; \mathsf{Val}^{\mathsf{L}}(H_2, \beta) \;\; \text{for each valuation } \; \beta\,.$$

# Relationships between those systems

**Theorem:** For all $m, n \in \mathbb{N}$ with $m, n \geq 2$ there hold true:

$(a)$ $\mathsf{taut}^{\mathsf{L}}_m \subseteq \mathsf{taut}^{\mathsf{L}}_n \;\Leftrightarrow\; \mathcal{W}_m \supseteq \mathcal{W}_n$ ,

$(b)$ $\mathsf{taut}^{\mathsf{L}}_m \subseteq \mathsf{taut}^{\mathsf{L}}_n \;\Leftrightarrow\; n-1$ divides $m-1$ ,

$(c)$ $\mathsf{taut}^{\mathsf{L}}_m \not\subseteq \mathsf{taut}^{\mathsf{L}}_{m+1}$ and $\mathsf{taut}^{\mathsf{L}}_{m+2} \not\subseteq \mathsf{taut}^{\mathsf{L}}_{m+1}$ ,

$(d)$ $\mathsf{taut}^{\mathsf{L}}_\infty = \mathsf{taut}^{\mathsf{L}}_0$ ,

$(e)$ $\mathsf{taut}^{\mathsf{L}}_\infty = \bigcap_{m=3}^{\infty} \mathsf{taut}^{\mathsf{L}}_m$ .

**Corollary:** For all integers $m, n > 2$ one has

$(a)$ $\mathsf{taut}^{\mathsf{L}}_\infty \subset \mathsf{taut}^{\mathsf{L}}_m \subset \mathsf{taut}^{\mathsf{L}}_2$ ,

$(b)$ $m < n \Rightarrow \mathsf{taut}^{\mathsf{L}}_m \not\subseteq \mathsf{taut}^{\mathsf{L}}_n$ .

# Axiomatizability Results

**Theorem:** An adequate, i.e. sound and complete, axiomatization of $\mathsf{L}_3$ is given by the rule of detachment w.r.t. $\to_{\mathsf{L}}$ and:

$(\mathsf{L}_3 1)$ $\quad H_1 \to_{\mathsf{L}} (H_2 \to_{\mathsf{L}} H_1)$ ,

$(\mathsf{L}_3 2)$ $\quad (H_1 \to_{\mathsf{L}} H_2) \to_{\mathsf{L}} ((H_2 \to_{\mathsf{L}} H_3) \to_{\mathsf{L}} (H_1 \to_{\mathsf{L}} H_3))$ ,

$(\mathsf{L}_3 3)$ $\quad (\neg H_2 \to_{\mathsf{L}} \neg H_1) \to_{\mathsf{L}} (H_1 \to_{\mathsf{L}} H_2)$ ,

$(\mathsf{L}_3 4)$ $\quad ((H_1 \to_{\mathsf{L}} \neg H_1) \to_{\mathsf{L}} H_1) \to_{\mathsf{L}} H_1$ .

**Theorem:** A sound and complete axiomatization of $\mathsf{L}_\infty$ is given by the rule of detachment w.r.t. $\to_{\mathsf{L}}$ together with:

$(\mathsf{L}_\infty 1)$ $\quad H_1 \to_{\mathsf{L}} (H_2 \to_{\mathsf{L}} H_1)$ ,

$(\mathsf{L}_\infty 2)$ $\quad (H_1 \to_{\mathsf{L}} H_2) \to_{\mathsf{L}} ((H_2 \to_{\mathsf{L}} H_3) \to_{\mathsf{L}} (H_1 \to_{\mathsf{L}} H_3))$ ,

$(\mathsf{L}_\infty 3)$ $\quad (\neg H_2 \to_{\mathsf{L}} \neg H_1) \to_{\mathsf{L}} (H_1 \to_{\mathsf{L}} H_2)$ ,

$(\mathsf{L}_\infty 4)$ $\quad ((H_1 \to_{\mathsf{L}} H_2) \to_{\mathsf{L}} H_2) \to_{\mathsf{L}} ((H_2 \to_{\mathsf{L}} H_1) \to_{\mathsf{L}} H_1)$ .

# Some Further Results

**Theorem:** The infinitely many-valued Łukasiewicz system $\mathsf{L}_\infty$ is decidable.

**McNaughton-Theorem:** Let $f : [0,1]^n \to [0,1]$ be any $n$-ary function. The function $f$ is a truth degree function determined by some sentence of the Łukasiewicz system $\mathsf{L}_\infty$ iff $f$ is continuous and there exist a finite number of polynomials

$$g_i(x_1, \ldots, x_n) = b_i + \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, \ldots, m$$

with integer coefficients $a_{ij}, b_i$, such that for all $t_1, \ldots, t_n \in [0,1]$ there exists some $1 \leq k \leq m$ with

$$f(t_1, \ldots, t_n) = g_k(t_1, \ldots, t_n)\,.$$

**Proposition:** Suppose $m \geq 3$. Then $f : \mathcal{W}_m^n \to \mathcal{W}_m$ is the truth degree function represented by some wff $H \in \mathcal{L}_{\mathsf{L}}$ iff for each $n$-tuple $(t_1, \ldots, t_n)$ of truth degrees from $\mathcal{W}_m$ the product $(m-1) \cdot f(t_1, \ldots, t_n)$ is an integer and also a multiple of the greatest common divisor of all the integers $t_1 \cdot (m-1), \ldots, t_n \cdot (m-1), m-1$.

**Corollary:** Suppose $m \geq 3$. Then a function $f : \mathcal{W}_m^n \to \mathcal{W}_m$ is the truth degree function represented by some wff $H \in \mathcal{L}_{\mathsf{L}}$ of $\mathsf{L}_m$ iff for each $\mathcal{W}_r \subseteq \mathcal{W}_m$ the restriction $f \restriction \mathcal{W}_r$ is a function from $\mathcal{W}_r^n$ into $\mathcal{W}_r$.

**Theorem:** If one adds to one of the Łukasiewicz systems $\mathsf{L}_m$ for $m \geq 3$ either a truth degree constant $\mathsf{t}_m^*$ denoting the truth degree $\tau_2 = \frac{m-2}{m-1}$ or a unary connective $\mathsf{T}_m$ with its truth degree function satisfying

$$\mathsf{ver}^{\mathsf{L}}_{\mathsf{T}_m}(x) = \tau_2 = \tfrac{m-2}{m-1} \qquad \text{for all } x \in \mathcal{W}_m\,,$$

then the resulting extension of $\mathsf{L}_m$ is functionally complete.

# Algebraic Structures for Łukasiewicz Systems

## MV-algebras

MV-algebras have been introduced by Chang in 1958 in a completeness proof for the infinite-valued Łukasiewicz system.

They are particular cases of BL-algebras.

The MV-algebras have for $\text{Ł}_\infty$ essentially the same rôle as the Boolean algebras have for classical logic: they offer an adequate algebraic semantics.

The first extended monograph devoted to MV-algebras is:

> CIGNOLI, R. – D'OTTAVIANO, I.M.L. – MUNDICI, D.: *Algebraic Foundations of Many-Valued Reasoning.* Trends in Logic, vol. 7, Kluwer, Dordrecht 2000.

The following definition is a simplification of the original definition of Chang.

**Definition:** An algebraic structure $\mathcal{A} = \langle A, \oplus, \neg, 0 \rangle$ is an *MV-algebra* iff $\langle A, \oplus, 0 \rangle$ is an **abelian monoid** with neutral element 0 such that always

$$
\begin{array}{rl}
(i) & \neg\neg x = x \, , \\
(ii) & x \oplus \neg 0 = \neg 0 \, , \\
(iii) & \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x \, .
\end{array}
$$

**Proposition:** Each MV-algebra is (isomorphic to) a subdirect product of MV-chains.

**Proposition:** An MV-equation is valid in all MV-algebras iff it is valid in all MV-chains.

**Algebraic Completeness Theorem:** An MV-equation is valid in all MV-algebras iff it is valid in the MV-algebra $\langle \mathcal{W}_\infty, \mathsf{vel}_L, \mathsf{non}_L, 0 \rangle$.

**Corollary:** A wff $H$ of the language $\mathcal{L}_{\text{Ł}}$ of the Łukasiewicz systems is an $\text{L}_\infty$-tautology iff $H$ is valid in all MV-chains, and hence in all MV-algebras.

# Relations to other algebraic structures

**Definition:** A *bounded commutative BCK-algebra* is such an algebraic structure $\mathcal{B} = \langle B, \ominus, \mathbf{0}, \mathbf{1} \rangle$ of similarity type $\langle 2, 0, 0 \rangle$ which satisfies for all $a, b, c \in B$ the equations

$$(a \ominus b) \ominus c = (a \ominus c) \ominus b, \qquad a \ominus (a \ominus b) = b \ominus (b \ominus a),$$
$$a \ominus a = 0, \qquad a \ominus \mathbf{0} = a, \qquad a \ominus \mathbf{1} = \mathbf{0}.$$

**Theorem:** (i) If $\langle A, \oplus, \neg, 0 \rangle$ is an MV-algebra, then the algebraic structure $\langle A, \ominus, 0, \neg 0 \rangle$ with the operation $\ominus$ defined as $a \ominus b =_{\mathrm{def}} a \oplus \neg b$ is a bounded commutative BCK-algebra.

(ii) If $\mathcal{B} = \langle B, \ominus, \mathbf{0}, \mathbf{1} \rangle$ is a bounded commutative BCK-algebra, then the structure $\langle B, \oplus, \neg, \mathbf{0} \rangle$ with the operations $\neg, \oplus$ defined as $\neg a =_{\mathrm{def}} \mathbf{1} \ominus a$ and $a \oplus b =_{\mathrm{def}} \mathbf{1} \ominus ((\mathbf{1} \ominus a) \ominus b)$ is an MV-algebra.

(iii) These mappings between the classes of MV-algebras and of bounded commutative BCK-algebras are inverse to one another and hence bijections.

**Definition:** An algebraic structure $\mathcal{G} = \langle G, +, \boldsymbol{o}, \leqslant, u \rangle$ is an $\ell$**-group with strong order unit** iff $\langle G, +, \boldsymbol{o}, \leqslant \rangle$ is a lattice ordered group, such that the **strong order unit** $\boldsymbol{o} \leqslant u \in G$ has the property that for each $a \in G$ there is an integer $n \geq 0$ with $a \leqslant nu$ and also $-a \leqslant nu$ for the inverse $-a$ of $a$ characterized by $a + (-a) = \boldsymbol{o}$.

**Theorem** (Mundici): The operator $\Gamma$ defined by

$$\Gamma(\mathcal{G}, u) = \langle [\boldsymbol{o}, u], \oplus, \neg, \boldsymbol{o} \rangle \quad \text{with} \quad x \oplus y = u \sqcap (x + y)\,, \ \ \neg x = u - x$$

maps abelian lattice-ordered groups $\mathcal{G}$ with strong order unit to MV-algebras and is even a natural equivalence between the categories of abelian lattice-ordered groups with a strong order unit and of MV-algebras (both with the respective natural homomorphisms as morphisms).

Besides its algebraic content, this last result may be read as stating that MV-algebras are not only important for Łukasiewicz's many-valued logics and interesting algebraic objects but also provide an equational formulation of the theory of magnitudes with an Archimedean unit.

# The Finitely-Valued Łukasiewicz Systems

**Definition:** For each integer $m \geq 3$ an $\mathrm{MV}_m$-*algebra* is a MV-algebra $\mathcal{A}$ which additionally satisfies for each $a \in A$ the conditions:

$$\begin{array}{ll} (\mathsf{MV}_m 1) & (m-1)a \oplus a = (m-1)a\,, \\ (\mathsf{MV}_m 2) & a^{m-1} \otimes a = a^{m-1}\,, \end{array}$$

and for each integer $1 < k < m-1$ which does not divide $m-1$ also the conditions:

$$\begin{array}{ll} (\mathsf{MV}_m 3) & [ka \otimes (\neg a \oplus \neg (k-1)a)]^{m-1} = \mathsf{0}\,, \\ (\mathsf{MV}_m 4) & (m-1)[a^k \oplus \neg a \otimes \neg a^{k-1}] = \mathsf{1}\,. \end{array}$$

Obviously, here the conditions ($\mathsf{MV}_m 3$) and ($\mathsf{MV}_m 4$) become empty in the case $m = 3$.

**Theorem:** Each $\mathrm{MV}_m$-algebra is isomorphic to a subdirect product of MV-algebras $\mathcal{W}_n$ for which $n-1$ divides $m-1$.

