Mathematical fuzzy logic: first-order and beyond

Part II

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Some more notions

Definition 1 (Conservative expansion)

Let $\mathcal{P}_1 \subseteq \mathcal{P}_2$. $\mathcal{P}_2$-theory T_2 is a conservative extension of $\mathcal{P}_1$-theory T_1 iff for each $\mathcal{P}_1$-formula, $T_2 \vdash \varphi$ iff $T_1 \vdash \varphi$.

Definition 2

Let $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M} \rangle$ be a $\mathcal{P}$-model. Then $Alg(\mathfrak{M})$ is the subalgebra of $\boldsymbol{A}$ with the domain

$$\{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}} \mid \varphi \text{ a } \mathcal{P}\text{-formula and v an } \mathfrak{M}\text{-evaluation}\}.$$

Definition 3 (Exhaustive model)

A model $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M} \rangle$ is *exhaustive* if $\boldsymbol{A} = Alg(\mathfrak{M})$.

We write:

$$\|\varphi(a_1, \dots, a_n)\|^{\mathfrak{M}} \text{ instead of } \|\varphi(x_1, \dots, x_n)\|^{\mathfrak{M}}_{\mathrm{v}} \text{ for } \mathrm{v}(x_i) = a_i.$$

Skolemization

A theory T is Henkin if it is $\forall$-Henkin and for each φ such that $T \vdash (\exists x)\varphi(x)$ there is a constant such that $T \vdash \varphi(c)$.

Henkin theories, preSkolem logics, and Skolemization

A theory T is Henkin if it is $\forall$-Henkin and for each φ such that $T \vdash (\exists x)\varphi(x)$ there is a constant such that $T \vdash \varphi(c)$.

$\mathrm{L}\forall$ is preSkolem if $T \cup \{\varphi(c)\}$ is a conservative expansion of $T \cup \{(\exists x)\varphi(x)\}$ for each $\mathcal{P}$-theory $T \cup \{\varphi\}$, and a constant $c \notin \mathcal{P}$

Theorem 4

1. $\mathrm{L}\forall$ *is preSkolem.*
2. *For each* $\mathcal{P}$*-theory* $T \cup \{\varphi\}$ *such that* $T \nvdash \varphi$ *there is* $\mathcal{P}' \supseteq \mathcal{P}$ *and a linear Henkin* $\mathcal{P}'$*-theory* $T' \supseteq T$ *such that* $T' \nvdash \varphi$*.*
3. $T \cup \{(\forall\vec{y})\varphi(f_\varphi(\vec{y}), \vec{y})\}$ *is a conservative expansion of* $T \cup \{(\forall\vec{y})(\exists x)\varphi(x, \vec{y})\}$ *for each* $\mathcal{P}$*-theory* $T \cup \{\varphi(x, \vec{y})$*, and a functional symbol* $f_\varphi \notin \mathcal{P}$ *of the proper arity.*

The notion of preSkolem logic and Henkin theory could be relativized to a chosen class of formulae while keeping the theorem provable

(A bit of) model theory

Definition 5

An *elementary embedding* of a $\mathcal{P}_1$-model $\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle$ into a $\mathcal{P}_2$-model $\langle \boldsymbol{B}_2, \mathbf{M}_2\rangle$ is a pair (f, g) such that:

1. f is an embedding of $\boldsymbol{B}_1$ into $\boldsymbol{B}_2$.
2. g is a one-one mapping of M_1 into M_2
3. $f(\|\varphi(a_1, \dots, a_n)\|^{\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle}) = \|\varphi(g(a_1), \dots, g(a_n))\|^{\langle \boldsymbol{B}_2, \mathbf{M}_2\rangle}$ holds for each $\mathcal{P}_1$-formula $\varphi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in \mathfrak{M}$.

We use the denotation: $\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle \overset{(f,g)}{\hookrightarrow} \langle \boldsymbol{B}_2, \mathbf{M}_2\rangle$

If $\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle$ is *exhaustive* than in the condition 1 it is sufficient to assume that f is a one-one mapping

Conservative extension

Lemma 6

Let $\text{L}\forall$ is preSkolem, T_2 be a conservative extension of T_1 and $\mathfrak{M}$ an exhaustive model of T_1. Then there exists a linear Henkin theory T extending T_2 such that $\mathfrak{M}$ can be elementarily embedded into $\mathbf{CM}(T)$.

