Mathematical fuzzy logic: first-order and beyond

Part I

Petr Cintula

Institute of Computer Science
Academy of Sciences of the Czech Republic

The goal of this first lecture is modest . . .

Generalize completeness theorem for CFOL

Classical first-order logic CFOL: $\vdash_{\mathrm{CFOL}}$ its provability relation
$\models_{\mathrm{CFOL}}$ the semantical consequence

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Problem of completeness of CFOL: formulated by Hilbert and Ackermann (1928) and solved by Gödel (1929):

Theorem 1 (Gödel's completeness theorem)

For every set of first-order formulae $T \cup \{\varphi\}$:

$$T \vdash_{\text{CFOL}} \varphi \qquad \textit{iff} \qquad T \models_{\text{CFOL}} \varphi$$

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First we have to define 'fuzzy' analogs of $\vdash_{\mathrm{CFOL}}$ and $\models_{\mathrm{CFOL}}$ ...

Some history

1947 Henkin: alternative proof of Gödel's completeness theorem

1961 Mostowski: interpretation of existential (resp. universal) quantifiers as suprema (resp. infima)

1963 Rasiowa, Sikorski: first-order intuitionistic logic

1963 Hay: infinitary standard Łukasiewicz first-order logic

1969 Horn: first-order Gödel–Dummett logic

1974 Rasiowa: first-order implicative logics

1990 Novák: first-order Pavelka logics

1992 Takeuti, Titani: first-order Gödel–Dummett logic with additional connectives

1998 Hájek: first-order axiomatic extensions of HL

2005 Cintula, Hájek: first-order core fuzzy logics

2011 Cintula, Noguera: first-order semilinear logics

Any lecture about first-order fuzzy logics has to start with . . .

Propositional fuzzy logics

$\mathcal{L}_{\mathrm{FL_e}}$: propositional language $(\to, \&, \wedge, \vee, \overline{0}, \overline{1}, \top, \bot)$

$\mathrm{FL_e}$: Full Lambek logic with exchange

Propositional fuzzy logics

$\mathcal{L}_{\mathrm{FL_e}}$: propositional language $(\to, \&, \wedge, \vee, \overline{0}, \overline{1}, \top, \bot)$

$\mathrm{FL_e}$: Full Lambek logic with exchange has axioms:

(id)	$\varphi \to \varphi$	(identity)
(pf)	$(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$	(prefixing)
(per)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	(permutation)
$(\&\wedge)$	$[(\varphi \wedge \overline{1})(\psi \wedge \overline{1})] \to (\varphi \wedge \psi)$	(fusion conjunction)
$(\wedge\to)$	$(\varphi \wedge \psi) \to \varphi$	(conjunction implication)
$(\wedge\to)$	$(\varphi \wedge \psi) \to \psi$	(conjunction implication)
$(\to\wedge)$	$[(\varphi \to \psi) \wedge (\varphi \to \chi)] \to [\varphi \to (\psi \wedge \chi)]$	(implication conjunction)
$(\to\vee)$	$\varphi \to (\varphi \vee \psi)$	(implication disjunction)
$(\to\vee)$	$\psi \to (\varphi \vee \psi)$	(implication disjunction)
$(\vee\to)$	$[(\varphi \to \chi) \wedge (\psi \to \chi)] \to [(\varphi \vee \psi) \to \chi]$	(disjunction implication)
$(\to\&)$	$\psi \to (\varphi \to \varphi \& \psi)$	(division fusion)
$(\&\to)$	$[\psi \to (\varphi \to \chi)] \to (\varphi \& \psi \to \chi)$	(fusion implication)
$(\overline{1})$	$\overline{1}$	(unit)
$(\overline{1}\to)$	$\overline{1} \to (\varphi \to \varphi)$	(unit implication)

Propositional fuzzy logics

$\mathcal{L}_{\mathrm{FL_e}}$: propositional language $(\rightarrow, \&, \wedge, \vee, \overline{0}, \overline{1}, \top, \bot)$

$\mathrm{FL_e}$: Full Lambek logic with exchange has rules:

(mp)	$\varphi, \varphi \rightarrow \psi \vdash \psi$	(*modus ponens*)
(adj$_u$)	$\varphi \vdash \varphi \wedge \overline{1}$	(adjunction unit)

Propositional fuzzy logics

$\mathcal{L}_{\mathrm{FL_e}}$: propositional language $(\rightarrow, \&, \wedge, \vee, \overline{0}, \overline{1}, \top, \bot)$

