SET FUNCTIONS, CAPACITIES AND GAMES ON FINITE SETS

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Outline

Set functions and games

- X: finite universe. *Set function on X:* $\xi : 2^X \to \mathbb{R}$.
- A set function can be
  1. *Additive* if $\xi(A \cup B) = \xi(A) + \xi(B)$ for every disjoint $A, B \in 2^X$;
  2. *Monotone* if $\xi(A) \leqslant \xi(B)$ whenever $A \subseteq B$;
  3. *Grounded* if $\xi(\varnothing) = 0$;
  4. *Normalized* if $\xi(X) = 1$.
- A *game* $v : 2^X \to \mathbb{R}$ is a grounded set function.
- $\mathcal{G}(X)$: set of games on X
- *conjugate* $\overline{\xi}$ of ξ:

$$\overline{\xi}(A) = \xi(X) - \xi(A^c) \qquad (A \in 2^X).$$

- Note that
  1. If $\xi(\varnothing) = 0$, then $\overline{\xi}(X) = \xi(X)$ and $\overline{\overline{\xi}} = \xi$;
  2. If ξ is additive, then $\overline{\xi} = \xi$ (ξ is *self-conjugate*).

Measures and capacities

- A *measure* is a nonnegative and additive set function
- A normalized measure is called a *probability measure*.
- A *capacity* $\mu : 2^X \to \mathbb{R}$ is a grounded monotone set function, i.e., $\mu(\varnothing) = 0$ and $\mu(A) \leqslant \mu(B)$ whenever $A \subseteq B$.
- $\mathcal{MG}(X)$: set of capacities on X
- $\mathcal{MG}_0(X)$: set of normalized capacities on X

Outline

Two different interpretations

- **capacities/games as a means to represent the importance/power/worth of a group:**
  - X: set of persons, usually called players, agents, voters, experts, decision makers, etc.
  - $A \subseteq X$: *coalition*, group of persons, who cooperate to achieve some common goal
  - $\mu(A)$: to what extent the goal is achieved by A
- **capacities as a means to represent uncertainty:**
  - X: set of possible outcomes of some experiment. It is supposed that X is exhaustive, and that each experiment produces a single outcome.
  - $A \subseteq X$: event
  - $\mu(A)$: uncertainty that the event A contains the outcome of an experiment, with $\mu(A) = 0$ indicating total uncertainty, and $\mu(A) = 1$ indicating that there is no uncertainty.

Examples

Example

Let X be a set of firms. Certain firms may form a coalition in order to control the market for a given product. Then $\mu(A)$ may be taken as the annual benefit of the coalition A. $\diamondsuit$

Example

Let X be a set of voters in charge of electing a candidate for some important position (president, director, etc.) or voting a bill by a yes/no decision. Before the election, groups of voters may agree to vote for the same candidate (or for yes or no). In many cases (presidential elections, parliament, etc.), these coalitions correspond to the political parties or to alliances among them. If the result of the election is in accordance with the wish of coalition A, the coalition is said to be winning, and we set $\mu(A) = 1$, otherwise it is loosing and $\mu(A) = 0$. $\diamondsuit$

Example

Let X be a set of workers in a factory, producing some goods. The aim is to produce these goods as much as possible in a given time (say, in one day). Then $\mu(A)$ is the number of goods produced by the group A in one day. Since the production needs in general the collaboration of several workers with different skills, it is likely that $\mu(A) = 0$ if A is a singleton or a too small group. $\diamondsuit$

Examples

Example

David throws a dice, and wonders what number will show. Here $X = \{1, 2, 3, 4, 5, 6\}$, and $\mu(\{1, 3, 5\})$ quantifies the uncertainty of obtaining an odd number. $\Diamond$

Example

A murder has been committed. After some investigation, it is found that the guilty is either Alice, Bob or Charles. Then $X = \{$Alice, Bob, Charles$\}$, and $\mu(\{$Bob, Charles$\})$ quantifies the degree to which it is "certain" (the precise meaning of this word being conditional on the type of capacity used) that the guilty is Bob or Charles. $\Diamond$

Example

Glenn is an amateur of antique chinese porcelain. He enters a shop and sees a magnificent vase, wondering how old (and how expensive) this vase could be. Then X is the set of numbers from, say -3000 to 2012, i.e., the possible date expressed in years A.C. when the vase was created. For example, $\mu([1368, 1644])$ indicates tho what degree it is certain that it is a vase of the Ming period. $\Diamond$