Theorem 7

Let $\text{L}\forall$ is preSkolem. Then the following claims are equivalent:

1. *$\mathcal{P}_2$-theory T_2 is a conservative extension of $\mathcal{P}_1$-theory T_1*
2. *each exhaustive model of T_1 can be elementarily embedded into a model of T_2.*

The condition of exhaustiveness cannot be omitted!

Witnessed semantics

Witnessed models

In CFOL: predicate logic with two variables has finite model property. Fuzzy logic: not even for one variable (monadic logic).

Example in (G∀) over standard G-algebra:

$$\varphi = \neg(\forall x)P(x) \,\&\, \neg(\exists x)\neg P(x)$$

Evidently φ has no finite model. But consider $\mathfrak{M}$ with domain $\mathbb{N}$, where $P^{\mathfrak{M}}(n) = \frac{1}{n+1}$. Then clearly for each $i \in \mathbb{N}$: $\|P(i)\| > 0$ and $\inf \|P(i)\| = 0$, i.e., $\mathfrak{M} \models \varphi$

The infimum is not the minimum, is not *witnessed*.

Definition 8

A $\mathcal{P}$-model $\mathfrak{M}$ is *witnessed* if for each $\mathcal{P}$-formula $\varphi(x, \vec{y})$ and for each $\vec{a} \in M$ there are $b_s, b_i \in M$ st.

$$\|(\forall x)\varphi(x, \vec{a})\|^{\mathfrak{M}} = \|\varphi(b_i, \vec{a})\|^{\mathfrak{M}} \qquad \|(\exists x)\varphi(x, \vec{a})\|^{\mathfrak{M}} = \|\varphi(b_s, \vec{a})\|^{\mathfrak{M}}.$$

Witnessing axioms and witnessed logic

Consider the axiom schemas (Baaz)

$$(C\exists) \quad (\exists y)((\exists x)\varphi(x) \to \varphi(y))$$

$$(C\forall) \quad (\exists y)(\varphi(y) \to (\forall x)\varphi(x))$$

Note that both $(C\exists)$ and $(C\forall)$ are provable in Ł∀; only $(C\exists)$ is provable in $\Pi\forall$, and none is provable in G∀

Definition 9

The *witnessed* predicate logic $\mathrm{L}\forall^{\mathrm{w}}$ extends $\mathrm{L}\forall$ by $(C\exists)$ and $(C\forall)$

Theorem 10

For each formula φ there is a formula φ' in a prenex form s.t.

$$\vdash_{\mathrm{L}\forall^{\mathrm{w}}} \varphi \leftrightarrow \varphi'$$

Witnessed completeness

Not each $\mathrm{L}\forall^{\mathrm{w}}$-model of T is witnessed. But we can prove:

Lemma 11

If $\mathrm{L}\forall$ is preSkolem, T a theory, and $\mathfrak{M}$ an exhaustive model of T. Then $\mathfrak{M}$ is a $\mathrm{L}\forall^{\mathrm{w}}$-model of T iff it can be elementarily embedded into a witnessed model of T.

Theorem 12 (Completeness of witnessed logics)

If $\mathrm{L}\forall$ is preSkolem, T a theory and φ a formula, TFAE:

- $T \vdash_{\mathrm{L}\forall^{\mathrm{w}}} \varphi$.
- $\mathfrak{M} \models \varphi$ *for each* witnessed *linear model* $\mathfrak{M}$ *of* T.

How do we show that a logic is preSkolem?