$\mathrm{FL_e}$: Full Lambek logic with exchange

UL: Uninorm logic (also denoted as $\mathrm{FL}_{\mathrm{e}}^{\ell}$), extension of $\mathrm{FL_e}$ by:

$$
\begin{array}{rll}
\text{(pre)} & (\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi) & \textit{(prelinearity)} \\
\text{(1-distr)} & (\varphi \vee \psi) \wedge \overline{1} \rightarrow (\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1}) & \text{(1-distributivity)}
\end{array}
$$

Let L be either $\mathrm{FL_e}$ or UL: we write $T \vdash_{\mathrm{L}} \varphi$ if there is proof of φ in logic L from theory T

Definition 2

A *bounded pointed commutative residuated lattice,* or an $\mathrm{FL_e}$-algebra, is an algebra $\boldsymbol{A} = \langle A, \&, \rightarrow, \wedge, \vee, \overline{0}, \overline{1}, \bot, \top \rangle$ s.t.:

1. $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice
2. $\langle A, \&, \overline{1} \rangle$ is a commutative monoid
3. $\rightarrow$ is the residuum of $\&$, i.e., for each $x, y, z \in A$ holds:

$$x \,\&\, y \leq z \text{ iff } x \leq y \rightarrow z$$

A $\mathrm{FL_e}$-algebra $\boldsymbol{A}$ is a UL-algebra if for all $x, y \in A$:

$$(x \rightarrow y) \vee (y \rightarrow x) \geq \overline{1} \quad \text{and} \quad (x \vee y) \wedge \overline{1} = (x \wedge \overline{1}) \vee (y \wedge \overline{1})$$

Note that

$$x \leq y \quad \text{iff} \quad x \wedge y = x \quad \text{iff} \quad x \rightarrow y \geq \overline{1}$$

Completeness

Let L be either $\mathrm{FL_e}$ or UL and $\mathbb{L}$ be class of all L-algebras and $\mathbb{L}^\ell$ the class of linearly ordered L-algebras

Let $\mathbb{K} \subseteq \mathbb{L}$, we write $T \models_{\mathbb{K}} \varphi$, if for each $\boldsymbol{A} \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e s.t. $e(\psi) \geq \overline{1}$ for each $\psi \in T$ we have $e(\varphi) \geq \overline{1}$.

Theorem 3 (General completeness of $\mathrm{FL_e}$ and UL)

$$T \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad T \models_{\mathbb{L}} \varphi$$

Theorem 4 (Linear completeness of UL)

$$T \vdash_{\mathrm{UL}} \varphi \quad \textit{iff} \quad T \models_{\mathbb{UL}^\ell} \varphi$$

Note that $\mathbb{FL}_{\mathrm{e}}^\ell = \mathbb{UL}^\ell$ and Theorem 3 is not valid for $\mathrm{FL_e}$

‘Basic’ semilinear logic SL^{ℓ}

Definition 5

An SL-algebra, is an algebra $\boldsymbol{A} = \langle A, \&, \to, \rightsquigarrow, \wedge, \vee, \bar{0}, \bar{1}, \bot, \top \rangle$:

1. $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice
2. $\langle A, \&, \bar{1} \rangle$ is a unital groupoid
3. $\to$ is the *left* residuum of $\&$ and $\rightsquigarrow$ is the *right* one, i.e., for each $x, y, z \in A$ holds:

$$x \,\&\, y \leq z \text{ iff } y \leq x \to z \text{ iff } x \leq y \rightsquigarrow z$$

Definition 6

The logics SL and SL^{ℓ} are the logics of SL-algebras or linearly ordered SL-algebras respectively, i.e., the logics for which:

$$T \vdash_{\mathrm{SL}} \varphi \text{ iff } T \models_{\mathbb{SL}} \varphi \qquad\qquad T \vdash_{\mathrm{SL}^{\ell}} \varphi \text{ iff } T \models_{\mathbb{SL}^{\ell}} \varphi$$

A logic L expands the logic SL if for each set formulae $T \cup \{\varphi\}$ in the language of SL we have:

$$T \vdash_{\mathrm{SL}} \varphi \quad \text{implies} \quad T \vdash_{\mathrm{L}} \varphi$$

For any such logic we define the class of (linearly ordered) L-algebras (denoted as $\mathbb{L}$ or $\mathbb{L}^{\ell}$ resp.) such that if define the semantical consequence $\models_{\mathbb{K}}$ as before we get:

$$T \vdash_{\mathrm{L}} \varphi \quad \text{iff} \quad T \models_{\mathbb{L}} \varphi$$