**Axiomatizability Theorem for $\mathsf{L}_m$:** For each integer $m \geq 3$ an adequate axiomatization of the $m$-valued Łukasiewicz system $\mathsf{L}_m$ is given by the axiom schemata

$(\mathsf{L}_m 1)$ $\quad H_1 \to_{\mathsf{L}} (H_2 \to_{\mathsf{L}} H_1)$ ,

$(\mathsf{L}_m 2)$ $\quad (H_1 \to_{\mathsf{L}} H_2) \to_{\mathsf{L}} ((H_2 \to_{\mathsf{L}} H_3) \to_{\mathsf{L}} (H_1 \to_{\mathsf{L}} H_3))$ ,

$(\mathsf{L}_m 3)$ $\quad (\neg H_2 \to_{\mathsf{L}} \neg H_1) \to_{\mathsf{L}} (H_1 \to_{\mathsf{L}} H_2)$ ,

$(\mathsf{L}_m 4)$ $\quad ((H_1 \to_{\mathsf{L}} H_2) \to_{\mathsf{L}} H_2) \to_{\mathsf{L}} ((H_2 \to_{\mathsf{L}} H_1) \to_{\mathsf{L}} H_1)$ ,

$(\mathsf{L}_m 5)$ $\quad \sum_{i=1}^{m} A \to_{\mathsf{L}} \sum_{i=1}^{m-1} A$ ,

$(\mathsf{L}_m 6)$ $\quad \sum_{i=1}^{m-1}\Big(\prod_{j=1}^{k} A \veebar \big(\neg A \mathbin{\&} \sum_{i=1}^{k-1} A\big)\Big)$ for each $1 < k < m-1$ for which $k-1$ does not divide $m-1$

together with the rule of detachment (MP) w.r.t. the implication $\to_{\mathsf{L}}$ as inference rule.

## Łukasiewicz Algebras

**Definition:** Given any $2 \leq m \in \mathbb{N}$, an *$m$-valued Łukasiewicz algebra* is an algebraic structure $\mathcal{A} = \langle A, \cup, \cap, \sim, \mathsf{s}_1^m, \ldots, \mathsf{s}_{m-1}^m, \mathsf{0}, \mathsf{1} \rangle$ of similarity type $\langle 2, 2, 1, 1, \ldots, 1, 0, 0 \rangle$ such that $\langle A, \cup, \cap, \mathsf{0}, \mathsf{1} \rangle$ is a distributive lattice with zero and unit, and such that for all $x, y \in A$ and all $1 \leq i, j \leq m-1$ there hold true:

$$
\begin{array}{ll}
(\mathsf{LA}1) & \sim(\sim(x)) = x\,, \\
(\mathsf{LA}2) & \sim(x \cup y) = \sim(x) \wedge \sim(y)\,, \\
(\mathsf{LA}3) & \mathsf{s}_i^m(x \cup y) = \mathsf{s}_i^m(x) \cup \mathsf{s}_i^m(y)\,, \\
(\mathsf{LA}4) & \mathsf{s}_i^m(x) \cup \sim \mathsf{s}_i^m(x) = \mathsf{1}\,, \\
(\mathsf{LA}5) & \mathsf{s}_j^m(\mathsf{s}_i^m(x)) = \mathsf{s}_i^m(x)\,, \\
(\mathsf{LA}6) & \mathsf{s}_i^m(\sim(x)) = \sim \mathsf{s}_{m-i}^m(x)\,, \\
(\mathsf{LA}7) & \mathsf{s}_i^m(x) \cup \mathsf{s}_{i+1}^m(x) = \mathsf{s}_{i+1}^m(x) \quad \text{for} \;\; i < m-1\,, \\
(\mathsf{LA}8) & x \cup \mathsf{s}_{m-1}^m(x) = \mathsf{s}_{m-1}^m(x)\,, \\
(\mathsf{LA}9) & (x \cap \sim(\mathsf{s}_i^m(x)) \cap \mathsf{s}_{i+1}^m(x)) \cup y = y \quad \text{for} \;\; i < m-1\,.
\end{array}
$$

Only for $m = 3, 4$ are these $m$-valued Łukasiewicz algebras really algebraic counterparts to the $m$-valued Łukasiewicz system $\mathsf{L}_m$ – because only in these cases the corresponding Łukasiewicz implication $\mathsf{seq}_2$ becomes definable in these structures.

A way out is to further enrich these $m$-valued Łukasiewicz algebras with a family of binary operations.

# The Gödel Systems

Gödel's family of finitely many-valued propositional logics $\mathsf{G}_m$ with $m \geq 2$ was introduced in 1932 in the context of investigations aimed to understand intuitionistic logic. Accordingly they have been formulated as systems in conjunction, disjunction, negation, and implication $\{\wedge, \vee, \sim, \rightarrow_{\mathsf{G}}\}$ over the truth degree sets $\mathcal{W}_m$.

The truth degree functions are for all systems, respectively, the functions

$$\min, \max, \mathsf{non}_G, \mathsf{seq}_G$$

It is a routine matter to extend these Gödel systems $\mathsf{G}_m$ with $m \geq 2$ to infinitely many-valued systems $\mathsf{G}_0$ and $\mathsf{G}_\infty$ with truth degree sets $\mathcal{W}_0, \mathcal{W}_\infty$, respectively.

The Gödel systems $\mathsf{G}_\nu$ with $\nu \neq 2$ are not functionally complete, because all the basic connectives satisfy the normal condition.

Contrary to the situation for the Łukasiewicz systems, the truth degree functions $\mathsf{seq}_G$ and $\mathsf{non}_G$ are discontinuous functions over the real unit interval $[0, 1]$.

It is an interesting fact that all the truth degree functions of the Gödel systems can be characterized in purely order theoretic terms. This is obvious for $\min$, $\max$, and $\mathsf{seq}_G$, but also for $\mathsf{non}_G$ because one obviously has

$$\mathsf{non}_G(x) = \begin{cases} \max \mathcal{W}_\nu\,, & \text{if } x = \min \mathcal{W}_\nu \\ \min \mathcal{W}_\nu\,, & \text{otherwise}\,. \end{cases}$$

Therefore the truth degree of a given wff $H$ depends not on the particular choice of some valuation $\beta$, but on the order type of the (essential initial part of the) sequence of values $(\beta(p^{(n)}))_{n\geq 1}$.

Even more: Different, but order isomorphic truth degree set generate the same sets of G-tautologies.

**Theorem:** A finitely many-valued GÖDEL system is completely characterized by its number of truth degrees, i.e. any two $n$-valued Gödel systems have coinciding entailment relations.

**Theorem:** One has for all $2 \leq m \in \mathbb{N}$ the relationships:

$(a)$ $\mathsf{taut}^{\mathsf{G}}_{\infty} \subset \mathsf{taut}^{\mathsf{G}}_{m+1} \subset \mathsf{taut}^{\mathsf{G}}_{m}$ ,

$(b)$ $\mathsf{taut}^{\mathsf{G}}_{\infty} = \mathsf{taut}^{\mathsf{G}}_{0} = \bigcap_{m=2}^{\infty} \mathsf{taut}^{\mathsf{G}}_{m}$ .

**Theorem:** Each infinitely many-valued (propositional) Gödel system $\mathsf{G}(\mathcal{W})$ has $\mathsf{taut}^{\mathsf{G}}_{\infty}$ as its set of tautologies.

**Theorem:** $\mathsf{G}_{\infty}$ is adequately axiomatized by any suitable axiom system $\mathsf{Ax}_{\mathsf{Int}}$ for **intuitionistic** propositional logic together with the schema $\mathsf{Ax}_{\mathsf{prelin}}$ of **prelinearity**:

$$(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)\,.$$

**Definition:** A **Heyting algebra**, or **pseudo-**$\textsc{Boole}$**an algebra**, is an algebraic structure $\mathbf{A} = \langle A, \sqcap, \sqcup, \rightarrowtail, \mathbf{0}, \mathbf{1} \rangle$ such that

(HA1) $\langle A, \sqcup, \sqcap, \mathbf{0}, \mathbf{1} \rangle$ is a distributive lattice with zero and unit elements,

and that $\sqcap, \rightarrowtail$ form an adjoint pair, i.e. that one has for all $a, b, c \in A$:

(HA2) $\quad a \sqcap b \leqslant c \quad \text{iff} \quad b \leqslant (a \rightarrowtail c)\,.$

Such a Heyting algebra is a **Heyting chain** iff it satisfies additionally for all $a, b \in A$ the **linearity condition**

(HA$_{\mathrm{lin}}$) $\quad (a \rightarrowtail b) = \mathbf{1} \quad \text{or} \quad (b \rightarrowtail a) = \mathbf{1}$ ,

and it is **prelinear** iff it satisfies the **prelinearity condition**

(HA$_{\mathrm{prelin}}$) $\quad (a \rightarrowtail b) \sqcup (b \rightarrowtail a) = \mathbf{1} \quad .$

**Theorem:** For a wff $H$ of the language of intuitionistic propositional logic there are equivalent:

$(a)$ $H$ is derivable from $\mathsf{Ax}_{\mathsf{Int}} + \mathsf{Ax}_{\mathsf{prelin}}$ using $(\mathsf{MP})$ ,

$(b)$ $H$ is valid in each prelinear Heyting-algebra ,

$(c)$ $H$ is valid in each $\textsc{Heyting}$ chain ,

$(d)$ $H$ is valid in some infinite $\textsc{Heyting}$ chain ,

$(e)$ $H$ is valid in each finite $\textsc{Heyting}$ chain ,

$(f)$ $H$ is valid in each $\textsc{Heyting}$ chain with at most $n+2$ elements, $n$ being the number of propositional variables which occur in $H$ .

# The Post Systems

Systems of finitely-valued propositional logic have been considered independent of J. Łukasiewicz and almost at the same time also by E.L. Post.

Contrary to the mainly philosophical motivations of Łukasiewicz, Post developed his systems in the context of studies toward classical propositional logic, e.g. toward functional completeness.

Post did not care for adequate axiomatizations of his systems.

The Post systems have been originally formulated uniformly in negation $\sim$ and disjunction $\vee$ as basic connectives.

$$\mathsf{ver}^{\mathsf{P}}_{\vee}(x,y) = \max\{x,y\}\,, \quad \mathsf{ver}^{\mathsf{P}}_{\sim}(x) = \mathsf{non}_P(x) = \begin{cases} 1\,, & \text{if } x = 0 \\ x - \frac{1}{m-1}\,, & \text{if } x \neq 0 \end{cases}$$

For the designated truth degrees Post gives no canonical choice. Nevertheless, $1$ has become a kind of standard for the only designated truth degree.

**Definition:** A distributive lattice $\boldsymbol{A} = \langle A, +, \cdot, \mathit{0}, \mathit{1} \rangle$ with a zero and a unit element is a **Post algebra of order** $n$, $n \geq 2$, iff:

(PA1) There exist elements $\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1} \in A$ with the property

$$\mathit{0} = \boldsymbol{e}_0 \leqq \boldsymbol{e}_1 \leqq \cdots \leqq \boldsymbol{e}_{n-1} = \mathit{1}$$

with respect to the lattice ordering $\leqq$ of $\boldsymbol{A}$.

(PA2) Each $a \in A$ has a unique representation $a = \sum_{i=1}^{n-1} a_i \cdot \boldsymbol{e}_i$ such that each of the coefficients $a_i$, $1 \leq i < n$, has in the lattice $\boldsymbol{A}$ a complement and such that there hold true

$$a_1 \geqq a_2 \geqq \cdots \geqq a_{n-1} \, .$$

**Remark:** Rosenbloom (1942) gave a more complicated definition; the present one goes back to Epstein (1960) and Dwinger (1977).

# 3 & 4

# Three-Valued Systems

I'll restrict my attention to two strongly related systems introduced by D.A. Bočvar 1938 and S.C. Kleene 1938, and a system designed as the "true" logic of the natural language by U. Blau 1978.

The intentions of these authors, connected with their systems, have been quite different despite some strong similarities of the systems.

The main problem of Bočvar has been the philosophical and logical analysis of logical and semantical antinomies as they appear in first-order and higher-order logic, often in connection with some lack of care e.g. in the use of the comprehension principle or of metatheoretical notions.

Therefore his interpretation of the additional truth degree $\frac{1}{2}$ was its reading as "meaningless", "paradoxical", or "senseless".

The starting point of KLEENE was his research on partial recursive relations. Such relations sometimes may be undefined.

So his intended reading of the degree $\frac{1}{2}$ was “undefined” or “indetermined”.

Both these systems consider the truth degrees $1, 0$ just as the classical truth-values $\top, \bot$. Thus for both systems the degree $1$ is the only designated one.

The core idea for the Kleene approach, to understand the third truth degree as “undefined”, sometimes is also realized in another way: via truth value gaps.

**NB:** If such truth value gaps behave truth-functionally it is a purely formal change either to have truth value gaps, or to have a third truth degree indicating such gaps.

# Bočvar

Bočvar subdivides his truth degree functions and thus also his connectives into **internal** and **external** ones.

The internal truth degree functions have always a function value different from $0, 1$ if some argument value differs from $0, 1$.

The external truth degree functions map into $\{0, 1\}$.