Deduction theorems

DT: a set of propositional formulae of a single variable $\star$

Definition 13

L enjoys the *almost implicational deduction theorem* with a set of *deductive terms* DT if for each fle $T \cup \{\varphi\}$:

$$T, \varphi \vdash_{\mathrm{L}} \psi \qquad \text{iff} \qquad T \vdash_{\mathrm{L}} \delta(\varphi) \to \psi \qquad \text{for some } \delta \in \mathrm{DT}.$$

Theorem 14 (Deduction theorem of $\mathrm{L}\forall$)

If L enjoys the almost implicational deduction theorem w.r.t. DT, then for each theory T, formula ψ and sentence φ:

$$T, \varphi \vdash_{\mathrm{L}\forall} \psi \qquad \textit{iff} \qquad T \vdash_{\mathrm{L}\forall} \delta(\varphi) \to \psi \qquad \textit{for some } \delta \in \mathrm{DT}.$$

preSkolem logics – characterization

Theorem 15

Let L *enjoy the* almost implicational deduction theorem *w.r.t.* DT. *Then the following are equivalent:*

- $\mathrm{L}\forall$ *is preSkolem*
- *For each* $\mathcal{P}$ *and each* $\delta \in \mathrm{DT}$ *there is* $\delta_0 \in \mathrm{DT}$ *such that*

$$\vdash_{\mathrm{L}\forall} \delta_0((\exists x)\varphi(x)) \to (\exists x)\delta(\varphi(x))$$

Proof.

Recall: preSkolem means that $T \cup \{\varphi(c)\}$ is a conservative extension of $T \cup \{(\exists x)\varphi(x)\}$.
From $\vdash \delta(\varphi(c)) \to (\exists x)\delta(\varphi(x))$ we get $\varphi(c) \vdash (\exists x)\delta(\varphi(x))$ using DT and so by conservativity: $(\exists x)\varphi(x) \vdash (\exists x)\delta(\varphi(x))$. DT again completes the proof.

preSkolem logics – characterization

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Proof.

Recall: preSkolem means that $T \cup \{\varphi(c)\}$ is a conservative extension of $T \cup \{(\exists x)\varphi(x)\}$.
Assume $T \cup \{\varphi(c)\} \vdash \psi$. Thus by DT: $T \vdash \delta(\varphi(c)) \to \psi$ for some is $\delta \in \mathrm{DT}$. By Constants Theorem also: $T \vdash \delta(\varphi(x)) \to \psi$. By (gen), $(\exists 2)$, and (mp) we have: $T \vdash (\exists x)\delta(\varphi(x)) \to \psi$. The assumption and DT complete the proof. □

preSkolem logics – examples

All axiomatic extensions of UL have the almost implicational deduction theorem for $\mathrm{DT} = \{(\star \wedge \overline{1})^n \mid n \in \mathbb{N}\}$

If L is axiomatic extension of UL, then $\mathrm{L}\forall$ is preSkolem because $\vdash_{\mathrm{UL}\forall} (((\exists x)\varphi(x)) \wedge \overline{1})^n \to (\exists x)((\varphi(x) \wedge \overline{1})^n)$

preSkolem logics – examples

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If L is axiomatic extension of UL, then $\mathrm{L}\forall$ is preSkolem because $\vdash_{\mathrm{UL}\forall} (((\exists x)\varphi(x)) \wedge \overline{1})^n \to (\exists x)((\varphi(x) \wedge \overline{1})^n)$

All axiomatic extensions of $\mathrm{MTL}_\triangle$ have the almost implicational deduction theorem for $\mathrm{DT} = \{\triangle(\star)\}$

If L is axiomatic extension of $\mathrm{MTL}_\triangle$, then $\mathrm{L}\forall$ is not preSkolem because $\nvdash_{\mathrm{MTL}_\triangle\forall} \triangle(\exists x)\varphi(x) \to (\exists x)\triangle\varphi(x)$

But note that $\vdash_{\mathrm{MTL}_\triangle\forall} \triangle(\exists x)\triangle\varphi(x) \to (\exists x)\triangle\triangle\varphi(x)$

First-order fuzzy logics with identity

First-order fuzzy logics with identity

Axioms of $\text{Ł}\forall_{=}$: those of Ł∀+

- $x = x$ (reflexivity—all things are identical to themselves)
- $x = y \to (\varphi(x) \to \varphi(y))$
  (Leibniz identity law—indiscernibility of identicals)

In sufficiently strong logics (eg, with $\triangle$), $=$ comes out crisp:
$$\vdash x = y \vee \neg(x = y)$$

In weaker logics we can add it as an additional axiom

Models can then be factorized so that $=_{\mathbf{M}}$ is interpreted as the identity of individuals

The logic where fuzzy identity is the only predicate (not necessarily satisfying the second condition) are studied in: Bělohlávek, Vychodil: Fuzzy Equational Logic. Springer, 2005.