Core semilinear logics – 1

A logic L expands the logic SL if for each set formulae $T \cup \{\varphi\}$ in the language of SL we have:

$$T \vdash_{\mathrm{SL}} \varphi \quad \text{implies} \quad T \vdash_{\mathrm{L}} \varphi$$

For any such logic we define the class of (linearly ordered) L-algebras (denoted as $\mathbb{L}$ or $\mathbb{L}^{\ell}$ resp.) such that if define the semantical consequence $\models_{\mathbb{K}}$ as before we get:

$$T \vdash_{\mathrm{L}} \varphi \quad \text{iff} \quad T \models_{\mathbb{L}} \varphi$$

Definition 7

A logic L is core semilinear logic whenever:

- L expands the logic SL
- L is complete w.r.t. linearly ordered L-algebras, i.e.,

$$T \vdash_{\mathrm{L}} \varphi \quad \text{iff} \quad T \models_{\mathbb{L}^{\ell}} \varphi$$

Core semilinear logics – 2

Theorem 8 (Central characterization theorem)

Let L *be a logic expanding* SL. *TFAE:*

- L *is core semilinear logic*
- $\mathbb{L}^{\ell} = \mathbb{L}_{\mathrm{RFSI}}$
- *whenever* $T \nvdash_{\mathrm{L}} \varphi$ *then there is linear* $T' \supseteq T$ *s.t.* $T' \nvdash_{\mathrm{L}} \varphi$
  T *is linear if for each* φ, ψ*:* $T \vdash \varphi \to \psi$ *or* $T \vdash \psi \to \varphi$
- L *enjoys the Semilinearity Property*
  $$T, \varphi \to \psi \vdash_{\mathrm{L}} \chi \text{ and } T, \psi \to \varphi \vdash_{\mathrm{L}} \chi \text{ implies } T \vdash_{\mathrm{L}} \chi \qquad \text{(SLP)}$$
- L *proves* (pre) *and enjoys the Proof by Cases Property*
  $$T, \varphi \vdash_{\mathrm{L}} \chi \text{ and } T, \psi \vdash_{\mathrm{L}} \chi \text{ implies } T, \varphi \vee \psi \vdash_{\mathrm{L}} \chi \qquad \text{(PCP)}$$
- L *proves* (pre) *and* $T \vdash_{\mathrm{L}} \varphi$ *implies* $T \vee \chi \vdash_{\mathrm{L}} \varphi \vee \chi$
  where $T \vee \chi = \{\psi \vee \chi \mid \psi \in T\}$

Let us fix a semilinear logic L in a language $\mathcal{L}$ ...

Predicate languages, formulas, etc.

Predicate language: $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$

Object variables: denumerable set OV

$\mathcal{P}$-terms, atomic $\mathcal{P}$-formulae, $\langle \mathcal{L}, \mathcal{P} \rangle$-formulae: as in CFOL

free/bounded variables, substitutable terms, sentences:
as in CFOL

$\mathcal{P}$-theory: set of $\mathcal{P}$-formulae

First-order semantics – 1

$\mathcal{P}$-structure $\mathfrak{M}$: a pair $\langle \boldsymbol{A}, \mathbf{M} \rangle$ where

- $\boldsymbol{A} \in \mathbb{L}$
- $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle f_{\mathbf{M}} \rangle_{f \in \mathbf{F}} \rangle$ $\qquad (M \neq \emptyset)$
- $P_{\mathbf{M}} \colon M^n \to A$, for each n-ary $P \in \mathbf{P}$
- $f_{\mathbf{M}} \colon M^n \to M$ for each n-ary $f \in \mathbf{F}$.

$\mathfrak{M}$-evaluation v: a mapping $\mathrm{v} \colon \mathrm{OV} \to M$

For $x \in \mathrm{OV}, m \in M$, and $\mathfrak{M}$-evaluation v, we define $\mathrm{v}[x{\to}m]$ as

$$\mathrm{v}[x{\to}m](x) = m \text{ and } \mathrm{v}[x{\to}m](y) = \mathrm{v}(y) \text{ for } y \neq x$$