The basic connectives of the 3-valued system $\boldsymbol{B}_3$ of Bočvar are

$$\neg,\ \wedge_+,\ \mathsf{J}_0,\ \mathsf{J}_1$$

for internal negation, internal conjunction, external negation, and external assertion.

They have, respectively, the truth degree functions

$$\mathsf{non}_L,\ \mathsf{et}^*,\ \mathsf{j}_0,\ \mathsf{j}_1\,.$$

with $\mathsf{et}^*$ defined by

| $\mathsf{et}^*$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $\frac{1}{2}$ | $0$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $1$ | $0$ | $\frac{1}{2}$ | $1$ |

or by the formula

$$\mathsf{et}^*(u,v) = \begin{cases} \mathsf{et}(u,v), & \text{if } u,v \in \{0,1\} \\ \frac{1}{2}, & \text{if } u = \frac{1}{2} \text{ or } v = \frac{1}{2} \end{cases}$$

with et indicating the truth value function of classical conjunction.

All these connectives obviously satisfy the normal condition. Hence the 3-valued Bočvar system is not functionally complete.

Further internal versions of disjunction, implication and biimplication can be defined as

$$\begin{aligned} H_1 \vee_+ H_2 &=_{\text{def}} \neg(\neg H_1 \wedge_+ \neg H_2)\,, \\ H_1 \rightarrow_+ H_2 &=_{\text{def}} (H_1 \wedge_+ \neg H_2), \\ H_1 \leftrightarrow_+ H_2 &=_{\text{def}} (H_1 \rightarrow_+ H_2) \wedge_+ (H_2 \rightarrow_+ H_1)\,, \end{aligned}$$

and get truth degree functions which coincide over $\{0,1\}$ with their classical counterparts and are internal truth degree functions.

Corresponding external versions of these connectives result by a uniform approach which e.g. for external conjunction and external disjunction reads as

$$\begin{aligned} H_1 \Cap H_2 &=_{\text{def}} \mathsf{J}_1(H_1) \wedge_+ \mathsf{J}_1(H_2)\,, \\ H_1 \Cup H_2 &=_{\text{def}} \mathsf{J}_1(H_1) \vee_+ \mathsf{J}_1(H_2)\,. \end{aligned}$$

For the 3-valued Bočvar system, $\neg$ is the negation of $\mathsf{L}_3$, and $\mathsf{J}_0, \mathsf{J}_1$ are definable in $\mathsf{L}_3$.

Also the internal conjunction $\wedge_+$ is $\mathsf{L}_3$-definable, e.g. as

$$H_1 \vee_+ H_2 = (H_1 \wedge H_2) \vee (H_1 \wedge \neg H_1) \vee (H_2 \wedge \neg H_2)\,.$$

Thus $\boldsymbol{B}_3$ is a subsystem of $\mathsf{L}_3$.

It is a proper subsystem because each wff $H$ of $\boldsymbol{B}_3$ in which only the propositional variables $p_1, \ldots, p_n$ occur can be written semantically equivalent in the "normal form"

$$\mathsf{J}_1(H) \vee_+ \mathsf{U}(p_{i_1}) \vee_+ \cdots \vee_+ \mathsf{U}(p_{i_k})$$

for some suitable subsequence $(i_1, \ldots, i_k)$ of $(1, \ldots, n)$, using the shorthand notation

$$\mathsf{U}(p) = p \wedge_+ \neg p \, .$$

This representability result means that the Łukasiewicz implication $\rightarrow_{\mathsf{L}}$ is not $\boldsymbol{B}_3$-definable.

Otherwise $p_1 \rightarrow_{\mathsf{L}} p_2$ would be representable by a wff $H_0$ of such a form. But $\mathsf{seq}_{\,L}(\frac{1}{2}, 0) = \frac{1}{2}$ would force that $\mathsf{U}(p_1)$ had to appear as a disjunct in $H_0$.

Hence $\mathsf{Val}^{\boldsymbol{B}}(H_0, \beta) = \frac{1}{2}$ for each valuation $\beta$ with $\beta(p_1) = \beta(p_2) = \frac{1}{2}$, contrary to $\mathsf{seq}_{\,L}(\frac{1}{2}, \frac{1}{2}) = 1$.

# Kleene

The Kleene system $\boldsymbol{K}_3$ has the so-called **strong** connectives

$$\neg, \wedge, \vee, \rightarrow_{\mathrm{K}}, \leftrightarrow_{\mathrm{K}}$$

with corresponding truth degree functions, respectively:

$$\mathsf{non}_L, \min, \max, \mathsf{seq}', \mathsf{eq}',$$

with:

| $\mathsf{seq}'$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1$ |
| $1$ | $0$ | $\frac{1}{2}$ | $1$ |

| $\mathsf{eq}'$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $1$ | $\frac{1}{2}$ | $0$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $1$ | $0$ | $\frac{1}{2}$ | $1$ |

Also for this system $\boldsymbol{K}_3$ the connectives $\neg, \wedge, \vee$ are just the same as in $\mathsf{L}_3$, and for the biimplication $\leftrightarrow_{\mathrm{K}}$ one has

$$H_1 \leftrightarrow_{\mathrm{K}} H_2 \quad \text{semantically equivalent} \quad (H_1 \rightarrow_{\mathrm{K}} H_2) \wedge (H_2 \rightarrow_{\mathrm{K}} H_1)\,.$$

$\boldsymbol{K}_3$ has also **weak** connectives $\wedge_+, \vee_+, \rightarrow_+$ which coincide with the equally denoted connectives of the Bočvar system $\boldsymbol{B}_3$, i.e. which yield wffs which have truth degree $\frac{1}{2}$ iff one of their constituents has truth degree $\frac{1}{2}$.

These weak connectives are $\boldsymbol{K}_3$-definable.

Because the implication $\rightarrow_{\mathrm{K}}$ of the Kleene system is $\mathsf{L}_3$-definable, e.g. by

$$H_1 \rightarrow_{\mathrm{K}} H_2 = (H_1 \rightarrow_{\mathsf{L}} H_2) \wedge (H_1 \vee \neg H_1 \vee H_2 \vee \neg H_2)$$

also the 3-valued Kleene system $\boldsymbol{K}_3$ is a subsystem of $\mathsf{L}_3$.

Again, however, $\rightarrow_{\mathsf{L}}$ is not definable in the Kleene system and this system therefore a proper subsystem of $\mathsf{L}_3$.

This undefinability follows from the fact that each (binary) connective $\triangle$ which is definable from the basic connectives of $\boldsymbol{K}_3$ has a corresponding truth degree function ver$_\triangle$ satisfying ver$_\triangle(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$ and hence cannot be seq$\,_L$.

If one combines the basic connectives of the Bočvar and Kleene systems, then $\rightarrow_{\mathsf{L}}$ becomes definable e.g. as

$$H_1 \rightarrow_{\mathsf{L}} H_2 = (H_1 \rightarrow_{\mathrm{K}} H_2) \vee (\mathsf{J}_{\frac{1}{2}}(H_1) \wedge \mathsf{J}_{\frac{1}{2}}(H_2))$$

with

$$\mathsf{J}_{\frac{1}{2}}(H) = \neg \mathsf{J}_0(H) \wedge \neg \mathsf{J}_1(H)\,.$$

**Proposition:** Neither are all the connectives of the Kleene system definable in the Bočvar system, nor conversely are all the connectives of the Bočvar system definable in the Kleene system.

In
L. Bolc – P. Borovik: Many-Valued Logics, vols. 1-2, Springer: Berlin 1992, 2003
a longer, but very concise survey of such and other 3-valued systems is given.

# Blau

This approach from 1978 is based on discussions in the philosophy of language. They give the fundamental motivations for the intuition behind the three truth degrees $0, \frac{1}{2}, 1$, which he denotes $w, u, f$, respectively.

A core point is that Blau identifies the standard truth value $\top$ with the truth degree 1, and that he **splits** the truth value $\perp$ into the two degrees $0$ and $\frac{1}{2}$.

The truth degree $0$ becomes a modified version of the (usual) truth value $\perp$, and the intended reading for the degree $u = \frac{1}{2}$ is "undetermined", combined with the understanding that the appearance of this degree is caused either by the use of vague predicates, or by the use of non-denoting names, i.e. by reference to unsatisfied presuppositions.

On the propositional level Blau's considerations are based on three primitive connectives, two kinds of negation connectives $\neg, \approx$ and a conjunction connective $\wedge$, characterized by

$$\mathsf{ver}_{\neg}^{\mathsf{Bl}} = \mathsf{non}_L\,, \qquad \mathsf{ver}_{\approx}^{\mathsf{Bl}} = \mathsf{non}_L \circ \mathsf{j}_1\,, \qquad \mathsf{ver}_{\wedge}^{\mathsf{Bl}} = \min\ .$$

All these connectives satisfy the normal condition, hence this system is not functionally complete.

However, these connectives suffice to introduce some further, interesting connectives, e.g. the GÖDEL negation $\sim$ as

$$\sim H = \neg \approx \neg H \,,$$

which additionally is the J-connective $\mathsf{J}_0$, to introduce the max-disjunction $\vee$ as

$$H_1 \vee H_2 = \neg(\neg H_1 \wedge \neg H_2) \,,$$

and to introduce an implication connective $\rightarrow_{\mathsf{BI}}$ characterized by:

| $\mathsf{ver}^{\mathsf{BI}}_{\rightarrow}$ | 0 | $\frac{1}{2}$ | 1 |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| $\frac{1}{2}$ | 1 | 1 | 1 |
| 1 | 0 | $\frac{1}{2}$ | 1 |

Blau claims that only this implication connective is suitable for a three-valued modeling of (two-valued) sentences of the form "All $A$ are $B$" in natural language.

This claim is essentially based upon the idea that for the truth of a sentence of this form "all objects which have property $A$ also have property $B$" it is completely out of any rational interest to allow the antecedent "a particular object has property $A$" to be undetermined.

Formally this means that inside a universally quantified sentence the case that the antecedent has truth degree $\frac{1}{2}$ should not be a reason that the whole sentence may become not true.

This implication connective may be defined from the basic connectives by

$$H_1 \rightarrow_{\mathsf{Bl}} H_2 = \neg H_1 \vee H_2 \,.$$

The implication connective $\to_{\mathsf{Bl}}$ proves to be quite useful for a lot of discussions. We present only one result which shows some "relative" functional completeness of the set $\neg, \approx, \wedge$ of connectives.

**Proposition:** Each three-valued connective which satisfies the normal condition can be defined from the basic connectives $\neg, \approx, \wedge$ of Blau's system.

The particular applicational interest of this result inside the Blau approach comes from the idea that natural language uses only such connectives which satisfy the normal condition, because the appearance of the truth degree $u = \frac{1}{2}$ for some sentence $H$ (in some particular situation) is simply an **unintended mistake** caused either by the use of vague predicates, or by the use of non-denoting names.

The extension of the system $\neg, \approx, \wedge$ of basic connectives with the connective $\mathsf{T}_3$ as done for the Łukasiewicz systems by J. Słupecki makes $\neg, \approx, \wedge, \mathsf{T}_3$ into a functionally complete system of connectives.

# Four-Valued Systems

There are only a few approaches toward four-valued systems.

One of the rare exceptions is Łukasiewicz who, in his later years, preferred $\mathrm{L}_4$ for a modal reading.

Another four-valued systems has become prominent more recently which makes essential use of the natural partial ordering of the truth degree set $\mathcal{W}_4^* = \{0,1\}^2 = \{(0,0),(0,1),(1,0),(1,1)\}$, and which connects a very natural interpretation with these degrees.

This approach was inspired by theoretical work on systems of relevance logic and later on also applied to considerations on how to treat – possibly inconsistent – information in computers.

In data and knowledge bases information is often stored in the form of “facts”, i.e. sentences which are marked as true or false. This information may have been collected from different sources, and at different times.

The crucial point is that this information usually is incomplete – and often even inconsistent.

Therefore one should allow a computer to answer not only “true” or “false”, but also “I don't know” – and even “true and false”.

The answer “I don't know” indicates incompleteness of the information stored in the data or knowledge base, and the answer “true and false” indicates that this information is inconsistent – in the simplest case because conflicting facts have been stored in the form of two sentences of the forms $H$ and $\neg H$.

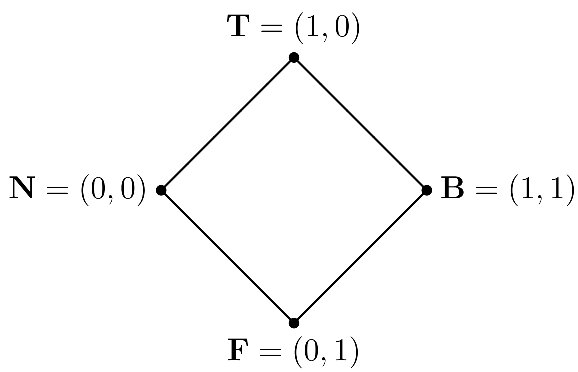
With these four “values” the computer should also be able to “reason” internally.