Completeness
w.r.t. special classes of algebras

We want 'stronger' completeness theorem

Definition 16

$\mathrm{L}\forall_{(=)}$ has the $\mathrm{S}\mathbb{K}\mathrm{C}$ if for each countable $\mathcal{P}$, theory T, and formula φ the following are equivalent:

- $T \vdash_{\mathrm{L}\forall_{(=)}} \varphi$.
- $\langle \boldsymbol{A}, \mathbf{M} \rangle \models \varphi$ for each $\boldsymbol{A} \in \mathbb{K}$ and each *countable* model $\langle \boldsymbol{A}, \mathbf{M} \rangle$ of T.

$\mathrm{L}\forall_{(=)}$ has the $\mathbb{K}\mathrm{C}$ if the above condition holds for the empty theory.

General result

Theorem 17

Let L *be a core fuzzy logic. The following are equivalent:*

1. $\mathrm{L}\forall_=$ *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$.
2. *For every countable model* $\langle \boldsymbol{A}, \mathbf{M}\rangle$ *there is an* L*-chain* $\boldsymbol{B} \in \mathbb{K}$ *and a countable model* $\langle \boldsymbol{B}, \mathbf{M}'\rangle$ *s.t.*

$$\langle \boldsymbol{A}, \mathbf{M}\rangle \overset{(f,g)}{\hookrightarrow} \langle \boldsymbol{B}, \mathbf{M}'\rangle.$$

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$$\langle \boldsymbol{A}, \mathbf{M}\rangle \overset{(f,g)}{\hookrightarrow} \langle \boldsymbol{B}, \mathbf{M}'\rangle.$$

The following condition is clearly sufficient for $\mathrm{S}\mathbb{K}\mathrm{C}$, is it necessary?

3. Every countable L-chain $\boldsymbol{A}$ can be σ-embedded into some L-chain $\boldsymbol{B} \in \mathbb{K}$.

We can find an example of a logic K and semantics $\mathbb{K}$ showing that the condition 3 is not necessary.

General result

Theorem 18

Let L be a core fuzzy logic. The following are equivalent:

3. *Every countable L-chain $\boldsymbol{A}$ can be σ-embedded into some L-chain $\boldsymbol{B} \in \mathbb{K}$.*
4. *For every countable model $\langle \boldsymbol{A}, \mathbf{M} \rangle$ there is an L-chain $\boldsymbol{B} \in \mathbb{K}$ and a countable model $\langle \boldsymbol{B}, \mathbf{M}' \rangle$ s.t.*

$$\langle \boldsymbol{A}, \mathbf{M} \rangle \overset{(f,g)}{\hookrightarrow} \langle \boldsymbol{B}, \mathbf{M}' \rangle \text{ and } f \text{ is an isomorphism}$$

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Theorem 19

The following are equivalent:

(i) L *enjoys the* $\mathrm{S}\mathcal{F}\mathrm{C}$,

(i) $\mathrm{L}\forall$ *enjoys the* $\mathrm{S}\mathcal{F}\mathrm{C}$,

(iii) *all* L*-chains are finite,*

(iv) *there is a natural number* n *such that the length of each* L*-chain is less or equal than* n*, and*

(v) *there is a natural number* n *such that* $\vdash_{\mathrm{L}} \bigvee_{i<n}(x_i \to x_{i+1})$.

Theorem 20

If $\mathrm{L}\forall$ *enjoys the* $\mathcal{F}\mathrm{C}$*, then* $\mathrm{L}\forall = \mathrm{L}\forall^{\mathrm{w}}$.