First-order semantics – 2

Definition 9 (Tarski style truth definition)

$$
\begin{array}{rcll}
\|x\|^{\mathfrak{M}}_{\mathrm{v}} & = & \mathrm{v}(x) & \text{for } x \in \mathrm{OV} \\
\|f(t_1,\dots,t_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & f_{\mathbf{S}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}},\ \dots,\ \|t_n\|^{\mathfrak{M}}_{\mathrm{v}}) & \text{for } f \in \mathbf{F} \\
\|P(t_1,\dots,t_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & P_{\mathbf{S}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}},\ \dots,\ \|t_n\|^{\mathfrak{M}}_{\mathrm{v}}) & \text{for } P \in \mathbf{P} \\
\|\circ(\varphi_1,\dots,\varphi_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & \circ^{\boldsymbol{A}}(\|\varphi_1\|^{\mathfrak{M}}_{\mathrm{v}},\ \dots,\ \|\varphi_n\|^{\mathfrak{M}}_{\mathrm{v}}) & \text{for } \circ \in \mathcal{L} \\
\|(\forall x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} & = & \inf_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x\to m]} \mid m \in M\} & \\
\|(\exists x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} & = & \sup_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x\to m]} \mid m \in M\} &
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\end{array}
$$

If the infimum/supremum does not exist, the value is undefined.
A $\mathcal{P}$-structure $\mathfrak{M}$ is safe if $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}}$ is defined for each φ and v.

Definition 10 (Model)

A safe structure $\mathfrak{M} = \langle \mathbf{A}, \mathbf{M} \rangle$ is a $\mathcal{P}$-model of T, $\mathfrak{M} \models T$ in symbols, if $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}} \geq \overline{1}^{\boldsymbol{A}}$ for each $\varphi \in T$ and each $\mathfrak{M}$-evaluation v.

Definition 11 (Semantical consequence)

A $\mathcal{P}$-formula φ is a semantical consequence of a $\mathcal{P}$-theory T w.r.t. the class $\mathbb{K}$ of $\mathbb{L}$-algebras, $T \models_{\mathbb{K}} \varphi$ in symbols, if for each $\boldsymbol{A} \in \mathbb{K}$ and each $\mathcal{P}$-model $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M} \rangle$ of T we have $\mathfrak{M} \models \varphi$

Proposition 12 (Assume that x is not free in ψ . . .)

$$\varphi \models_{\mathbb{L}} (\forall x)\varphi \quad \textit{thus} \quad \varphi \models_{\mathbb{K}} (\forall x)\varphi$$

$$\varphi \vee \psi \models_{\mathbb{L}^\ell} ((\forall x)\varphi) \vee \psi \quad \textit{BUT} \quad \varphi \vee \psi \not\models_{\mathbb{G}} ((\forall x)\varphi) \vee \psi$$

First-order semantics – 3

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Thus $\models_{\mathbb{L}} \subsetneq \models_{\mathbb{L}^\ell}$ even though in propositional logic: $\models_{\mathbb{L}} = \models_{\mathbb{L}^\ell}$

Axiomatization: two first-order logics

Minimal predicate logic $\mathrm{L}\forall^{\mathrm{m}}$:

(P)	first-order substitutions of axioms and rules of L	
$(\forall 1)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\varphi(x,\vec{z}) \to \varphi(t,\vec{z})$	t substitutable for x in φ
$(\exists 1)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi(t,\vec{z}) \to (\exists x)\varphi(x,\vec{z})$	t substitutable for x in φ
$(\forall 2)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$	x not free in χ
$(\exists 2)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$	x not free in χ
(gen)	$\varphi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\varphi$	

Predicate logic $\mathrm{L}\forall$: an the extension of $\mathrm{L}\forall^{\mathrm{m}}$ by:

$(\forall 3)$	$\vdash_{\mathrm{L}\forall} (\forall x)(\varphi \vee \chi) \to ((\forall x)\varphi) \vee \chi$	x not free in χ

Theorems (for x not free in χ)

The logic $\mathrm{L}\forall^{\mathrm{m}}$ proves:

$\chi \leftrightarrow (\forall x)\chi$

$(\exists x)\chi \leftrightarrow \chi$

$(\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi)$

$(\forall x)(\forall y)\varphi \leftrightarrow (\forall y)(\forall x)\varphi$

$(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to (\exists x)\psi)$

$(\exists x)(\exists y)\varphi \leftrightarrow (\exists y)(\exists x)\varphi$

$(\forall x)(\chi \to \varphi) \leftrightarrow (\chi \to (\forall x)\varphi)$

$(\forall x)(\varphi \to \chi) \leftrightarrow ((\exists x)\varphi \to \chi)$

$(\exists x)(\chi \to \varphi) \to (\chi \to (\exists x)\varphi)$

$(\exists x)(\varphi \to \chi) \to ((\forall x)\varphi \to \chi)$

$(\exists x)(\varphi \vee \psi) \leftrightarrow (\exists x)\varphi \vee (\exists x)\psi$