Example

Leonard is planning to go to the countryside tomorrow for a picnic. He wonders if the wheather will be favorable or not. Here X is the set of possible states of the weather, like "sunny", "rainy", "cloudy", and so on. For example, $\mu(\{$sunny, cloudy$\})$ indicates to what degree of certainty it will not rain, and so if the picnic is conceivable or not. $\Diamond$

Outline

Derivative of a set function

Definition

Let ξ be a set function on X, and consider $A \subseteq X$, $i \in X$. The *derivative of ξ at A w.r.t. i* is defined by

$$\Delta_i \xi(A) := \xi(A \cup \{i\}) - \xi(A \setminus \{i\}).$$

Note that $\Delta_i \xi$ is a set function on X. One can therefore take the derivative of a derivative (*second order derivative*):

$$\begin{aligned}\Delta_i(\Delta_j \xi(A)) = \xi(A \cup \{i,j\}) - \xi(A \cup \{i\} \setminus \{j\}) - \xi(A \cup \{j\} \setminus \{i\}) + \xi(A \setminus \{i,j\}) \\ = \Delta_j(\Delta_i \xi(A)).\end{aligned}$$

We denote it by $\Delta_{ij}\xi(A)$.

Derivative of a set function

More generally:

Definition

Consider subsets $A, K \subseteq X$ and a set function ξ on X. The *derivative* of ξ at A w.r.t. K is defined inductively by

$$\Delta_K \xi(A) := \Delta_{K \setminus \{i\}}(\Delta_{\{i\}}\xi(A)),$$

with the convention $\Delta_{\varnothing}\xi = 0$, and $\Delta_{\{i\}}\xi = \Delta_i \xi$.

When $K \cap A = \varnothing$, it is easy to obtain the following useful formula:

$$\Delta_K \xi(A) = \sum_{L \subseteq K} (-1)^{|K \setminus L|} \xi(A \cup L). \qquad (1)$$

Outline

Properties

1. v is *superadditive* if for any $A, B \in 2^X$, $A \cap B = \varnothing$,
$$v(A \cup B) \geqslant v(A) + v(B).$$
(*subadditive* if the reverse inequality holds)
2. v is *supermodular* if for any $A, B \in 2^X$,
$$v(A \cup B) + v(A \cap B) \geqslant v(A) + v(B).$$
(*submodular* if the reverse inequality holds)
3. v *k-monotone* ($k \geqslant 2$) if for $A_1, \dots, A_k \in 2^X$,
$$v\Big(\bigcup_{i=1}^{k} A_i\Big) \geqslant \sum_{\substack{I \subseteq \{1,\dots,k\} \\ I \neq \varnothing}} (-1)^{|I|+1} v\Big(\bigcap_{i \in I} A_i\Big).$$
v is *totally monotone* if it is k-monotone for any $k \geqslant 2$
4. v *k-alternating* ($k \geqslant 2$) if for $A_1, \dots, A_k \in 2^X$,
$$v\Big(\bigcap_{i=1}^{k} A_i\Big) \leqslant \sum_{\substack{I \subseteq \{1,\dots,k\} \\ I \neq \varnothing}} (-1)^{|I|+1} v\Big(\bigcup_{i \in I} A_i\Big).$$
v is *totally alternating* if it is k-alternating for any $k \geqslant 2$.

Theorem

Let v be a game on X. The following holds.

1. v *superadditive* $\Rightarrow \overline{v} \geqslant v$.
2. v *is* k*-monotone (resp.,* k*-alternating) for some* $k \geqslant 2$ *if and only if* $\overline{v}$ *is* k*-alternating (resp.,* k*-monotone). In particular,* v *is supermodular (resp., submodular) if and only if* $\overline{v}$ *is submodular (resp., supermodular).*
3. $v \geqslant 0$ *and supermodular implies that* v *is monotone.*
4. v *is* k*-monotone for some* $k \geqslant 2$ *if and only if for every disjoint* $S, K \subseteq X$ *with* $2 \leqslant |K| \leqslant k$,

$$\Delta_K v(S) := \sum_{T \subseteq K} (-1)^{|K \setminus T|} v(S \cup T) \geqslant 0.$$

5. v *is totally monotone if and only if* v *is* $(2^n - 2)$*-monotone, with* $|X| = n$.

Properties

Corollary

Let v be a game on X. The following holds.

