THE CORE OF GAMES WITH RESTRICTED COOPERATION

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Part I: Basic Notions

Part II: Games with Restricted Cooperation

Part III: The Core of Games with Restricted Cooperation

Part IV: Bounded Faces of the Core

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- A game is *convex (or supermodular)* if for all $S, T \subseteq N$,

$$v(S \cup T) + v(S \cap T) \geqslant v(S) + v(T)$$

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- A collection $\mathcal{B}$ of nonempty sets is *balanced* if there exist $\lambda_S > 0$ for all $S \in \mathcal{B}$ such that

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- (Bondareva 63, Shapley 67) $\mathrm{core}(v)$ is nonempty iff v is balanced:

$$v(N) \geqslant \sum_{S \in \mathcal{B}} \lambda_S v(S)$$

for all balanced collections $\mathcal{B}$ with weights λ_S.

Marginal vectors and the Weber set

- To each permutation σ on N we assign the sequence of sets $\varnothing = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_n = N$ (*maximal chain*) defined by

$$S_1 = \{\sigma(1)\}$$
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- Let v be a game. To each permutation σ we assign a *marginal worth vector* m^σ in $\mathbb{R}^N$ by:

$$\begin{aligned} m^\sigma_{\sigma(i)} &:= v(S_i) - v(S_{i-1}) \\ &= v(S_{i-1} \cup \sigma(i)) - v(S_{i-1}) \end{aligned}$$

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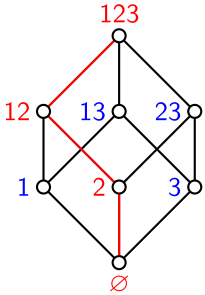
- The *Weber set* is the convex hull of all marginal vectors
$$\mathcal{W}(v) := \mathrm{conv}(m^\sigma \mid \sigma \in \mathfrak{S}(N))$$

Example: $N = \{1, 2, 3\}$,
maximal chain $C = \varnothing, \{2\}, \{1, 2\}, \{1, 2, 3\}$ (denoted $\varnothing, 2, 12, 123$),
hence permutation σ is 2,1,3



$m_1^\sigma = v(12) - v(2)$,
$m_2^\sigma = v(2)$,
$m_3^\sigma = v(123) - v(12))$

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Theorem

(Shapley 1971, Edmonds 1970, Ichiishi 1981) The following are equivalent.

1. v *is convex*
2. *All marginal vectors* m^σ, $\sigma \in \mathfrak{S}(N)$ *belong to the core*
3. $\mathrm{core}(v) = \mathrm{conv}(\{m^\sigma\}_{\sigma \in \mathfrak{S}(N)})$
4. $\mathrm{ext}(\mathrm{core}(v)) = \{m^\sigma\}_{\sigma \in \mathfrak{S}(N)}$.

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- A *game with restricted cooperation* $(\mathcal{F}, v)$ is a mapping $v : \mathcal{F} \rightarrow \mathbb{R}$ satisfying $v(\varnothing) = 0$, and $\mathcal{F}$ is a subcollection of 2^N, which contains $\varnothing$ and N (*set system*).

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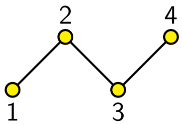
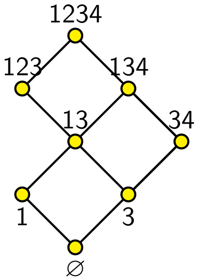
- Many structures are possible for $\mathcal{F}$: antimatroids, convex geometries, lattices, regular set systems, weakly union-closed set systems, etc.
- We mainly focus on distributive lattices.

(considered by Faigle & Kern 92, Derks et al.)