$(\exists x)(\varphi \,\&\, \chi) \leftrightarrow (\exists x)\varphi \,\&\, \chi$

If L is associative, then $\mathrm{L}\forall^{\mathrm{m}}$ proves:

$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)(\varphi^n) \leftrightarrow ((\exists x)\varphi)^n$

The logic $\mathrm{L}\forall$ furthermore proves:

$(\forall x)\varphi \vee \chi \leftrightarrow (\forall x)(\varphi \vee \chi)$

$(\exists x)(\varphi \wedge \chi) \leftrightarrow (\exists x)\varphi \wedge \chi$

Let $\vdash$ be either $\vdash_{\mathrm{L}\forall^{\mathrm{m}}}$ or $\vdash_{\mathrm{L}\forall}$

Theorem 13 (Congruence Property)

Let φ, ψ be sentences, χ a formula, and $\hat{\chi}$ a formula resulting from χ by replacing some occurrences of φ by ψ. Then

$$\vdash \varphi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi \vdash \psi \leftrightarrow \varphi$$

$$\varphi \leftrightarrow \psi \vdash \chi \leftrightarrow \hat{\chi} \qquad \varphi \leftrightarrow \delta, \delta \leftrightarrow \psi \vdash \varphi \leftrightarrow \psi$$

Theorem 14 (Constants Theorem)

Let $T \cup \{\varphi(x, \vec{z})\}$ be a theory and c a constant not occurring there. Then $\Sigma \vdash \varphi(c, \vec{z})$ iff $\Sigma \vdash \varphi(x, \vec{z})$

Theorem 15 (Proof by Cases Property)

For a $\mathcal{P}$-theory T and $\mathcal{P}$-sentences φ, ψ, χ:

$$\frac{T, \varphi \vdash_{\mathrm{L}\forall} \chi \qquad T, \psi \vdash_{\mathrm{L}\forall} \chi}{T, \varphi \vee \psi \vdash_{\mathrm{L}\forall} \chi} \tag{PCP}$$

Syntactical properties of $\vdash_{L\forall}$

Theorem 15 (Proof by Cases Property)

For a $\mathcal{P}$-theory T and $\mathcal{P}$-sentences φ, ψ, χ:

$$\frac{T, \varphi \vdash_{L\forall} \chi \qquad T, \psi \vdash_{L\forall} \chi}{T, \varphi \vee \psi \vdash_{L\forall} \chi} \qquad \text{(PCP)}$$

Proof.

We show by induction $T \vee \chi \vdash \varphi \vee \chi$ whenever $T \vdash \varphi$: trivial if $\varphi \in T$ or φ is an axiom; assume that φ follows using rule $\Gamma \vdash \varphi$; using IH we have $T \vee \chi \vdash \gamma \vee \chi$ for each $\gamma \in \Gamma$; to finish this part of the proof we show $\Gamma \vee \chi \vdash \varphi \vee \chi$: for propositional rules using Theorem 8, for (gen) using (gen), $(\forall 3)$, and (mp).

Now: from $T, \psi \vdash_{L\forall} \chi$ we get $T \vee \chi, \psi \vee \chi \vdash_{L\forall} \chi$ and from $T, \varphi \vdash_{L\forall} \chi$ we get $T \vee \psi, \varphi \vee \psi \vdash_{L\forall} \psi \vee \chi$

Thus $T \vee \psi, T \vee \chi, \varphi \vee \psi \vdash \chi$ and so $T, \varphi \vee \psi \vdash \chi$ □

Syntactical properties of $\vdash_{\text{Ł}\forall}$

Theorem 15 (Proof by Cases Property)

For a $\mathcal{P}$-theory T and $\mathcal{P}$-sentences φ, ψ, χ:

$$\frac{T, \varphi \vdash_{\text{Ł}\forall} \chi \qquad T, \psi \vdash_{\text{Ł}\forall} \chi}{T, \varphi \vee \psi \vdash_{\text{Ł}\forall} \chi} \tag{PCP}$$

Theorem 16 (Semilinearity Property)

For a $\mathcal{P}$-theory T and $\mathcal{P}$-sentences φ, ψ, χ:

$$\frac{T, \varphi \to \psi \vdash_{\text{Ł}\forall} \chi \qquad T, \psi \to \varphi \vdash_{\text{Ł}\forall} \chi}{T \vdash_{\text{Ł}\forall} \chi} \tag{SLP}$$

Proof.