One interprets such a truth degree $\mathsf{Val}^4(H, \beta) = (a, b)$ that $a$ says whether $H$ is given (inside the data or knowledge base) the truth degree $\top$, and that $b$ says whether $H$ is given the truth degree $\bot$.

So $\mathsf{Val}^4(H, \beta) = (0, 1)$ signalizes that $H$ is given the truth value $\bot$, but not the truth value $\top$, and $\mathsf{Val}^4(H, \beta) = (0, 0)$ signalizes that $H$ is given no truth value at all.

The truth degree $\mathbf{N}$ (for "none") indicates "underdetermination" or a **gap**, i.e. the lack of information on the truth value of $H$, and the truth degree $\mathbf{B}$ (for "both") indicates "overdetermination" or a **glut**, i.e. the presence of contradictory information on the truth value of $H$.

The truth degree set $\mathcal{W}_4^*$ comes with a natural lattice structure and two types of orderings:




left $\Longrightarrow$ right is the **knowledge ordering**,
bottom $\Longrightarrow$ up is the **truth ordering**.

In $\mathcal{W}_4^*$ the standard lattice ordering $\leqq$ goes "bottom-up", i.e. lattice elements which are on a lower level position are smaller ones.

With the corresponding lattice operations $\sqcap, \sqcup$ one has natural candidates for truth degree functions for a conjunction and a disjunction connective $\curlywedge, \curlyvee$ of a (propositional) system $\mathsf{D}_4$ of four-valued logic which is to be based on the intuitions discussed up to now.

These truth degree functions fit well into the intuitive picture, as does the following negation $\neg$:

| $H$ | $\mathbf{F}$ | $\mathbf{N}$ | $\mathbf{B}$ | $\mathbf{T}$ |
|---|---|---|---|---|
| $\neg H$ | $\mathbf{T}$ | $\mathbf{N}$ | $\mathbf{B}$ | $\mathbf{F}$ |

Thus one has for the basic vocabulary of $\mathsf{D}_4$ the set of connectives

$$\mathcal{J}^{\mathsf{D}} = \{\curlywedge, \curlyvee, \neg\}.$$

Straightforward calculations show that for these connectives their truth degree functions w.r.t. the truth degree set $\mathcal{W}^{\mathsf{D}} = \mathcal{W}_4^*$ have the characterizing equations

$$\begin{aligned}
\mathsf{ver}^{\mathsf{D}}_{\curlywedge}((x_1,y_1),(x_2,y_2)) &= (\mathsf{et}(x_1,x_2), \mathsf{vel}(y_1,y_2))\,,\\
\mathsf{ver}^{\mathsf{D}}_{\curlyvee}((x_1,y_1),(x_2,y_2)) &= (\mathsf{vel}(x_1,x_2), \mathsf{et}(y_1,y_2))\,,\\
\mathsf{ver}^{\mathsf{D}}_{\neg}(x_1,y_1) &= (y_1,x_1)\,,
\end{aligned}$$

for all $x_i, y_i \in \{0,1\}$, with $\mathsf{et}, \mathsf{vel}$ here the truth value functions of classical conjunction and classical disjunction, respectively.

All these connectives of $\mathsf{D}_4$ satisfy the normal condition, so $\mathsf{D}_4$ is not functionally complete.

Both three-element sets $\{\mathbf{T},\mathbf{F},\mathbf{N}\}$ and $\{\mathbf{T},\mathbf{F},\mathbf{B}\}$ are closed under these truth degree functions.

They are even isomorphic as algebraic structures with $\mathsf{ver}^{\mathsf{D}}_{\curlywedge}, \mathsf{ver}^{\mathsf{D}}_{\curlyvee}, \mathsf{ver}^{\mathsf{D}}_{\neg}$ as operations – and isomorphic to the algebraic structure $\langle \mathcal{W}_3, \min, \max, \mathsf{non}_L\rangle$.

Therefore the two three-valued subsystems of $\mathsf{D}_4$, which are constituted by the restrictions of the truth degrees to $\{\mathbf{T}, \mathbf{F}, \mathbf{N}\}$ or $\{\mathbf{T}, \mathbf{F}, \mathbf{B}\}$, coincide as three-valued systems, and are subsystems of $\mathsf{L}_3$.

The crucial point now is to define a suitable entailment relation $\models_{\mathsf{D}}$ which fits well into the intuitive realm discussed previously.

A natural approach is to say that a set of wffs $\Sigma$ entails a wff $H$ iff each model of $\Sigma$ is also a model of $H$.

And a model of a set $\Sigma$ of wffs should be some valuation which gives to all the wffs of $\Sigma$ a designated truth degree.

So the problem arises what the designated truth degrees should be. Up to now there is no agreement on this point. One of the possible approaches is to take only $\mathbf{T}$ as a designated truth degree.

Then for our computer a model of a set of formulas is any (partial and non-functional) two-valued valuation which makes all the formulas of $\Sigma$ "definitely true", i.e. **true but not false or valueless**.

Thus we get:

$$\Sigma \models_{\mathsf{D}} H \;\Leftrightarrow\; \mathsf{Mod}^{\mathsf{D}}(\Sigma) \subseteq \mathsf{Mod}^{\mathsf{D}}(H)\,.$$

With applications to relevance logic in mind, J.M. Dunn considers instead both truth degrees $\mathbf{T}, \mathbf{B}$ as designated. The intuition behind this choice is that a wff which has such a designated truth degree is considered as "at least true".

The models in the sense of Dunn for this situation are the $(\geqq \mathbf{B})$-models, with $\leqq$ for the lattice ordering of the truth degree lattice $\mathcal{W}_4^*$. Let us call them **weak models** for the moment, and denote the class of all weak models of some set $\Sigma$ of wffs by $\mathsf{Mod}_0^{\mathsf{D}}(\Sigma)$.

Then Dunn discusses the following notion $\models_{\mathsf{D}}^0$ of **weak entailment**:

$$\Sigma \models_{\mathsf{D}}^0 H \;\Leftrightarrow\; \mathsf{Mod}_0^{\mathsf{D}}(\Sigma) \subseteq \mathsf{Mod}_0^{\mathsf{D}}(H)\,,$$

which is more general than the previous notion, as the following result shows.

**Proposition:** For each set $\Sigma$ of wffs of $\mathsf{D}_4$ and each wff $H$ one has

$$\Sigma \models_{\mathsf{D}}^0 H \quad \Rightarrow \quad \Sigma \models_{\mathsf{D}} H\,.$$

There are a lot of open problems here. Some of them discussed in

Y. Shramko / H. Wansing: Truth and Falsehood; Trends in Logic 36; Springer: Berlin 2011.

# Rosser-Turquette Axiomatization for Finitely-Valued Logics

Let S be an arbitrary propositional system with a characteristic matrix over the finite truth degree set $\mathcal{W}^{\mathsf{S}}$.

Assume that S has a kind of implication connective $\to$, and unary connectives $\mathsf{J}_s$ for each $s \in \mathcal{W}^{\mathsf{S}}$ – at least that as definable ones.

These connectives shall satisfy the following conditions:

(RT1) The truth degree function of $\to$ assumes a non-designated truth degree just in the case that its first argument is a designated truth degree and its second argument is a non-designated one, i.e. $\to$ satisfies the standard condition of an implication connective.

(RT2) The truth degree function of $\mathsf{J}_s$, $s \in \mathcal{W}^{\mathsf{S}}$, has a designated truth degree just in the case that their argument value is $s$, and a non-designated truth degree otherwise.

Define for wff $H_1, H_2, \ldots, G$ recursively:

$$\bigodot_{i=1}^{0}{}^{\!\!\to}(H_i, G) =_{\mathrm{def}} G \,,$$
$$\bigodot_{i=1}^{k+1}{}^{\!\!\to}(H_i, G) =_{\mathrm{def}} \Big(H_{k+1} \to \bigodot_{i=1}^{k}{}^{\!\!\to}(H_i, G)\Big) \,.$$

Additionally, for each connective $\varphi \in \mathcal{J}^{\mathsf{S}}$ its corresponding truth degree function shall be $\mathsf{ver}_\varphi$.

Let the axiom set $\mathsf{Ax}_{\mathrm{RT}}$ be determined by the following schemata:

$\mathsf{Ax}_{\mathrm{RT}}1 :\; A \to (B \to A)$ ,

$\mathsf{Ax}_{\mathrm{RT}}2 :\; (A \to (B \to C)) \to (B \to (A \to C))$ ,

$\mathsf{Ax}_{\mathrm{RT}}3 :\; (A \to B) \to ((B \to C) \to (A \to C))$ ,

$\mathsf{Ax}_{\mathrm{RT}}4 :\; (\mathsf{J}_s(A) \to (\mathsf{J}_s(A) \to B)) \to (\mathsf{J}_s(A) \to B)$ for each $s \in \mathcal{W}^{\mathsf{S}}$ ,

$\mathsf{Ax}_{\mathrm{RT}}5: \ \ominus_{i=1}^{m}(\mathsf{J}_{\frac{i-1}{m-1}}(A) \to B, B)$ ,

$\mathsf{Ax}_{\mathrm{RT}}6: \ \mathsf{J}_s(\mathsf{s})$ for each truth degree $s$ and each truth degree constant s denoting it,

$\mathsf{Ax}_{\mathrm{RT}}7: \ \mathsf{J}_t(A) \to A$ for each designated truth degree $t$ ,

$\mathsf{Ax}_{\mathrm{RT}}8: \ \ominus_{i=1}^{n}(\mathsf{J}_{s_i}(A_i), \mathsf{J}_t(\varphi(A_1, \ldots, A_n)))$ for each $n$-ary connective $\varphi \in \mathcal{J}^{\mathsf{S}}$, for all $s_1, \ldots, s_n \in \mathcal{W}^{\mathsf{S}}$, and for the particular truth degree $t = \mathsf{ver}_{\varphi}(s_1, \ldots, s_n)$.

This logical calculus $\mathbb{K}^m_{\mathrm{RT}}$ has a single rule of inference, the **rule of detachment**

$$\text{(MP)} \qquad \frac{A, A \to B}{B} \, .$$

A wff $H$ is **derivable** in $\mathbb{K}^m_{\mathrm{RT}}$ or a **theorem** of $\mathbb{K}^m_{\mathrm{RT}}$, denoted: $\vdash_{\mathrm{RT}} H$, iff there exists a finite sequence $H_1, H_2, \dots, H_n$ of wffs such that

$\mathbf{(D1)}$ $H_n$ is the wff $H$;

$\mathbf{(D2)}$ each wff $H_k$ of this sequence is either an axiom of $\mathbb{K}^m_{\mathrm{RT}}$ or the result of an application of the inference rule MP to preceding wffs of this sequence.

And each such sequence is called a **derivation** or a **proof** of $H$ (in $\mathbb{K}^m_{\mathrm{RT}}$).

Thus axiom schema (Ax$_{\mathrm{RT}}8$) codes for each connective $\varphi$ its truth degree behavior, i.e. its truth degree function.

Similarly (Ax$_{\mathrm{RT}}7$) codes that each wff with (provably) designated truth degree is derivable in $\mathbb{K}^m_{\mathrm{RT}}$.

And axiom schema (Ax$_{\mathrm{RT}}6$) codes for a truth degree constant t that one can derive that t denotes $t$.

**Soundness Theorem** Suppose that some propositional system S of many-valued logic satisfies the conditions (RT$1$) and (RT$2$). Then the logical calculus $\mathbb{K}^m_{\mathrm{RT}}$ is sound w.r.t. this system S, i.e. each $\mathbb{K}^m_{\mathrm{RT}}$-derivable wff is an S-tautology.

**Completeness Theorem** Suppose that some propositional system S of many-valued logic satisfies the conditions (RT$1$) and (RT$2$). Then the logical calculus $\mathbb{K}^m_{\mathrm{RT}}$ is complete w.r.t. this system S, i.e. each S-tautology is $\mathbb{K}^m_{\mathrm{RT}}$-derivable.

Looking at suitable proofs of these theorems, one recognizes that there are proofs of the Completeness Theorem which do not use any semantical property of the connective $\rightarrow$, particularly not use the standard condition (**I**). And they additionally do not use any semantical property of $\mathsf{J}_t$.

There are proofs of the Soundness Theorem which only use the conditions:

**(E1)** An implication $H \to G$ has a non-designated truth degree provided that $H$ has a designated and $G$ an undesignated truth degree.

**(E2)** Each axiom from ($\mathsf{Ax}_{\mathrm{RT}}1$),..., ($\mathsf{Ax}_{\mathrm{RT}}8$) is an S-tautology.

Thus one even has

**Theorem:** $\mathbb{K}^m_{\mathrm{RT}}$ provides an adequate axiomatization for each propositional system S′ of many-valued logic with $m$ truth degrees which has in its alphabet connectives $\to$ and $\mathsf{J}_t$, for each $t \in \mathcal{W}^{\mathsf{S}'}$, which satisfy conditions (E1) and (E2) – or in which such connectives are definable.