Logic	$\mathcal{RC}$	$\mathcal{SRC}$	(S)$\mathcal{QC}$	(S)$\mathcal{FC}$
$\mathrm{SL}^{\ell}_{\mathrm{S}}\forall$, for each $\mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$	Yes	Yes	Yes	No
$\mathrm{SL}^{\ell}_{\mathrm{a}}\forall$	No	No	No	No
$\mathrm{SL}^{\ell}_{\mathrm{aw}}\forall$	Yes	Yes	Yes	No
MTL∀, IMTL∀, SMTL∀	Yes	Yes	Yes	No
WCMTL∀, ΠMTL∀	?	No	?	No
BL∀, SBL∀	No	No	No	No
Ł∀, Π∀	No	No	Yes	No
G∀, WNM∀, NM∀	Yes	Yes	Yes	No
C_nMTL∀, C_nIMTL∀	Yes	Yes	Yes	No
CFOL	No	No	No	Yes

Decidability and arithmetical hierarchy

Logic	THM	stTAUT	stSAT	$\text{stTAUT}_{\text{pos}}$	$\text{stSAT}_{\text{pos}}$
BL∀	Σ_1-compl.	Non-arit.	Non-arit.	Non-arit.	Non-arit.
SBL∀	Σ_1-compl.	Non-arit.	Non-arit.	Non-arit.	Non-arit.
Ł∀	Σ_1-compl.	Π_2-compl.	Π_1-compl.	Σ_1-compl.	Σ_2-compl.
G∀	Σ_1-compl.	Σ_1-compl.	Π_1-compl.	Σ_1-compl.	Π_1-compl.
Π∀	Σ_1-compl.	Non-arit.	Non-arit.	Non-arit.	Non-arit.
(Ł⊕)∀	Σ_1-compl.	Π_2-hard	Π_1-compl.	Σ_1-compl.	Σ_2-compl.
(G⊕)∀	Σ_1-compl.	Σ_1-hard	Π_1-compl.	Σ_1-compl.	Π_1-compl.
(Π⊕)∀	Σ_1-compl.	Non-arit.	Non-arit.	Non-arit.	Non-arit.

Formal fuzzy mathematics

Formal fuzzy mathematics

First-order fuzzy logic is strong enough to support non-trivial formal mathematical theories

Mathematical concepts in such theories show gradual rather than bivalent structure

Examples:

- Skolem, Hájek (1960, 2005): naïve set theory over Ł
- Takeuti–Titani (1994): ZF-style fuzzy set theory
  in a system close to Gödel logic ($\Rightarrow$ contractive)
- Restall (1995), Hájek–Paris–Shepherdson (2000):
  arithmetic with the truth predicate over Ł
- Hájek–Haniková (2003): ZF-style set theory over $\mathrm{BL}_\triangle$
- Novák (2004): Church-style fuzzy type theory over $\mathrm{IMTL}_\triangle$
- Běhounek–Cintula (2005): higher-order fuzzy logic

Hájek–Haniková fuzzy set theory

Logic: First-order $BL_{\triangle}$ with identity

Language: $\in$

Axioms: (z not free in φ)

- $\triangle(\forall u)(u \in x \leftrightarrow u \in y) \to x = y$ (extensionality)
- $(\exists z)\triangle(\forall y)\neg(y \in z)$ (empty set $\emptyset$)
- $(\exists z)\triangle(\forall u)(u \in z \leftrightarrow (u = x \vee u = y))$ (pair $\{x, y\}$)
- $(\exists z)\triangle(\forall u)(u \in z \leftrightarrow (\exists y)(u \in y \,\&\, y \in x))$ (union $\bigcup$)
- $(\exists z)\triangle(\forall u)(u \in z \leftrightarrow \triangle(\forall u \in x)(u \in y))$ (weak power)
- $(\exists z)\triangle(\emptyset \in z \,\&\, (\forall x \in z)(x \cup \{x\} \in z))$ (infinity)
- $(\exists z)\triangle(\forall u)(u \in z \leftrightarrow (u \in x \,\&\, \varphi(u, x))$, z not free in φ (separation)
- $(\exists z)\triangle[(\forall u \in x)(\exists v)\varphi(u, v) \to (\forall u \in x)(\exists v \in z)\varphi(u, v)]$ (collection)
- $\triangle(\forall x)((\forall y \in x)\varphi(y) \to \varphi(x)) \to \triangle(\forall x)\varphi(x)$ ($\in$-induction)
- $(\exists z)\triangle((\forall u)(u \in z \vee \neg(u \in z)) \,\&\, (\forall u \in x)(u \in y))$ (support)