Easy using PCP and $\vdash_{\text{Ł}\forall} (\varphi \to \psi) \vee (\psi \to \varphi)$ □

Soundness

It is straightforward to check that:

$$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} \subseteq \models_{\mathbb{L}} \qquad \text{and} \qquad \vdash_{\mathrm{L}\forall} \subseteq \models_{\mathbb{L}^{\ell}}$$

However recall that in general: $\vdash_{\mathrm{L}\forall} \not\subseteq \models_{\mathbb{L}}$

Failure of certain classical theorems (for x not free in χ)

Recall:

$\vdash_{\mathrm{L}\forall} (\forall x)\varphi \vee \chi \leftrightarrow (\forall x)(\varphi \vee \chi)$ $\quad$ $\vdash_{\mathrm{L}\forall} (\exists x)(\varphi \wedge \chi) \leftrightarrow (\exists x)\varphi \wedge \chi$

$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)(\chi \to \varphi) \leftrightarrow (\chi \to (\forall x)\varphi)$ $\quad$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)(\varphi \to \chi) \leftrightarrow ((\exists x)\varphi \to \chi)$

$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)(\chi \to \varphi) \to (\chi \to (\exists x)\varphi)$ $\quad$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)(\varphi \to \chi) \to ((\forall x)\varphi \to \chi)$

But we have:

- the first row's formulas are not provable in $\mathrm{L}\forall^{\mathrm{m}}$
  (except for extensions of Łukasiewicz logic)
- the converse directions of the last row's formulas are not provable in $\mathrm{L}\forall$ (except for extensions of Łukasiewicz logic)

Let $\vdash$ be either $\vdash_{\mathrm{L}\forall^{\mathrm{m}}}$ or $\vdash_{\mathrm{L}\forall}$

Lindenbaum–Tarski algebra of T ($\mathbf{LindT}_T$):

- domain $L_T = \{[\varphi]_T \mid \varphi$ a $\mathcal{P}$-sentence$\}$ where

$$[\varphi]_T = \{\psi \mid \psi \text{ a } \mathcal{P}\text{-sentence and } T \vdash \varphi \leftrightarrow \psi\}.$$

- operations:

$$\circ^{\mathbf{LindT}_T}([\varphi_1]_T, \ldots, [\varphi_n]_T) = [\circ(\varphi_1, \ldots, \varphi_n)]_T$$

Towards completeness – Lindenbaum–Tarski algebra

Let $\vdash$ be either $\vdash_{\mathrm{L}\forall^{\mathrm{m}}}$ or $\vdash_{\mathrm{L}\forall}$

Lindenbaum–Tarski algebra of T ($\mathbf{LindT}_T$):

- domain $L_T = \{[\varphi]_T \mid \varphi$ a $\mathcal{P}$-sentence$\}$ where

$$[\varphi]_T = \{\psi \mid \psi \text{ a } \mathcal{P}\text{-sentence and } T \vdash \varphi \leftrightarrow \psi\}.$$

- operations:

$$\circ^{\mathbf{LindT}_T}([\varphi_1]_T, \ldots, [\varphi_n]_T) = [\circ(\varphi_1, \ldots, \varphi_n)]_T$$

Proposition 17

- $\mathbf{LindT}_T \in \mathbb{L}$
- $[\varphi]_T \leq_{\mathbf{LindT}_T} [\psi]_T$ *iff* $T \vdash \varphi \to \psi$
- $\mathbf{LindT}_T \in \mathbb{L}^\ell$ *if, and only if,* T *is linear.*

Towards completeness – Canonical model 1

Canonical model ($\mathfrak{CM}_T$) of a $\mathcal{P}$-theory T (in $\vdash$): $\mathcal{P}$-structure $\langle \mathbf{LindT}_T, \mathbf{M}\rangle$ such that

- domain of $\mathbf{M}$: the set CT of closed $\mathcal{P}$-terms
- $f_{\mathbf{M}}(t_1,\dots,t_n) = f(t_1,\dots,t_n)$ for each n-ary $f \in \mathbf{F}$, and
- $P_{\mathbf{M}}(t_1,\dots,t_n) = [P(t_1,\dots,t_n)]_T$ for each n-ary $P \in \mathbf{P}$.