**Example:** The implication of the Łukasiewicz $m$-valued propositional systems does not satisfy the standard condition (I), but these systems can be adequately axiomatized by the corresponding calculi $\mathbb{K}^m_{\mathrm{RT}}$.

**Theorem:** The axiom schemata $(\mathsf{Ax}_{\mathrm{RT}}1), \ldots, (\mathsf{Ax}_{\mathrm{RT}}8)$ constitute together with the rule of detachment $(\mathsf{MP})$ an adequate axiomatization for each one of the finitely many-valued systems $\mathsf{L}_m$.

## Axiomatizing First-Order Systems

This type of axiomatization has been extended by Rosser-Turquette also to first-order systems.

Crucial points of this extension are that the truth degree behavior of the quantifiers as well as that one of the basic predicate symbols has to be describable in the language of the corresponding systems.

**Remark:**

The Hilbert-type axiomatization method of Rosser-Turquette is suitable for a large class of finitely many-valued propositional logics. Its main drawback is that it needs besides the implication connective $\rightarrow$ the whole class of all the connectives $\mathsf{J}_t$ for each truth degree $t$.

This is not a severe restriction for functionally complete system. However, a lot of interesting systems of propositional MVL are not functionally complete.

Therefore it would be quite welcome to have (also) a method of axiomatization which does not presuppose these restrictions.

Such an approach was first described by K. Schröter in 1955. And his Gentzen-type system is also suited to formalize the relation of entailment. We discuss it later on.

## Axiomatizing the Entailment Relation

The knowledge of the entailment relation for a system S of MVL is at least as basic as the knowledge of its logically valid wffs.

Hence look for a complete axiomatization of the entailment for S: given the system S we ask for a logical calculus $\mathbb{K}$ with a notion $\vdash_{\mathbb{K}}$ of $\mathbb{K}$-derivability such that for any (suitable) set $\Sigma$ of wffs that a wff $H$ is entailed by the set $\Sigma$ of premises iff $H$ is $\mathbb{K}$-derivable from this set $\Sigma$.

For any logical calculus $\mathbb{K}$ which completely formalizes the notion of entailment for a system S, i.e. which satisfies

$$\Sigma \vdash_{\mathbb{K}} H \quad \text{iff} \quad \Sigma \models_{\mathsf{S}} H$$

for any wff $H$ and each set $\Sigma$ of wffs, one says that $\mathbb{K}$ **adequately axiomatizes** the entailment relation $\models_{\mathsf{S}}$.

There is a close connection between both kinds of axiomatization: that of the entailment relation for S and that of the logical validity for S. For, having a logical calculus $\mathbb{K}$ which satisfies this last condition, then one has immediately

$$\emptyset \vdash_{\mathbb{K}} H \quad \text{iff} \quad H \in \mathsf{taut}^{\mathsf{S}},$$

which means, that in this case the logical calculus $\mathbb{K}$ also adequately axiomatizes the class of S-tautologies.

Therefore also for $\mathbb{K}$-derivability one usually writes

$$\vdash_{\mathbb{K}} H \quad \text{instead of} \quad \emptyset \vdash_{\mathbb{K}} H\,.$$

Some care is necessary with this type of notation. Having a logical calculus $\mathbb{K}$ to formalize the notion of entailment, then "$\Sigma \vdash_{\mathbb{K}} H$" as well as "$\vdash_{\mathbb{K}} H$" are suitably defined notions. Having, however, a logical calculus $\mathbb{K}_0$ to formalize the notion of logical validity, one usually has only "$\vdash_{\mathbb{K}_0} H$" as a suitable notation and has "$\Sigma \vdash_{\mathbb{K}_0} H$" undefined.

For a wide class of logical calculi there is a simple standard way to extend logical calculi which derive wffs to logical calculi which allow to derive wffs from given sets of premises.

The extension considers as a $\mathbb{K}$-derivation from a given set $\Sigma$ of premises any finite sequence $H_1, H_2, \ldots, H_n$ which satisfies the conditions

$(\mathbf{D1})$ $H_n$ is the wff $H$;

$(\mathbf{D}2')$ each wff $H_k$ of this sequence is either an axiom of $\mathbb{K}_0$, or an element of $\Sigma$, or the result of an application of one of the inference rules of $\mathbb{K}_0$ to preceding wffs of this sequence.

This extended notion of $\mathbb{K}$-derivability will be called **standard extension** of $\vdash_{\mathbb{K}}$ and denoted by $\vdash^{\star}_{\mathbb{K}}$.

This standard extension allows not only to derive wffs "relative" to some given set of premises, it allows also to "shorten" derivations by using new inference rules, which have to be "suitable" in some appropriate sense.

An inference rule

$$\frac{H_1, H_2, \ldots, H_n}{G}$$

is **admissible**, or **sound**, for $\mathbb{K}$ iff $\{H_1, \ldots, H_n\} \vdash^{\star}_{\mathbb{K}} G$.

Intuitively, such an admissible inference rule symbolizes a whole derivation which leads from $\{H_1, \ldots, H_n\}$ to $G$.

Some few additional assumptions suffice to prove that the standard extension of a logical calculus $\mathbb{K}$ adequately axiomatizes the notion of entailment, provided $\mathbb{K}$ adequately axiomatizes logical validity. The additional properties to be considered are:

$(\mathrm{FIN}_{\models})$ For each wff $H$ and all sets $\Sigma$ of wffs it holds true

$$\Sigma \models H \quad \text{iff} \quad \exists\, \Sigma' \subseteq \Sigma(\Sigma' \text{ finite } \Lleftarrow\!\!\!\!\wedge\; \Sigma' \models H)\,.$$

$(\mathrm{FIN}_{\vdash})$ For each wff $H$ and all sets $\Sigma$ of wffs it holds true

$$\Sigma \vdash H \quad \text{iff} \quad \exists\, \Sigma' \subseteq \Sigma(\Sigma' \text{ finite } \Lleftarrow\!\!\!\!\wedge\; \Sigma' \vdash H)\,.$$

$(\mathrm{DED}_{\models})$ For all wffs $H_1, H_2$ and all sets $\Sigma$ of wffs it holds true

$$\Sigma \cup \{H_1\} \models H_2 \quad \text{iff} \quad \Sigma \models (H_1 \rightarrow H_2)\,.$$

$(\mathrm{DED}_{\vdash})$ For all wffs $H_1, H_2$ and all sets $\Sigma$ of wffs it holds true

$$\Sigma \cup \{H_1\} \vdash H_2 \quad \text{iff} \quad \Sigma \vdash (H_1 \rightarrow H_2)\,.$$

Property $(\mathrm{FIN}_{\models})$ is an important property of the logical system under consideration and of its entailment relation. In the case that $(\mathrm{FIN}_{\models})$ holds true for a logical system S one says that for S the **finiteness theorem for entailment** holds true.

Property $(\mathrm{FIN}_{\vdash})$ is a property of the logical calculus under consideration. It usually holds true in all the cases where all the inference rules of this logical calculus have only finitely many premises, i.e. are **finitary**.

Property $(\mathrm{DED}_{\models})$ is essentially a condition which is related to the set $\mathcal{J}^{\mathsf{S}}$ of connectives of the logical system S under consideration and to an implication connective (resp. its truth degree function) this set should contain.

Property $(\mathrm{DED}_{\vdash})$ is again a property of the logical calculus under consideration and amounts to formal properties regarding the derivability of implications within this calculus. In the case that $(\mathrm{DED}_{\vdash})$ holds true for a logical calculus $\mathbb{K}$ one says that for $\mathbb{K}$ (or for its notion $\vdash_{\mathbb{K}}$ of derivability) the **deduction theorem** holds true.

The properties $(\mathrm{FIN}_{\vdash})$ and $(\mathrm{DED}_{\models})$ are not too strong and important ones, properties $(\mathrm{FIN}_{\models})$ and $(\mathrm{DED}_{\vdash})$ on the other hand are strong and important properties.

All together, these four properties allow to characterize such cases in which the standard extension of a derivability relation provides an adequate axiomatization of the entailment relation.

**Theorem:** Let S be a propositional system of MVL and $\mathbb{K}$ a logical calculus which adequately axiomatizes the set of S-tautologies. Suppose that the entailment relation $\models_{\mathsf{S}}$ of S satisfies properties $(\mathrm{FIN}_{\models})$ and $(\mathrm{DED}_{\models})$, and that $\vdash^{\star}_{\mathbb{K}}$ satisfies properties $(\mathrm{FIN}_{\vdash})$ and $(\mathrm{DED}_{\models})$. Then this standard extension $\vdash^{\star}_{\mathbb{K}}$ adequately axiomatizes the entailment relation of S.

Because of this result, for any logical system S with an adequately axiomatizable set $\mathrm{taut}^{\mathsf{S}}$ of logically valid wffs, and with an entailment relation for which the finiteness theorem holds true, one usually asks whether the deduction theorem holds true for suitable logical calculi adequately axiomatizing $\mathrm{taut}^{\mathsf{S}}$.

**Proposition:** If the implication connective which appears in $(\mathrm{DED}_{\models})$ satisfies the standard condition $(\mathrm{I})$ of an implication connective then property $(\mathrm{DED}_{\models})$ holds true.

So property $(\mathrm{DED}_{\vdash})$ needs to be considered.

**Proposition:** In each one of the Rosser-Turquette calculi $\mathbb{K}^m_{\mathrm{RT}}$ it holds true that

$$(\mathrm{DED}_{\vdash}) \quad \text{iff} \quad \vdash_{\mathrm{RT}} ((A \to B) \to ((A \to (B \to C)) \to (A \to C))) \,.$$

Because of the Completeness Theorem $(\mathrm{DED}_{\vdash})$ is established if one is able to show

$$((A \to B) \to ((A \to (B \to C)) \to (A \to C))) \in \mathsf{taut}^{\mathsf{S}}$$

for the system S of MVL which is axiomatized by $\mathbb{K}^m_{\mathrm{RT}}$.

If $\to$ satisfies the standard condition this is a routine matter.

Thus we have also the following result.

**Proposition:** If the implication connective of a system S, which appears in $(\mathrm{DED}_\vdash)$, satisfies the standard condition $(\mathrm{I})$ of an implication connective then the deduction theorem $(\mathrm{DED}_\vdash)$ holds true for the logical calculus $\mathbb{K}^m_{\mathrm{RT}}$ which adequately axiomatizes S.

Both these propositions together establish the following main theorem.

**Strong Completeness Theorem:** Let S be a propositional system of many-valued logic with $m$ truth degrees which satisfies the conditions *(*RT*1)*, *(*RT*2)*. Then the logical calculus $\mathbb{K}^m_{\mathrm{RT}}$ with the standard extension $\vdash^\star_{\mathrm{RT}}$ as its derivability relation provides an adequate axiomatization of the entailment relation $\models_{\mathsf{S}}$, i.e. then there always holds true

$$\Sigma \models_{\mathsf{S}} H \quad \text{iff} \quad \Sigma \vdash^\star_{\mathrm{RT}} H \,.$$

A closer inspection of the proof of this theorem shows that one really only needs for the logical calculus under consideration that (MP) and ($\mathsf{Ax}_{\mathrm{RT}}1$) are available – either as a (primitive) rule of inference and an axiom schema, or as an admissible rule of inference and a derivable schema.

The Łukasiewicz systems show that there exist also $m$-valued propositional systems S which are adequately axiomatized by $\mathbb{K}^m_{\mathrm{RT}}$, which however have an implication connective for which the deduction theorem does not hold true, and which do not satisfy condition (RT1). The assumptions of the last theorem hence are sufficient ones, but not necessary ones.

# Tableau Calculi
# for Many-Valued Logic

Tableau calculi for classical logic start from given wff $H$, and form systematically series of tree-like graphs, the **tableaux** of the construction, up to some level of completeness of the construction.

Tableau constructions intend to find either a refutation of the starting wff $H$, or to find a valuation which gives $H$ the truth value $\top$.

The nodes of these tree-like graphs become decorated with wffs, or sets of wffs, and the edges indicate the succession of the construction steps toward the completed tableau. The most elementary ones of these tableaux are given by a graph with only one node (and no edges at all) which is decorated with the "initial" wff $H$.

The tableau constructions which are most suitable for a generalization to the MVL-setting "decorates" the nodes with signed formulas.

A **signed formula** is an ordered pair of a wff and some truth value as "sign".

**Example:**

$$\{\bot\} : (H_1 \Wedge H_2 \Rightarrow H_1) \Wedge (H_1 \Rightarrow H_1 \Vee H_2)$$

$$\begin{array}{cc}
\{\bot\} : (H_1 \Wedge H_2 \Rightarrow H_1) & \{\bot\} : (H_1 \Rightarrow H_1 \Vee H_2) \\
| & | \\
\{\top\} : H_1 \Wedge H_2 & \{\top\} : H_1 \\
\{\bot\} : H_1 & \{\bot\} : H_1 \Vee H_2 \\
| & | \\
\{\top\} : H_1 & \{\bot\} : H_1 \\
\{\top\} : H_2 & \{\bot\} : H_2
\end{array}$$

Each branch analyses one possibility to give to the "root formula" the truth value which is indicated in its sign.