1. *v is supermodular if and only if for any $A \subseteq X$, $i, j \notin A$, we have*

$$\Delta_{ij} v(A) := v(A \cup \{i,j\}) - v(A \cup \{i\}) - v(A \cup \{j\}) + v(A) \geqslant 0. \quad (2)$$

2. *v is supermodular if and only if for every $A \subseteq B \subseteq X$, for every $i \in X \setminus B$, we have*

$$\Delta_i v(A) \leqslant \Delta_i v(B). \quad (3)$$

Outline

0-1 capacities

- A *0-1-capacity* is a capacity valued on $\{0, 1\}$.
- Apart the null capacity 0, all 0-1-capacities are normalized.
- In game theory, 0-1-capacities are called *simple games*.
- A set A is a *winning coalition* for μ if $\mu(A) = 1$.
- A 0-1-capacity μ is uniquely determined by the antichain of its minimal winning coalitions.
- The number of antichains in 2^X with $|X| = n$ is the Dedekind number $M(n)$

n	$M(n)$
0	2
1	3
2	6
3	20
4	168
5	7581
6	7828354
7	2414682040998
8	56130437228687557907788

Unanimity games

- Let $A \subseteq X$, $A \neq \varnothing$. The *unanimity game centered on A* is the game u_A defined by

$$u_A(B) = \begin{cases} 1, & \text{if } B \supseteq A \\ 0, & \text{otherwise.} \end{cases}$$

- Unanimity games are 0-1-valued capacities. *Dirac measures*

$$\delta_{x_0}(A) = \begin{cases} 1, & \text{if } x_0 \in A \\ 0, & \text{otherwise.} \end{cases}$$

($x_o \in X$) are exactly the additive unanimity games

Possibility and necessity measures

- A *possibility measure* on a X is a normalized capacity Π on X satisfying

$$\Pi(A \cup B) = \max(\Pi(A), \Pi(B)) \text{ for all } A, B \in 2^X$$

- A *necessity measure* is a normalized capacity N satisfying

$$N(A \cap B) = \min(N(A), N(B)) \text{ for all } A, B \in 2^X$$

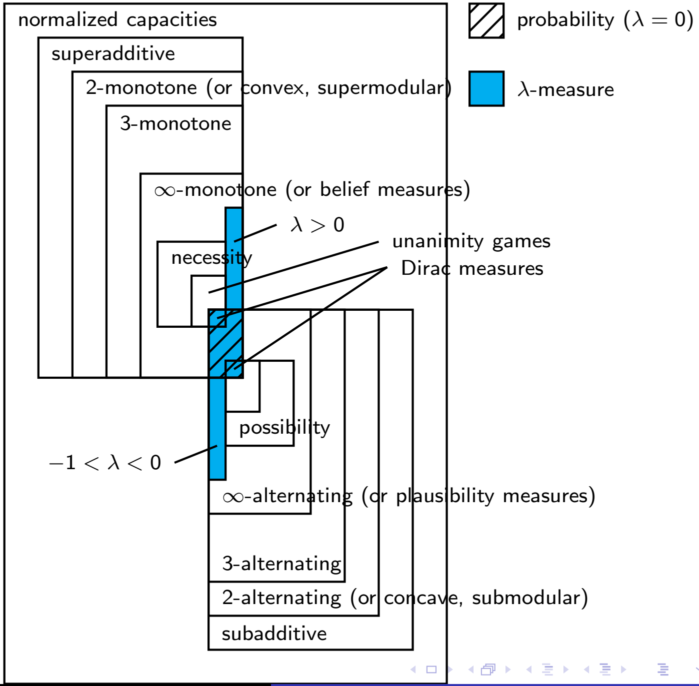
- The conjugate of a possibility measure (resp., a necessity measure) is a necessity measure (resp., a possibility measure).

Belief and plausibility measures

- A *belief measure* is a totally monotone normalized capacity.
- A *plausibility measure* is a totally alternating normalized capacity.
- the conjugate of a belief measure is a plausibility measure, and vice versa
- Possibility and necessity measures are particular cases of belief and plausibility measures
- A particular case: the λ-*measure* $(\lambda > -1)$ is a normalized capacity satisfying for every $A, B \in 2X$, $A \cap B = \varnothing$

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B)$$

- A λ-measure is a belief measure if and only if $\lambda \geqslant 0$, and is a plausibility measure otherwise


normalized capacities
superadditive
2-monotone (or convex, supermodular)
3-monotone
∞-monotone (or belief measures)
$\lambda > 0$
unanimity games
necessity
Dirac measures
possibility
$-1 < \lambda < 0$
∞-alternating (or plausibility measures)
3-alternating
2-alternating (or concave, submodular)
subadditive
probability ($\lambda = 0$)
λ-measure

Outline

The Möbius transform

Definition

Let ξ be a set function on X. The *Möbius transform* or *Möbius inverse* of ξ is a set function m^ξ on X defined by

$$m^\xi(A) = \sum_{B\subseteq A}(-1)^{|A\setminus B|}\xi(B) \qquad (4)$$

for every $A \subseteq X$.