Towards completeness – Canonical model 2

A $\mathcal{P}$-theory T is $\forall$-Henkin if for each $\mathcal{P}$-formula ψ such that $T \nvdash (\forall x)\psi(x)$ there is a constant c in $\mathcal{P}$ such that $T \nvdash \psi(c)$

Proposition 18

Let T be a $\forall$-Henkin $\mathcal{P}$-theory. Then for each $\mathcal{P}$-sentence φ we have $\|\varphi\|^{\mathfrak{CM}_T} = [\varphi]_T$ and so $\mathfrak{CM}_T \models \varphi$ iff $T \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi$.

Proof.

Let v be evaluation s.t. $\mathrm{v}(x) = t^x$ for some $t_x \in CT$. We show by induction that $\|\varphi(x_1, \ldots, x_n)\|_{\mathrm{v}}^{\mathfrak{CM}_T} = [\varphi(t_1^x, \ldots, t_n^x)]_T$.

□

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The base case and the induction step for connectives are just the definition.

□

Towards completeness – Canonical model 2

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Proof.

Let v be evaluation s.t. $\mathrm{v}(x) = t^x$ for some $t_x \in CT$. We show by induction that $\|\varphi(x_1, \ldots, x_n)\|^{\mathfrak{CM}_T}_{\mathrm{v}} = [\varphi(t^x_1, \ldots, t^x_n)]_T$.
Quantifiers: $[(\forall x)\varphi]_T = \|(\forall x)\varphi\|^{\mathfrak{CM}_T} = \inf_{\leq_{\mathbf{LindT}_T}}\{[\varphi(t)]_T \mid t \in CT\}$
From $T \vdash (\forall x)\varphi \to \varphi(t)$ we get that $[(\forall x)\varphi]_T$ is a lower bound.
We show it is the largest one: take any χ s.t. $[\chi]_T \not\leq_{\mathbf{LindT}_T} [(\forall x)\varphi]_T$; thus $T \nvdash x \to (\forall x)\varphi$, and so $T \nvdash (\forall x)(\chi \to \varphi)$. So there is $c \in CT$ s.t. $T \nvdash (\chi \to \varphi(c))$, i.e., $[\chi]_T \not\leq_{\mathbf{LindT}_T} [\varphi(t)]_T$. □

Completeness of $\mathrm{L}\forall^{\mathrm{m}}$

Theorem 19 (Completeness theorem for $\mathrm{L}\forall^{\mathrm{m}}$)

Let L *be a logic and* $T \cup \{\varphi\}$ *a* $\mathcal{P}$*-theory. Then*

$$T \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi \quad \textit{iff} \quad T \models_{\mathbb{L}} \varphi$$

Completeness of $L\forall^m$

Theorem 19 (Completeness theorem for $L\forall^m$)

Let L *be a logic and* $T \cup \{\varphi\}$ *a* $\mathcal{P}$*-theory. Then*

$$T \vdash_{L\forall^m} \varphi \quad \textit{iff} \quad T \models_{\mathbb{L}} \varphi$$

All we need is to prove this theorem is to show that:

Proposition 20

Let $T \cup \{\varphi\}$ *be a* $\mathcal{P}$*-theory such that* $T \nvdash_{L\forall^m} \varphi$*. Then there is a predicate language* $\mathcal{P}' \supseteq \mathcal{P}$ *and a* $\forall$*-Henkin* $\mathcal{P}'$*-theory* $T' \supseteq T$ *such that* $T' \nvdash_{L\forall^m} \varphi$*.*

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Proof.

$\mathcal{P}' = \mathcal{P}$ + countably many new object constants. Let T' be T as $\mathcal{P}'$-theory. Take any $\mathcal{P}'$-formula $\psi(x)$, such that $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\psi(x)$. Thus $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \psi(x)$ and so $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \psi(c)$ for some $c \in \mathcal{P}'$ not occurring in $T' \cup \{\psi\}$ (by Constants Theorem). □

Theorem 21 (Completeness theorem for $\text{L}\forall$)

Let L *be a logic and* $T \cup \{\varphi\}$ *a* $\mathcal{P}$*-theory. Then*

$$T \vdash_{\text{L}\forall} \varphi \quad \textit{iff} \quad T \models_{\mathbb{L}^{\ell}} \varphi$$

Theorem 21 (Completeness theorem for $\mathrm{L}\forall$)

Let L be a logic and $T \cup \{\varphi\}$ a $\mathcal{P}$-theory. Then

$$T \vdash_{\mathrm{L}\forall} \varphi \quad \textit{iff} \quad T \models_{\mathbb{L}^{\ell}} \varphi$$

All we need is to prove this theorem is to show that:

Proposition 22

Let $T \cup \{\varphi\}$ be a $\mathcal{P}$-theory such that $T \nvdash_{\mathrm{L}\forall} \varphi$. Then there is a predicate language $\mathcal{P}' \supseteq \mathcal{P}$ and a linear $\forall$-Henkin $\mathcal{P}'$-theory $T' \supseteq T$ such that $T' \nvdash_{\mathrm{L}\forall} \varphi$.