- Essentially, they are distributive lattices generated by a poset $(N, \preceq)$:
$$\mathcal{F} = \mathcal{O}(N, \preceq)$$
where $\mathcal{O}(\cdot)$ is the set of downsets of some poset (Birkhoff theorem)





Weakly union-closed set systems

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- The *basis* (collection of sets S in $\mathcal{F}$ which cannot be written as $S = A \cup B$, with $A, B \in \mathcal{F}$, $A, B \neq S$, $A \cap B \neq \varnothing$) permits to reconstruct $\mathcal{F}$.

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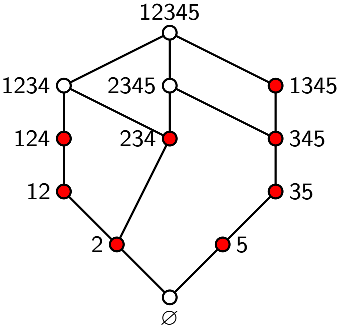

Regular set systems

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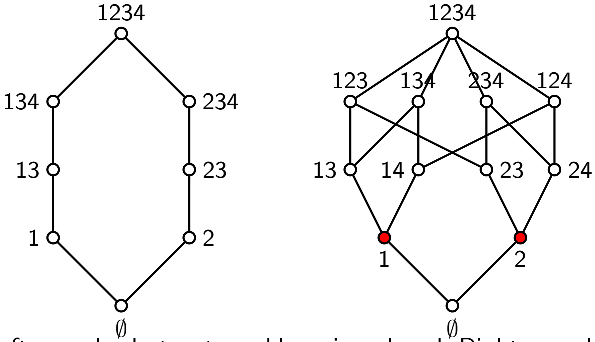
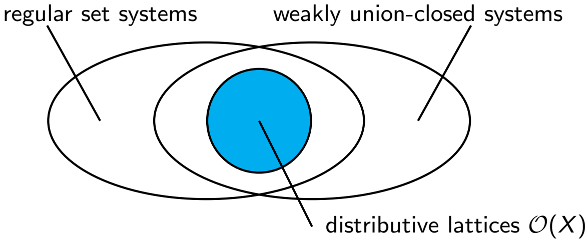


Figure: Left: regular but not weakly union-closed; Right: regular and weakly union-closed but not a lattice, since 1 and 2 have no supremum

Summary


regular set systems
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distributive lattices $\mathcal{O}(X)$

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The core of games on set systems

- Let $\mathcal{F}$ be a set system, v a game on $\mathcal{F}$. The core of v is defined by
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- The fundamental theorem on polyhedra asserts that a polyhedron P defined by $Ax \leqslant b, x \in \mathbb{R}^N$ has the following structure:
$$P = \mathrm{conv}(x^1, x^2, \dots, x^p) + \mathrm{cone}(r^1, \dots, r^q)$$
where $x^1, \dots, x^p$ are the extreme points (vertices) of P, and $r^1, \dots, r^q$ are the extremal rays (half-lines).

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- Moreover, the conic part (called the *recession cone*) is defined by $Ax \leqslant 0$.
- Therefore, we write
$$\mathrm{core}(v) = \mathrm{conv}(x^1, x^2, \dots, x^p) + \mathrm{core}(0)$$

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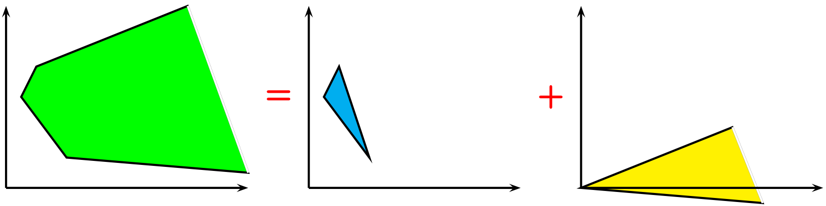