Given m^ξ, it is possible to recover ξ by the formula

$$\xi(A) = \sum_{B\subseteq A} m^\xi(B) \qquad (A \subseteq X). \qquad (5)$$

Lemma

For any set function ξ on X, any disjoint sets $K, A \subseteq X$, we have

$$\Delta_K\xi(A) = \sum_{L\in[K,A\cup K]} m^\xi(L).$$

Properties

1. v is additive if and only if $m^v(A) = 0$ for all $A \subseteq X$, $|A| > 1$. Moreover, we have $m^v(\{i\}) = v(\{i\})$ for all $i \in X$.
2. v is monotone if and only if
$$\sum_{i \in L \subseteq K} m^v(L) \geqslant 0 \qquad (K \subseteq X, \quad i \in K).$$
3. Let $k \geqslant 2$ be fixed. v is k-monotone if and only if
$$\sum_{L \in [A,B]} m^v(L) \geqslant 0 \qquad (A, B \subseteq X, \quad A \subseteq B, \quad 2 \leqslant |A| \leqslant k).$$
4. If v is k-monotone for some $k \geqslant 2$, then $m^v(A) \geqslant 0$ for all $A \subseteq X$ such that $2 \leqslant |A| \leqslant k$.
5. v is a nonnegative totally monotone game if and only if $m^v \geqslant 0$.
6. The Möbius transform of $\overline{v}$ is given by
$$m^{\overline{v}}(A) = (-1)^{|A|+1} \sum_{B \supseteq A} m^v(B).$$

The Möbius transform of some remarkable capacities

- **Unanimity games** For any $A \subseteq X$, $A \neq \varnothing$,

$$m^{u_A}(B) = \begin{cases} 1, & \text{if } B = A \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

- **Plausibility measures**

$$m^{\mathrm{Pl}}(A) = (-1)^{|A|+1} \sum_{B \supseteq A} m(B), \quad (7)$$

where m is the Möbius transform of Bel, conjugate of Pl

Outline

Definition

The *interaction transformation* I is a linear invertible transformation, defined for any set function ξ by

$$I^{\xi}(A) = \sum_{B\subseteq X\setminus A} \frac{(n-b-a)!b!}{(n-a+1)!}\Delta_A\xi(B)$$
$$= \sum_{K\subseteq X} \frac{|X\setminus(A\cup K)|!|K\setminus A|!}{(n-a+1)!}(-1)^{|A\setminus K|}\xi(K)$$

for all $A\subseteq X$, where a, b, k are cardinalities of subsets A, B, K, respectively.

Relation with Möbius transform

$$I^{\xi}(A) = \sum_{B \supseteq A} \frac{1}{b-a+1} m^{\xi}(B)$$

$$m^{\xi}(A) = \sum_{B \supseteq A} B_{a-b} I^{\xi}(B)$$

with $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, etc., the Bernoulli numbers

Inverse transform

$$\xi(A) = \sum_{D \subseteq X} \beta^d_{|A \cap D|} I^\xi(D)$$

with the coefficients β^ℓ_k defined by

$$\beta^\ell_k = \sum_{j=0}^{k} \binom{k}{j} B_{\ell-j} \qquad (k \leqslant \ell).$$

The first values of β^ℓ_k are given below:

$k \setminus \ell$	0	1	2	3	4
0	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$
1		$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{30}$
2			$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{2}{15}$
3				0	$-\frac{1}{30}$
4					$-\frac{1}{30}$

Outline

k-additive games

Definition

A game v on X is said to be *k-additive* for some integer $k \in \{1, \dots, |X|\}$ if $m^v(A) = 0$ for all $A \subseteq X$, $|A| > k$, and there exists some $A \subseteq X$ with $|A| = k$ such that $m^v(A) \neq 0$.