The tableau construction usually is a refutation procedure: to show that a wff is logically valid one discusses the possibilities to make it false.

A **complete** tableau shows that this is impossible – if each of its branches is **closed**, i.e. contains some wff with two different signs.

If a complete tableau has an **open** branch then one has a construction how to give the starting wff the truth value given by its sign.

This construction idea can also be applied for systems of MVL.

Now the signs are truth degrees, **or** sets of truth degrees, out of $\mathcal{W}^{\mathsf{S}}$.

And for a signed formula $\mathsf{s} : H$ the sign s indicates that the truth degree of $H$ should be s or belong to s.

Again, tableau constructions in MVL follow rules how to extend some given tableau at end nodes of its branches.

The form of these tableau extension rules, particularly the number of branches into which the branch under consideration has to split, also depends on the sign, this wff $H$ is attached with, as well as on the whole set of truth degrees $\mathcal{W}^{\mathsf{S}}$ of the system S under consideration.

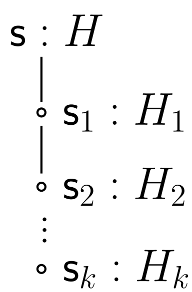
For $\mathcal{W}^{\mathsf{S}} = \mathcal{W}_m$ finite, the branching number of the tableau trees remains finite, and hence all the tableaux trees.

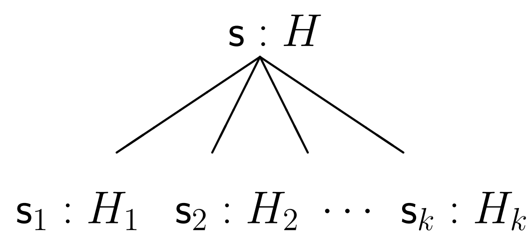
**Example:**

Consider some 3-valued system $\mathsf{S}_0$ which has a disjunction connective $\vee_*$ with truth degree function $\mathsf{vel}_L$.

$$\begin{array}{c} \{\tfrac{1}{2}\} : (H_1 \vee_* H_2) \\ \hline \begin{array}{cc} \{\tfrac{1}{2}\} : H_1 & \{0\} : H_1 \\ \{0\} : H_2 & \{\tfrac{1}{2}\} : H_2 \end{array} \end{array} \qquad \begin{array}{c} \{0, \tfrac{1}{2}\} : (H_1 \vee_* H_2) \\ \hline \begin{array}{cc} \{0, \tfrac{1}{2}\} : H_1 & \{0\} : H_1 \\ \{0\} : H_2 & \{0, \tfrac{1}{2}\} : H_2 \end{array} \end{array}$$

There are two main types of simple subtrees which can be considered as building blocks for the tableau extensions – finite chains and finite branchings:




**Rules:**

$$\frac{\mathsf{s} : H}{\mathsf{s}_1 : H_1 \circ \mathsf{s}_2 : H_2 \circ \cdots \circ \mathsf{s}_k : H_k} \qquad \frac{\mathsf{s} : H}{\mathsf{s}_1 : H_1 + \mathsf{s}_2 : H_2 + \cdots + \mathsf{s}_k : H_k}$$

or shorthand

$$\frac{\mathsf{s} : H}{\circ\, \{\mathsf{s}_i : H_i \,|\, 1 \le i \le k\}} \qquad \frac{\mathsf{s} : H}{+\, \{\mathsf{s}_i : H_i \,|\, 1 \le i \le k\}}\,.$$

Tableau extensions provide a kind of **complete**, but **irredundant** analysis of all the cases which lead to the situation that the root formula has one of the truth degrees which are given by its sign.

In general a tableau extension will proceed by some combination of extensions of these two simple types yielding a branching into linear pieces.

Particular tableau extension rules depend on the definition of the (truth degree functions of the) connectives of the particular system S of (propositional) MVL.

A tableau extension rule for an $n$-ary connective $\varphi$ of some system S of MVL is a (partial) function from the class of signed formulas into the class of tableau trees. Each such rule has the form

$$\frac{\mathsf{s} : \varphi(H_1, \ldots, H_n)}{+ \big\{ \circ \{ \hat{\mathsf{s}}_j : H_{i_j} \,|\, (j, \hat{\mathsf{s}}_j) \in M \} \,\big|\, M \in \mathcal{T}(\varphi; \mathsf{s}) \big\}} \; .$$

Here $\mathcal{T}(\varphi; \mathsf{s})$ is a suitable class of finite sets $M$ of ordered pairs $(j, \hat{\mathsf{s}}_j)$.

Assuming that sufficiently many signs exist, the following holds true.

**Proposition:** For a signed formula $\mathsf{s} : \varphi(H_1, \ldots, H_n)$ there does not exist a tableau extension rule iff one has $\mathsf{s} \cap \mathsf{rg}(\mathsf{ver}_\varphi^{\mathsf{S}}) = \emptyset$.

**Corollary:** If there exists a tableau extension rule for some signed formula $\mathsf{s} : \varphi(H_1, \ldots, H_n)$, and if $\bar{\mathsf{s}}$ is a sign such that $\mathsf{s} \subseteq \bar{\mathsf{s}}$, then there exists such a tableau extension rule also for the signed formula $\bar{\mathsf{s}} : \varphi(H_1, \ldots, H_n)$.

**Tableau construction process** for $\mathsf{s} : H$:

- Take the tree with only the source node decorated with the $\mathsf{s} : H$.
- For tableaux for $\mathsf{s} : H$ which are not closed, extend any one of its branches, which are not themselves closed, by an application of some tableau extension rule to some signed compound formula of this branch, which is not marked as "used" – and mark this signed formula in the extended tableau as **used**.
- A branch of a tableau is **closed** if it has nodes which are decorated with signed formulas $\mathsf{s}_1 : G, \ldots, \mathsf{s}_h : G$ such that $\bigcap_{j=1}^{h} \mathsf{s}_j = \emptyset$, or if it contains only unmarked signed formulas for which there do not exist tableau extension rules.

- Mark a tableau as **closed** if all its branches are marked as closed.
- A branch as well as a tableau is **open** iff it is not closed. A branch is **complete** iff all its signed compound formulas for which there exists a tableau extension rule are marked as used. A tableau is **complete** iff all its branches are complete.

This tableau construction for a signed formula $\mathsf{s} : H$ ends either if all branches of some tableau for $\mathsf{s} : H$ are closed, or if some complete tableau is reached.

This tableau extension method preserves satisfiability.

**Definition:** A valuation $\beta$ is a **model** of a signed formula $\mathsf{s} : H$ of iff $\mathsf{Val}^{\mathsf{S}}(H, \beta) \in \mathsf{s}$. And it is a model of some set $M$ of signed formulas iff it is a model for each signed formula $\mathsf{s} : H \in M$.

A signed formula, as well as a set of signed formulas, is **satisfiable** iff there exists some model of it.

A branch of some tableau is called **satisfiable** iff the set of all its signed formulas is satisfiable. And a tableau is **satisfiable** iff it has a satisfiable branch.

**Proposition:** Each extension of a satisfiable tableau for $\mathsf{s} : H$ which is constructed according to one of the tableau extension rules is a satisfiable tableau for $\mathsf{s} : H$.

Again as in the case of classical logic, the existence of some closed tableau for a signed formula $\mathsf{s} : H$ means that there does not exist a valuation $\beta$ with $\mathsf{Val}^{\mathsf{S}}(H, \beta) \in \mathsf{s}$, i.e. there does not exist a model of $\mathsf{s} : H$.

And if our tableau construction process stops with a complete tableau which is not closed, then one can construct from its non-closed branches valuations $\beta$ with $\mathsf{Val}^{\mathsf{S}}(H, \beta) \in \mathsf{s}$.

**Soundness Theorem:** Let S be some system of many-valued logic with finitely many truth degrees. Then the existence of a closed tableau for the signed formula $(\mathcal{W}^{\mathsf{S}} \setminus \mathcal{D}^{\mathsf{S}}) : H$ means that the wff $H$ of $\mathcal{L}_{\mathsf{S}}$ is S-logically valid.

**Completeness Theorem:** Let S be a system of many-valued logic with finitely many truth degrees. Then there exists a closed tableau for the signed formula $(\mathcal{W}^{\mathsf{S}} \setminus \mathcal{D}^{\mathsf{S}}) : H$ if the wff $H$ of $\mathcal{L}_{\mathsf{S}}$ is S-logically valid.

# Sequent Calculi for Many-Valued Logic

An adaptation of the GENTZEN sequent calculi to finitely-valued systems of MVL was introduced by K. Schröter 1955.

It does not have to refer to conditions (RT1), (RT2) for the Rosser-Turquette axiomatization.

An S-**sequent** $(\Sigma_1|\ldots|\Sigma_m)$ is a finite sequence $(\Sigma_1,\ldots,\Sigma_m)$ of sets of wff.

$(\Sigma_1|\ldots|\Sigma_m)$ is **finite** iff all $\Sigma_i$ are finite.

A valuation $\beta : \mathcal{V}_0 \to \mathcal{W}^{\mathsf{S}}$ is a **sequent model** of $(\Sigma_1|\ldots|\Sigma_m)$ iff $\beta$ is a $\tau_i$-model of $\Sigma_i$ for each $i = 1,\ldots,m$.

An S-sequent $(\Sigma_1|\ldots|\Sigma_m)$ is **valid**, written $\mathsf{S}\mathrm{val}(\Sigma_1|\ldots|\Sigma_m)$, iff there does **not** exist a sequent model of $(\Sigma_1|\ldots|\Sigma_m)$.

**Remark:** In principle also another, dual approach is possible here: to call a valuation $\beta$ a sequent-model of an S-sequent $(\Sigma_1|\ldots|\Sigma_m)$ iff there exists some $i \leq m$ and some $H \in \Sigma_i$ with $\mathsf{Val}^{\mathsf{S}}(H,\beta) = \tau_i$. Then, of course, $(\Sigma_1|\ldots|\Sigma_m)$ has to be taken as valid iff each valuation $\beta$ is a sequent-model of $(\Sigma_1|\ldots|\Sigma_m)$.

The present one, however, is the better suited one if one is interested in discussing sequent and tableau calculi in parallel.

For each truth degree $t \in \mathcal{W}^{\mathsf{S}}$ denote by $\mathcal{K}^{\mathsf{S}}_{\neq t}$ the set of all truth degree constants of S which do not denote $t$.

Consider for S-sequents $(\Sigma_1|\ldots|\Sigma_m)$ the condition

**(Disj)** There exist indices $i, j \leq m$ with $i \neq j$ such that

$$\Sigma_i \cap (\Sigma_j \cup \mathcal{K}^{\mathsf{S}}_{\neq \tau_i}) \neq \emptyset\,.$$

**Proposition:**

$(i)$ If $(\Sigma_1|\ldots|\Sigma_m)$ satisfies $(\mathrm{Disj})$ then $\mathsf{S}\mathrm{val}(\Sigma_1|\ldots|\Sigma_m)$.

$(ii)$ If each $\Sigma_i$ is a set of atomic wffs, i.e. of propositional variables or truth degree constants, then:

$$(\mathrm{Disj}) \quad \text{iff} \quad \mathsf{S}\mathrm{val}(\Sigma_1|\ldots|\Sigma_m).$$

Now consider extensions of sequents. To **introduce** a wff $H$ in position $k \leq m$ into a sequent $(\Sigma_1|\dots|\Sigma_m)$ means the transition

$$\text{from } (\Sigma_1|\dots|\Sigma_m) \text{ to } (\Sigma_1|\dots|\Sigma_{k-1}|\Sigma_k \cup \{H\}|\Sigma_{k+1}|\dots|\Sigma_m).$$

**Proposition:** For each $n$-ary connective $\varphi$, any wffs $H_1, \dots, H_n$, and each $1 \leq k \leq m$ one can formulate a necessary and sufficient condition $G_{\varphi,k}$ characterizing the introducibility of $\varphi(H_1, \dots, H_n)$ in position $k$ in such a way that the S-sequent, which results from any given S-sequent $(\Sigma_1|\dots|\Sigma_m)$ by introducing the wff $\varphi(H_1, \dots, H_n)$ in position $k$, is a valid S-sequent iff certain S-sequents are valid ones which result from $(\Sigma_1|\dots|\Sigma_m)$ via introduction of (some of the) wffs $H_1, \dots, H_n$ in suitable positions.

For completeness it should be noted that these conditions $G_{\varphi,k}$ in general are not uniquely determined. Nevertheless, to formulate such conditions often is a routine matter.

**Example:** Consider a MVL system $\mathsf{S}^*$ with truth degree set $\mathcal{W}_3$. Suppose that an implication connective $\to_*$ is present with truth degree function $\mathrm{seq}_G$.