Hájek–Haniková fuzzy set theory

Semantics: A cumulative hierarchy of BL-valued fuzzy sets

Features:

- Contains an inner model of classical ZF:
  (as the subuniverse of hereditarily crisp sets)
- Conservatively extends classical ZF with fuzzy sets
- Generalizes Takeuti–Titani's construction
  in a non-contractive fuzzy logic

Cantor–Łukasiewicz set theory

Logic: First-order Łukasiewicz logic Ł∀

Language: $\in$, set comprehension terms $\{x \mid \varphi\}$

Axioms:

- $y \in \{x \mid \varphi\} \leftrightarrow \varphi(y)$ (unrestricted comprehension)

Features:

- Non-contractivity of Ł blocks Russell's paradox
- Consistency conjectured by Skolem (1960—still open: in 2010 a gap found by Terui in White's 1979 proof)
- Adding extensionality is inconsistent with CŁ
- Open problem: define a reasonable arithmetic in CŁ (some negative results by Hájek, 2005)

Fuzzy class theory = (Henkin-style) higher-order MFL

Logic: Any first-order deductive fuzzy logic with $\triangle$ and $=$

Originally: ŁΠ for its expressive power

Language:

- Sorts of variables for atoms, classes, classes of classes…
- Subsorts for k-tuples of objects at each level
- $\in$ between successive sorts
- At all levels: $\{x \mid \dots\}$ for classes, $\langle \dots \rangle$ for tuples

Axioms (for all sorts):

- $\langle x_1, \dots, x_k \rangle = \langle y_1, \dots, y_k \rangle \to x_1 = y_1 \mathbin{\&} \dots \mathbin{\&} x_k = y_k$ (tuple identity)
- $(\forall x)\triangle(x \in A \leftrightarrow x \in B) \to A = B$ (extensionality)
- $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$ (class comprehension)

Fuzzy class theory = (Henkin-style) higher-order MFL

Semantics:

Fuzzy sets and relations of all orders over a crisp ground set
(Henkin-style $\Rightarrow$ non-standard models exist,
full higher-order fuzzy logic non-axiomatizable)

Features:

- Suitable for the reconstruction and graded generalization of large parts of traditional fuzzy mathematics
- Several mathematical disciplines have been developed within its framework, using it as a foundational theory: (eg, fuzzy relations, fuzzy numbers, fuzzy topology)
- The results obtained trivialize initial parts of traditional fuzzy set theory

References and further research topics

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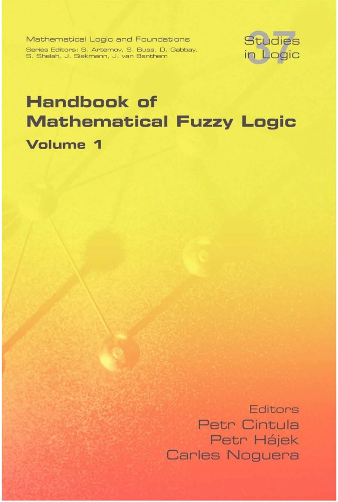

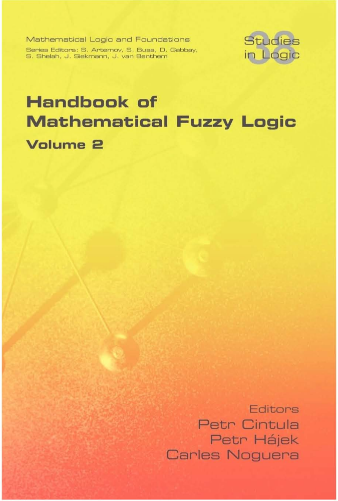

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Volume 1

Volume 2

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