The proof of Proposition 22, part 1

Let $\mathcal{P}'$ be the expansion of $\mathcal{P}$ by countably many new constants.

We enumerate all $\mathcal{P}'$-formulae with one free variable.

We construct a sequence of $\mathcal{P}'$-formulas φ_i and increasing chain of $\mathcal{P}'$-theories T_i s.t. $T_i \nvdash \varphi_j$ for each $j \leq i$.

Take $T_0 = T$ and $\varphi_0 = \varphi$, which fulfils our conditions.

In the induction step we distinguish two possibilities and show that the required conditions are met …

(H1) If $T_i \vdash \varphi_i \vee (\forall x)\chi_{i+1}(x)$: then we define $\varphi_{i+1} = \varphi_i$ and
$T_{i+1} = T_i \cup \{(\forall x)\chi_{i+1}(x)\}$

(H2) If $T_i \nvdash \varphi_i \vee (\forall x)\chi_{i+1}(x)$, then we define $T_{i+1} = T_i$ and
$\varphi_{i+1} = \varphi_i \vee \chi_{i+1}(c))$ for some c not occurring in $T_i \cup \{\varphi_j \mid j \leq i\}$.

Assume, for a contradiction, that $T_{i+1} \vdash \varphi_j$ for some $j \leq i + 1$.
Then also $T_{i+1} \vdash \varphi_{i+1}$

Thus in case (H1) we have $T_i \cup \{(\forall x)\chi_{i+1}(x)\} \vdash \varphi_i$. Because trivially $T_i \cup \{\varphi_i\} \vdash \varphi_i$ we obtain by Proof by Cases Property that $T_i \cup \{\varphi_i \vee (\forall x)\chi_{i+1}(x)\} \vdash \varphi_i$ and so $T_i \vdash \varphi_i$, a contradiction.

The proof of Proposition 22, part 2

(H1) If $T_i \vdash \varphi_i \vee (\forall x)\chi_{i+1}(x)$: then we define $\varphi_{i+1} = \varphi_i$ and
$$T_{i+1} = T_i \cup \{(\forall x)\chi_{i+1}(x)\}$$

(H2) If $T_i \nvdash \varphi_i \vee (\forall x)\chi_{i+1}(x)$, then we define $T_{i+1} = T_i$ and $\varphi_{i+1} = \varphi_i \vee \chi_{i+1}(c))$ for some c not occurring in $T_i \cup \{\varphi_j \mid j \leq i\}$.

Assume, for a contradiction, that $T_{i+1} \vdash \varphi_j$ for some $j \leq i + 1$. Then also $T_{i+1} \vdash \varphi_{i+1}$

Thus in case (H2) we have $T_i \vdash \varphi_i \vee \chi_{i+1}(c)$. Using Constants Theorem we obtain $T_i \vdash \varphi_i \vee \chi_{i+1}(x)$ and thus by (gen), $(\forall 3)$, and (mp) we obtain $T_\mu \vdash \varphi_i \vee (\forall x)\chi_{i+1}(x)$, a contradiction.

The proof of Proposition 22, part 3

Let T' be a maximal theory extending $\bigcup T_i$ s.t. $T' \nvdash \varphi_i$ for each i
Such T' exists due to Zorn's lemma: let $\mathcal{T}$ be a chain of such theories then clearly so is $\bigcup \mathcal{T}$.

T' is linear: assume that $(\psi \to \chi) \notin T'$ and $(\chi \to \psi) \notin T'$. Then there are i, j s.t. $T', \psi \to \chi \vdash \varphi_i$ and $T', \chi \to \psi \vdash \varphi_j$ Thus also

$$T', \psi \to \chi \vdash \varphi_{\max\{i,j\}} \text{ and } T', \chi \to \psi \vdash \varphi_{\max\{i,j\}}$$

Thus by Semilinearity property also $T' \vdash \varphi_{\max\{i,j\}}$,
a contradiction

T' is $\forall$-Henkin: if $T' \nvdash (\forall x)\chi_{i+1}(x)$, then we must have used case (H2); because $T' \nvdash \varphi_{i+1}$ we also have $T' \nvdash \chi_{i+1}(c)$