Then one has the following three introduction rules for $\to_*$ in the different possible places:

$$\frac{(\Sigma_1|\Sigma_2, A|\Sigma_3, A) \qquad (\Sigma_1', B|\Sigma_2'|\Sigma_3')}{(\Sigma_1, \Sigma_1', A \to_* B|\Sigma_2, \Sigma_2'|\Sigma_3, \Sigma_3')},$$

$$\frac{(\Sigma_1|\Sigma_2|\Sigma_3, A) \qquad (\Sigma_1'|\Sigma_2', B|\Sigma_3')}{(\Sigma_1, \Sigma_1'|\Sigma_2, \Sigma_2', A \to_* B|\Sigma_3, \Sigma_3')},$$

$$\frac{(\Sigma_1, A|\Sigma_2, A|\Sigma_3, B) \qquad (\Sigma_1', A|\Sigma_2', B|\Sigma_3', B)}{(\Sigma_1, \Sigma_1'|\Sigma_2, \Sigma_2'|\Sigma_3, \Sigma_3', A \to_* B)}.$$

Ask now for a suitable logical calculus $\mathbb{K}_{\mathrm{G}}^{m}$ which adequately formalizes this notion of validity.

The purpose of $\mathbb{K}_{\mathrm{G}}^{m}$ is to generate just the valid S-sequents.

Because $\mathbb{K}_{\mathrm{G}}^{m}$ has to generate S-sequents $(\Sigma_1|\ldots|\Sigma_m)$, the language of $\mathbb{K}_{\mathrm{G}}^{m}$ has to be more expressive than the language of S.

The crucial point is that the language for $\mathbb{K}_{\mathrm{G}}^{m}$ has to be able to treat sets of wffs, because after being able to do this it is a routine matter to treat also S-sequents.

The extended language for $\mathbb{K}_{\mathrm{G}}^{m}$ thus needs some kind of reference to a suitable set theoretic world. Fortunately the details of this world are not really important here because we are going to discuss problems of derivability in $\mathbb{K}_{\mathrm{G}}^{m}$ – but need not really produce particular derivations.

The **axioms** of $\mathbb{K}_{\mathrm{G}}^{m}$ shall be all those S-sequents $(\Sigma_1|\dots|\Sigma_m)$ which satisfy the condition (Disj).

And the **inference rules** of $\mathbb{K}_{\mathrm{G}}^{m}$ shall be the **thinning** rule together with an **introduction** rule for each connective $\varphi \in \mathcal{J}^{\mathsf{S}}$ and each position $k = 1, \dots, m$.

The **introduction rule** for the $n$-ary connective $\varphi$ in position $k$ is

$$\frac{(\Sigma_1^1|\dots|\Sigma_m^1), \dots, (\Sigma_1^n|\dots|\Sigma_m^n)}{(\Sigma_1|\dots|\Sigma_k \cup \{\varphi(H_1, \dots, H_n)\}|\dots|\Sigma_m)}$$

with the premises of this rule determined by the conditions $G_{\varphi,k}$ of the previous Proposition. And the **thinning rule** is

$$\frac{(\Sigma_1|\dots|\Sigma_m)}{(\Sigma_1 \cup \Theta_1|\dots|\Sigma_m \cup \Theta_m)}$$

for any S-sequents $(\Sigma_1|\dots|\Sigma_m), (\Theta_1|\dots|\Theta_m)$.

A **derivation** in $\mathbb{K}^m_{\mathrm{G}}$ is a finite sequence of S-sequents such that each one of them is either an axiom of $\mathbb{K}^m_{\mathrm{G}}$ or results from previous ones in the sequence by application of one of the inference rules of $\mathbb{K}^m_{\mathrm{G}}$.

Write $\vdash_{\mathrm{G}} (\Sigma_1|\ldots|\Sigma_m)$ to indicate that $(\Sigma_1|\ldots|\Sigma_m)$ is derivable in $\mathbb{K}^m_{\mathrm{G}}$, i.e. is the last sequent of some $\mathbb{K}^m_{\mathrm{G}}$-derivation.

**Soundness Theorem for $\mathbb{K}^m_{\mathrm{G}}$:** If a S-sequent is $\mathbb{K}^m_{\mathrm{G}}$-derivable then it is a valid S-sequent.

**Proposition:**

1. If $(\Sigma_1|\ldots|\Sigma_m)$ satisfies $(\mathrm{Disj})$ then $\vdash_{\mathrm{G}} (\Sigma_1|\ldots|\Sigma_m)$.
2. If each $\Sigma_i$ is a set of atomic wffs, then: $(\mathrm{Disj})$ iff $\vdash_{\mathrm{G}} (\Sigma_1|\ldots|\Sigma_m)$.

A **finite subsequent** of $(\Sigma_1|\ldots|\Sigma_m)$ is a S-sequent $(\Theta_1|\ldots|\Theta_m)$ such that each $\Theta_k \subseteq \Sigma_k$ is finite.

**Compactness Theorem:** If each finite subsequent of an S-sequent $(\Sigma_1|\ldots|\Sigma_m)$ has a sequent model, then also $(\Sigma_1|\ldots|\Sigma_m)$ has a sequent model.

**Finiteness Theorem for Entailment:** One has for all S-sequents $(\Sigma_1|\ \ldots\ |\Sigma_m)$:

1. If $\mathrm{Sval}(\Sigma_1|\ldots|\Sigma_m)$ then there exists a finite subsequent $(\Sigma_1^*|\ldots|\Sigma_m^*)$ of $(\Sigma_1|\ldots|\Sigma_m)$ with $\mathrm{Sval}(\Sigma_1^*|\ldots|\Sigma_m^*)$.
2. If $\vdash_{\mathrm{G}}\ (\Sigma_1|\ldots|\Sigma_m)$ then there exists a finite subsequent $(\Sigma_1^*|\ldots|\Sigma_m^*)$ of $(\Sigma_1|\ldots|\Sigma_m)$ with $\vdash_{\mathrm{G}} (\Sigma_1^*|\ldots|\Sigma_m^*)$ holds true.

**Completeness Theorem:** Every valid S-sequent is a derivable sequent in the calculus $\mathbb{K}_{\mathrm{G}}^m$.

For an adequate formalization of the entailment relation $\models_{\mathsf{S}}$ one has to distinguish between designated and non-designated truth degrees.

Suppose

$$\mathcal{D}^{\mathsf{S}} = \{\tau_{k_1}, \tau_{k_2}, \ldots, \tau_{k_s}\} .$$

**Theorem:** $\Sigma \models_{\mathsf{S}} H$ iff for each non-designated truth degree $t_0 \in \mathcal{W}^{\mathsf{S}} \setminus \mathcal{D}^{\mathsf{S}}$ and each partition $(\Theta_1)_{i \leq s}$ of $\Sigma$ it holds $\vdash_{\mathrm{G}} (\Sigma_1, \ldots, \Sigma_m)$ for the S-sequent $(\Sigma_1 | \ldots | \Sigma_m)$ with

$$\Sigma_j = \begin{cases} \Theta_i, & \text{if } j = k_i \text{ for some } i \leq s \\ \{H\}, & \text{if } j = m - t_0 \cdot (m-1) \\ \emptyset & \text{otherwise} \end{cases}$$

for each $j = 1, \ldots, m$.

This characterization of the entailment relation still has a disadvantage: if S has more than one designated truth degree, each infinite set $\Sigma_0$ of wffs of $\mathcal{L}_{\mathsf{S}}$ has infinitely many partitions $(\Theta_i)_{i \leq s}$, i.e. $\Sigma_0 \models_{\mathsf{S}} H$ is characterized by the $\mathbb{K}^m_{\mathrm{G}}$-derivability of infinitely many S-sequents.

If S has only one designated truth degree, i.e. if $\mathcal{D}^{\mathsf{S}} = \{1\} = \{\tau_1\}$, this difficulty disappears.

**Corollary:** Let the $m$-valued propositional system S have only $1$ as designated truth degree. Then always:

$$\begin{aligned} \Sigma \models_{\mathsf{S}} H \quad \text{iff} \quad & \vdash_{\mathrm{G}} (\Sigma|H|\emptyset|\dots|\emptyset) \\ \text{and} \quad & \vdash_{\mathrm{G}} (\Sigma|\emptyset|H|\dots|\emptyset) \\ \text{and} \quad & \vdash_{\mathrm{G}} (\Sigma|\emptyset|\emptyset|H|\dots|\emptyset) \\ & \vdots \\ \text{and} \quad & \vdash_{\mathrm{G}} (\Sigma|\emptyset|\dots|\emptyset|H)\,. \end{aligned}$$

Another reading of the S-sequents is possible which refers to the signed formulas used for tableau constructions, now with only truth degrees as signs.

An S-sequent $\mathcal{S} = (\Sigma_1| \dots |\Sigma_m)$ can be interpreted as the set

$$\mathcal{S}^{\sim} = \bigcup_{i=1}^{m} \{\tau_i : H \mid H \in \Sigma_i\}$$

of signed formulas.

Then a valuation $\beta$ is a sequent model of $\mathcal{S}$ iff it is a model of the set $\mathcal{S}^{\sim}$ of signed formulas:

$$\mathsf{S}\mathrm{val}(\mathcal{S}) \quad \Leftrightarrow \quad \mathcal{S}^{\sim} \text{ not } \mathsf{S}\text{-satisfiable}\,.$$

This point of view opens the way to form sequent calculi for sequents built up from signed formulas. These modified calculi have the advantage that their sequents can simply be taken as (finite) sequences of signed formulas.

The length of these sequences is no longer tied with the number of truth degrees here. The truth degrees are, instead, represented by the signs.

And the inference rules in the case of this approach also get a simpler shape which looks almost standard from the point of view of sequent calculi for classical logic.

Another kind of sequent calculi: **hypersequent calculi**, first used in the context of modal logic.

**Hypersequents** $\Gamma_1 \Vdash \Sigma_1 \,|\, \ldots \,|\, \Gamma_k \Vdash \Sigma_k$ are (finite) sequences – or: multisets – of sequents $\Gamma_i \Vdash \Sigma_i$.

Hypersequent calculi enrich the usual inference rules of sequent calculi with additional rules which allow to infer hypersequents from given hypersequents, and which cannot be reduced to componentwise applications of sequent rules).

**Example:** The rule of **external contraction** gives e.g. for hypersequents $\mathcal{G}, \mathcal{H}$:

$$\frac{\mathcal{G} \,|\, \mathcal{H} \,|\, \mathcal{H}}{\mathcal{G} \,|\, \mathcal{H}}\,.$$

Hypersequent calculi are useful for the infinite-valued **fuzzy logics**.

G. Metcalfe – N. Olivetti – D. Gabbay: Proof Theory for Fuzzy Logics. Springer 2009.

# Functional Completeness

It is interesting to characterize for each propositional system of MVL the class of connectives expressible, i.e. **definable** within this system.

By extensionality, this means to ask for the class of truth degree functions which are definable from the basic truth degree functions (of the primitive vocabulary).

An even more restricted question is to ask for criteria under which **all** (finitary) connectives are definable, i.e. under which conditions all (finitary) truth degree functions over $\mathcal{W}^{\mathsf{S}}$ are definable.

This is the problem of **functional completeness**.

Hence a set $\mathfrak{F}$ of (finitary) truth degree functions over $\mathcal{W}^{\mathsf{S}}$ is called *functionally complete* iff each (finitary) truth degree function over $\mathcal{W}^{\mathsf{S}}$ can be "combined" out of truth degree functions from $\mathfrak{F}$.

What has to be specified are the operations which are allowed for these "combinations" of truth degree functions, i.e. the methods for defining new connectives out of given ones.

For our propositional languages, to introduce a new $n$-ary connective $\psi$ using a "stock" of existing ones means to introduce a new **compound wff** of the form "$\psi(p_1, \dots, p_n)$", with pairwise different propositional variables $p_1, \dots, p_n$, as definitionally equivalent with some "more complicated" wff $H(p_1, \dots, p_n)$ which has all of its propositional variables among $p_1, \dots, p_n$, and which contains only connectives out of the "stock".

And each compound formula "$\psi(H_1, \dots, H_n)$" is then understood as the result of substituting simultaneously the wffs $H_1, \dots, H_n$ for the propositional variables $p_1, \dots, p_n$, respectively, in the wff $H(p_1, \dots, p_n)$.

For truth degree functions these definitional possibilities mean the following operations:

1. the **superposition** of truth degree functions;

2. the identification of argument places;

3. the addition of fictitious variables.

It is helpful to consider also some further, improper “definitional” operations: to “define” a propositional connective by itself, to choose the “compound” wff $H(p_1, \ldots, p_n)$ simply as the atomic wff $p_k$ for some $k = 1, ..., n$, and to exchange two of the argument places of a given connective. For truth degree functions this means:

4. to treat each truth degree function as a superposition of itself;
5. to accept as primitive each $n$-ary **projection** $\mathsf{pr}^n_k$ onto the $k$-th argument: $\mathsf{pr}^n_k(x_1, \ldots, x_n) = x_k$ ;
6. to exchange argument places.