- A game v is *at most k-additive* for some $1 \leqslant k \leqslant |X|$ if it is k'-additive for some $k' \in \{1, \dots, k\}$
- The set of k-additive games on X (resp., capacities, etc.,) is denoted by $\mathcal{G}^k(X)$ (resp., $\mathcal{MG}^k(X)$, etc.)
- We denote by $\mathcal{G}^{\leqslant k}(X), \mathcal{MG}^{\leqslant k}(X)$ the set of at most k-additive games and capacities.

$$\begin{aligned}\mathcal{G}(X) &= \mathcal{G}^1(X) \cup \mathcal{G}^2(X) \cup \cdots \cup \mathcal{G}^{|X|}(X)\\ &= \mathcal{G}^1(X) \cup \mathcal{G}^{\leqslant 2}(X) \cup \cdots \cup \mathcal{G}^{\leqslant |X|}(X)\end{aligned}$$

- m^v can be replaced by I^v without any change in the definition

Outline

The vector space of games

For any nonempty $A \subseteq X$ the *identity game δ_A centered at A* is the 0-1-game defined by

$$\delta_A(B) = \begin{cases} 1, & \text{if } A = B \\ 0, & \text{otherwise.} \end{cases}$$

Theorem

The set of identity games $\{\delta_A\}_{A \in 2^X \setminus \{\varnothing\}}$ and the set of unanimity games $\{u_A\}_{A \in 2^X \setminus \{\varnothing\}}$ are bases of $\mathcal{G}(X)$ of dimension $2^{|X|} - 1$.

- In the basis of identity games, the coordinates of a game v are simply $\{v(A)\}_{A \in 2^X \setminus \{\varnothing\}}$.
- We have for any game $v \in \mathcal{G}(X)$

  $$v(B) = \sum_{A \in 2^X \setminus \{\varnothing\}} \lambda_A u_A(B) = \sum_{A \subseteq B, A \neq \varnothing} \lambda_A \qquad (B \subseteq X).$$

  It follows that the coefficients of a game v in the basis of unanimity games are its Möbius transform: $\lambda_A = m^v(A)$ for all $A \subseteq X, A \neq \varnothing$

The cone of totally monotone nonnegative games

- The set of capacities $\mathcal{MG}(X)$ is a pointed cone.
- Totally monotone nonnegative games are exactly totally monotone capacities. We denote by $\mathcal{G}_+(X)$ the set of such capacities.

Theorem

$\mathcal{G}_+(X)$ is a pointed cone (subcone of $\mathcal{MG}(X)$), whose extremal rays are the unanimity games u_A, $A \in 2^X \setminus \{\varnothing\}$.

The polytope of normalized capacities

The set of normalized capacities reads

$$\mathcal{MG}_0(X) = \left\{ \mu \in \mathbb{R}^{2^{|X|}-1} : \begin{array}{lcll} \mu(A) & \geqslant & \mu(B), & \varnothing \neq B \subseteq A \subseteq X \\ \mu(A) & \geqslant & 0, & A \subseteq X \\ \mu(X) & = & 1 & \end{array} \right\}.$$

Removing redundant inequalities, we find

$$\mathcal{MG}_0(X) = \left\{ \mu \in \mathbb{R}^{2^{|X|}-1} : \begin{array}{lcll} \mu(A \cup \{i\}) & \geqslant & \mu(A), & \varnothing \neq A \subset X, i \notin A \\ \mu(\{i\}) & \geqslant & 0, & i \in X \\ \mu(X) & = & 1 & \end{array} \right\}$$

The polytope of normalized capacities

$$\mathcal{MG}_0(X) = \left\{ \mu \in \mathbb{R}^{2^{|X|}-1} : \begin{array}{lll} \mu(A \cup \{i\}) & \geqslant & \mu(A), \quad \varnothing \neq A \subset X, i \notin A \\ \mu(\{i\}) & \geqslant & 0, \quad i \in X \\ \mu(X) & = & 1 \end{array} \right\} \tag{8}$$

It follows that $\mathcal{MG}_0(X)$ is a polyhedron. Since for any set $A \in 2^X \setminus \{\varnothing\}$, we have $0 \leqslant \mu(A) \leqslant 1$, the polyhedron is bounded, hence it is a polytope.

Theorem

The set $\mathcal{MG}_0(X)$ of normalized capacities on X is a $(2^{|X|} - 2)$-dimensional polytope, whose extreme points are all 0-1-capacities, except the null capacity 0. Moreover, each inequality and equality in (8) defines a facet.