One is thus led to all those operations for truth degree functions which are considered in the theory of function algebras.

However, to a large amount even this general theory is restricted to the case that the functions under consideration map some given **finite** set into itself. Accordingly we again restrict here to the consideration of finitely many-valued propositional systems S, assuming $\mathcal{W}^{\mathsf{S}} = \mathcal{W}_m$.

Some further notation:

$$P_m^{(n)} = \{f \mid f : \mathcal{W}_m^n \to \mathcal{W}_m\}\,, \qquad\qquad P_m = \bigcup_{n=1}^{\infty} P_m^{(n)}\,.$$

Via the identification of each constant unary truth degree function with the truth degree which is the only value of this particular function: $\mathcal{W}_m \subseteq P_m$.

For any set $\mathfrak{F}$ of truth degree functions from $P_m$ let $\langle\mathfrak{F}\rangle$ be the **function algebra** generated by $\mathfrak{F}$, i.e. the $\subseteq$-smallest class $\mathfrak{F} \subseteq \mathfrak{G} \subseteq P_m$ of functions which is closed under all the above mentioned operations (1) to (6) for truth degree functions.

Hence for any $n$-ary function $f \in \langle \mathfrak{F} \rangle$ and all $k$-ary functions $g_1, \ldots, g_n \in \langle \mathfrak{F} \rangle$ also all functions $h_1, \ldots, h_6$ are members of $\langle \mathfrak{F} \rangle$:

$$
\begin{aligned}
h_1(x_1, \ldots, x_k) &= f(g_1(x_1, \ldots, x_k), \ldots, g_n(x_1, \ldots, x_k))\,, \\
h_2(x_2, \ldots, x_n) &= f(x_2, x_2, x_3, \ldots, x_n)\,, \\
h_3(x_0, x_1, \ldots, x_n) &= f(x_1, \ldots, x_n)\,, \\
h_4(x_1, \ldots, x_n) &= f(x_1, \ldots, x_n)\,, \\
h_5(x_1, \ldots, x_n) &= \mathrm{pr}_i^n(x_1, \ldots, x_n) \quad \text{for all } i = 1, \ldots, n\,, \\
h_6(x_1, \ldots, x_n) &= f(x_2, x_1, x_3, \ldots, x_n)\,.
\end{aligned}
$$

Obviously, these six possibilities realize the operations mentioned previously.

Thus one has for all classes $\mathfrak{F} \subseteq P_m$:

$$\mathfrak{F} \text{ functionally complete} \Leftrightarrow \langle \mathfrak{F} \rangle = P_m\,.$$

In any case, $P_m$ is infinite. Each $P_m^{(n)}$ however is finite.

Do there exist finite functionally complete sets?

**Theorem:** For each $k = 1, \ldots, m$ let $c_k$ and $\mathsf{j}_k^*$ be the unary functions characterized by the equations

$$c_k(x) =_{\text{def}} \tau_k, \qquad \mathsf{j}_k^* =_{\text{def}} \begin{cases} 0, & \text{if } x = \tau_k \\ 1 & \text{otherwise.} \end{cases}$$

Then the set of functions

$$\mathfrak{F}_P = \{\max, \min, c_1, \ldots, c_m, \mathsf{j}_1^*, \ldots, \mathsf{j}_m^*\}$$

is functionally complete.

**Corollary:** Each (finitary) truth degree function of a finitely many-valued propositional system S can be represented as a superposition of at most binary truth degree functions.

The knowledge of particular functionally complete sets of truth degree functions does not only mean to know interesting examples. It is a cornerstone for testing also further sets of truth degree functions for their functional completeness.

**Proposition:** Suppose that $\mathfrak{F} \subseteq P_m$ is a functionally complete set of (truth degree) functions and that each function $f \in \mathfrak{F}$ can be represented as a superposition of functions from $\mathfrak{G} \subseteq P_m$. Then also $\mathfrak{G}$ is functionally complete.

From this point of view it is particularly important to know small sets of truth degree functions which nevertheless are functionally complete.

**Theorem:** Functionally complete sets of truth degree functions are the sets

$$\{\max, \mathsf{non}_P\} \quad \text{and} \quad \{\mathsf{sh}\}$$

with the (generalized Sheffer-) function $\mathsf{sh} : \mathcal{W}_m^2 \to \mathcal{W}_m$ characterized by

$$\mathsf{sh}(x,y) = \begin{cases} 1, & \text{if } x = y = 0 \\ \max\{x,y\} - \frac{1}{m-1} & \text{otherwise.} \end{cases}$$

The idea to discuss the functions which can be represented via superpositions of given functions is not only useful for proving functional completeness, it can also be used to disprove this property.

**Proposition:** Suppose that all truth degree functions of some class $\mathfrak{F} \subseteq P_m$ of functions satisfy the normal condition and that $m > 2$ holds true. Then $\mathfrak{F}$ is not functionally complete.

Each one of the criteria for functional completeness discussed up to now does refer to particular sets of functions. What is lacking is some kind of "general" criterion.

**Theorem:** A class of functions $\mathfrak{G} \subseteq P_m$ is not functionally complete iff there exists some maximal function algebra $\mathfrak{F}$ such that $\mathfrak{G} \subseteq \mathfrak{F}$.

**Corollary:** If a function $f \in P_m$ does not belong to any maximal function algebra of $P_m$, then $\{f\}$ is functionally complete.

The last theorem is an acceptable characterization of all sets $\mathfrak{F} \subseteq P_m$ which are not functionally complete, and hence also of all functionally complete $\mathfrak{F} \subseteq P_m$, if one is able to determine all the maximal function algebras of $P_m$.

For the case $m = 2$ this was already done by Post 1920, for $m = 3$ the problem was solved by Jablonskij (1958), and in the general case finally by Rosenberg (1970).

To describe his general result, some further notions are needed.

A $k$-ary relation $\varrho$ in $\mathcal{W}_m$, i.e. a set of $k$-tuples with components from $\mathcal{W}_m$, is called **invariant** w.r.t. some $f \in P_m$ iff for any family of $k$-tuples

$$(s_1^i, \ldots, s_k^i) \in \varrho \quad \text{with} \;\; i = 1, \ldots, n$$

the “componentwise” application of $f$ again produces a $k$-tuple from $\varrho$:

$$(f(s_1^1, \ldots, s_1^n), f(s_2^1, \ldots, s_2^n), \ldots, f(s_k^1, \ldots, s_k^n)) \in \varrho\,.$$

The **polymorph** Pol$(\varrho)$ of such a $k$-ary relation $\varrho$ is the class

$$\mathrm{Pol}(\varrho) = \{f \in P_m \mid \varrho \text{ invariant w.r.t. } f\}\,.$$

The functions $f \in \mathrm{Pol}(\varrho)$ are the **polymorphisms** of $\varrho$.

An equivalence relation $\varrho$ in $\mathcal{W}_m$ is **nontrivial** iff $\varrho \neq \{(x,x) \mid x \in \mathcal{W}_m\}$ as well as $\varrho \neq \mathcal{W}_m \times \mathcal{W}_m$.

Consider

$$\begin{aligned} \mathcal{E}_m &= \text{all nontrivial equivalence relations in } \mathcal{W}_m. \\ \mathcal{O}_m &= \text{all partial orderings } \varrho \text{ in } \mathcal{W}_m \text{ which have a } \varrho\text{-min and a } \varrho\text{-max.} \end{aligned}$$

A **permutation** of $\mathcal{W}_m$ is a bijection from $\mathcal{W}_m$ onto itself. Each such permutation $g$ of $\mathcal{W}_m$ generates an equivalence relation $\sim_g$ in $\mathcal{W}_m$ by putting $a \sim_g b$ for $a, b \in \mathcal{W}_m$ iff there is a finite iteration of $g$ which "transforms" $a$ into $b$. Then consider

$$\mathcal{P}_m = \text{all permutations } g \text{ of } \mathcal{W}_m \text{ for which there exists a prime } p \text{ such that each equivalence class of } \sim_g \text{ has exactly } p \text{ elements.}$$

Because each permutation of $\mathcal{W}_m$ is a set of ordered pairs, it is also a binary relation in $\mathcal{W}_m$. Hence $\mathcal{P}_m$ is a set of relations.

A $k$-ary relation $\varrho$ in $\mathcal{W}_m$ is **central** iff (i) $\varrho \neq \mathcal{W}_m^k$ holds true and (ii) there exists some $\emptyset \neq C \subset \mathcal{W}_m$ such that for all $a_1, \ldots, a_k \in \mathcal{W}_m$ and all $1 \leq i < j \leq k$

$$\begin{aligned} a_i \in C \;&\Rightarrow\; (a_1, \ldots, a_k) \in \varrho\,, \\ a_i = a_j \;&\Rightarrow\; (a_1, \ldots, a_k) \in \varrho\,, \end{aligned}$$

and (iii) $\varrho$ is invariant under any permutation $g$ of the set $\{1, 2, \ldots, k\}$:

$$(a_1, \ldots, a_k) \in \varrho \Rightarrow (a_{g(1)}, \ldots, a_{g(k)}) \in \varrho\,.$$

Then let be

$$\mathcal{C}_m = \text{all central relations in } \mathcal{W}_m.$$

A nonempty family $(\sigma_i)_{i \leq k}$, $k \geq 1$, of equivalence relations in $\mathcal{W}_m$ is $h$-**regular** for $3 \leq h \leq m$ iff each equivalence relation $\sigma_1, \ldots, \sigma_k$ has exactly $h$ equivalence classes and furthermore each family $(A_i)_{i \leq k}$ of equivalence classes $A_i$ of $\sigma_i$, $i = 1, \ldots, k$, has a nonempty intersection: $\bigcap_{i=1}^{k} A_i \neq \emptyset$.

Each $h$-regular family $(\sigma_i)_{i \leq k}$ of equivalence relations **uniquely determines** an $h$-ary relation $\varrho^*$ characterized by

$$(a_1, \ldots, a_h) \in \varrho^* \Leftrightarrow \begin{array}{c}(a_1, \ldots, a_h) \text{ has at least two } \sigma_i\text{-equivalent} \\ \text{components for each } 1 \leq i \leq k\end{array}.$$

Let be

$$\mathcal{B}_m = \text{all relations in } \mathcal{W}_m \text{ which are determined by some } h\text{-regular family of equivalence relations in } \mathcal{W}_m.$$

For $p$ prime, an **abelian** $p$-**group** is an abelian group $(A, \otimes)$ for which each "$p$-th power" $a^p = a \otimes \cdots \otimes a$ of any $a \in G$ satisfies $b \otimes a^p = b$ for all $b \in G$. Then

$$\mathcal{G}_m =_{\text{def}} \emptyset \qquad \text{if } m \text{ is not a power of some prime number,}$$

and for $m = p^n$, $p$ prime and $n \geq 1$

$$\mathcal{G}_m = \text{all quaternary relations } R = \{(a_1, \ldots, a_4) \,|\, a_1 \otimes a_2 = a_3 \otimes a_4\} \text{ for a binary operation } \otimes \text{ such that } (\mathcal{W}_m, \otimes) \text{ is an abelian } p\text{-group.}$$

**Theorem:** A class $\mathfrak{F} \subseteq P_m$ of truth degree functions is a maximal function algebra iff it holds true

$$\mathfrak{F} = \mathsf{Pol}(\varrho) \quad \text{for some } \varrho \in \mathcal{E}_m \cup \mathcal{O}_m \cup \mathcal{P}_m \cup \mathcal{C}_m \cup \mathcal{B}_m \cup \mathcal{G}_m.$$

Because one has obviously $\mathcal{O}_m \neq \emptyset$ for each $m \geq 2$, this last theorem yields also that in $P_m$ there always exist maximal function algebras.

**Corollary:** A class $\mathfrak{F} \subseteq P_m$ of truth degree functions is functionally complete iff there exists for each relation $\varrho \in \mathcal{E}_m \cup \mathcal{O}_m \cup \mathcal{P}_m \cup \mathcal{C}_m \cup \mathcal{B}_m \cup \mathcal{G}_m$ some function $f \in \mathfrak{F}$ such that $\varrho$ is not invariant under $f$.

It is interesting to notice that $P_m$ has always only finitely many maximal function algebras despite the fact that there exist infinitely many function algebras $\mathfrak{F} \subseteq P_m$, viz. $\aleph_0$ for $m = 2$ and $2^{\aleph_0}$ for $m \geq 3$.

This number $\mu(m)$ of maximal function algebras in $P_m$ is growing rapidly with $m$ and has for small $m$ e.g. the following values:

| $m$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $\mu(m)$ | 5 | 18 | 82 | 643 | 15,182 | 7,848,984 | $> 549 \cdot 10^9$ |