Recalling that 0-1-capacities are in bijection with antichains in $(2^X, \subseteq)$, we see that the number of extreme points of $\mathcal{MG}_0(X)$ is extremely large: it is $M(|X|) - 2$

Bounds of the Möbius transform

Theorem

For any normalized capacity μ, its Möbius transform satisfies for any nonempty $A \subseteq X$

$$-\binom{|A|-1}{l_{|A|}-1} \leqslant m^{\mu}(A) \leqslant \binom{|A|-1}{l_{|A|}},$$

with

$$l_m = \begin{cases} \frac{m}{2}-1, & \text{if } m \equiv 0 \pmod 4 \\ \frac{m-1}{2}, & \text{if } m \equiv 1 \pmod 4 \\ \frac{m}{2}, & \text{if } m \equiv 2 \pmod 4 \\ \frac{m+1}{2}, & \text{if } m \equiv 3 \pmod 4, \end{cases}$$

and the convention $\binom{x}{-1} = 0$. These upper and lower bounds are attained by the normalized capacities μ^*_A, μ_{A*}, respectively:

$$\mu^*_A = \mathrm{mc}(v^*_A), \quad \mu_{A*} = \mathrm{mc}(v_{A*}),$$

with

$$v^*_A(B) = \begin{cases} 1, & \text{if } B \subseteq A, |B| \geqslant l_{|A|}+1 \\ 0, & \text{otherwise,} \end{cases}, \quad v_{A*}(B) = \begin{cases} 1, & \text{if } B \subseteq A, |B| \geqslant l_{|A|} \\ 0, & \text{otherwise.} \end{cases}$$

Bounds of the Möbius transform

$\|A\|$	1	2	3	4	5	6	7	8	9	10
u.b. of $m^\mu(A)$	1	1	1	3	6	10	15	35	70	126
l.b. of $m^\mu(A)$	0	-1	-2	-1	-4	-10	-20	-21	-56	-126

Table: Lower and upper bounds for the Möbius transform of a normalized capacity

The polytope of belief measures

Let us express the set $\mathrm{Bel}(X)$ using the basis of unanimity games:

$$\mathrm{Bel}(X) = \left\{ m \in \mathbb{R}^{2^{|X|}-1} : \begin{array}{rcl} m(A) & \geqslant & 0, \quad \varnothing \neq A \subseteq X \\ \sum_{A \subseteq X, A \neq \varnothing} m(A) & = & 1 \end{array} \right\}.$$

This is a $(2^{|X|} - 2)$-dimensional polytope, which is merely the intersection of the hyperplane $\sum_{A \subseteq X, A \neq \varnothing} m(A) = 1$ with the positive orthant.

The polytope of probability measures

= 1-additive normalized capacities.

$$\mathcal{MG}^1(X) = \left\{ m \in \mathbb{R}^{|X|} : \begin{array}{rcl} m_i & \geqslant & 0, \quad i \in X \\ \sum_{i\in X} m_i & = & 1 \end{array} \right\}$$

is a $(|X|-1)$-dimensional polytope, whose extreme points are all vectors of the form $(0,\dots,0,1,0,\dots,0)$. These correspond to the unanimity games $u_{\{i\}}$, $i \in X$, or put differently, the set of Dirac measures on X.

The polytope of at most 2-additive normalized capacities

$$\mathcal{MG}^{\leqslant 2}(X) =$$
$$\left\{ m \in \mathbb{R}^{\kappa(2)} : \begin{array}{lcl} m_i & \geqslant & 0, \quad i \in X \\ m_i + \sum_{j \in K} m_{ij} & \geqslant & 0, \quad i \in X, \varnothing \neq K \subseteq X \setminus i \\ \sum_{i \in X} m_i + \sum_{\{i,j\} \subseteq X} m_{ij} & = & 1 \end{array} \right.$$

with $\kappa(2) = \binom{|X|}{1} + \binom{|X|}{2}$.

Theorem

$\mathcal{MG}^{\leqslant 2}(X)$ is a $(\kappa(2) - 1)$-dimensional polytope, whose extreme points are all at most 2-additive 0-1-capacities. These 0-1-capacities are of three different types:

1. *The unanimity games $u_{\{i\}}$, $i \in X$ (these are the extreme points of $\mathcal{MG}^1(X)$);*
2. *The games $u_i + u_j - u_{\{i,j\}}$, $i, j \in X, i \neq j$;*
3. *The unanimity games $u_{\{i,j\}}$.*