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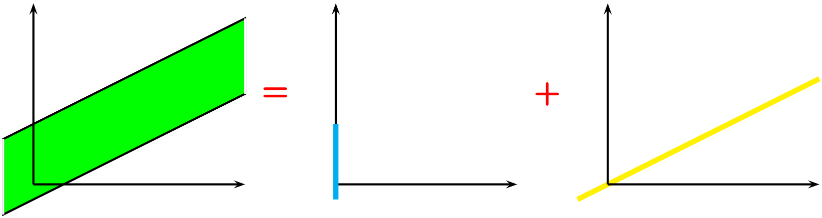

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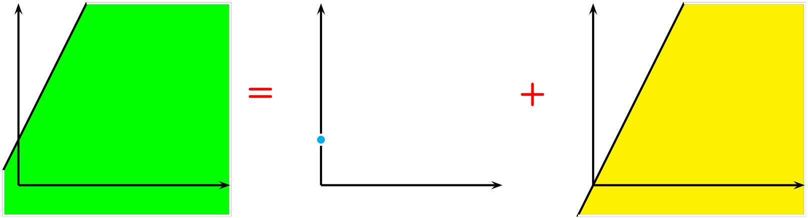

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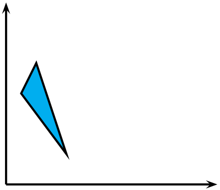
3. $\mathrm{core}(v)$ is a polytope (bounded polyhedron) if and only if $\mathrm{core}(0) = \{0\}$.



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Case of distributive lattices:

Theorem

(G. & Sudhölter 2012) Let $\mathcal{F} = \mathcal{O}(N, \preceq)$. The following holds.

1. *If $(N, \preceq)$ is connected, then for any game v on $\mathcal{F}$, $\mathrm{core}(v) \neq \varnothing$.*
2. *If $(N, \preceq)$ is not connected, then there exists a game v on $\mathcal{F}$ such that $\mathrm{core}(v) = \varnothing$.*

Pointedness of the core

By the theory of polyhedra, for a balanced game v, $\mathrm{core}(v)$ is pointed iff the system

$$x(S) = 0, \quad S \in \mathcal{F} \setminus \{\emptyset\}$$

has 0 as unique solution. If this condition is satisfied, we say that the set system $\mathcal{F}$ is *nondegenerate*. In particular:

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4. The core is pointed if $\mathcal{F}$ contains a chain of length n (e.g., if $\mathcal{F}$ is regular, in particular, if $\mathcal{F} = \mathcal{O}(N, \preceq)$)

Boundedness of the core

Lemma (Derks & Reijnierse 98)

The recession cone $\mathrm{core}(0)$ *of a game on* $\mathcal{F}$ *is a linear subspace if and only if* $\mathcal{F} \setminus \{\varnothing, N\}$ *is a balanced collection or is empty.*

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Since boundedness is equivalent to $\mathrm{core}(0) = \{0\}$, we get:

Theorem

Let v *be a game on a set system* $\mathcal{F}$. *Then* $\mathrm{core}(v)$ *is bounded if and only if* $\mathcal{F}$ *is nondegenerate and* $\mathcal{F} \setminus \{\varnothing, N\}$ *is balanced.*

Extremal rays

Extremal rays are known if $\mathcal{F} = \mathcal{O}(N)$ (distributive lattice).
Notation: $i \prec\cdot\, j$ means $i \prec j$ in $(N, \preceq)$, and there is no k such that $i \prec k \prec j$.

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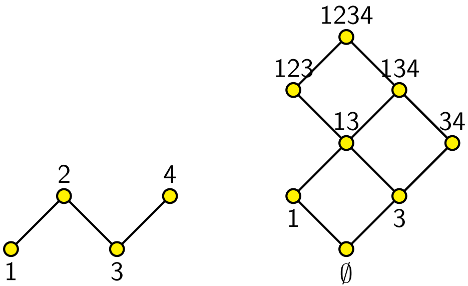
Consequence: when $\mathcal{F}$ is a distributive lattice, the core of a game is bounded if and only if $\mathcal{F} = 2^N$.

Extremal rays: example



Extremal rays are $(0, 0, 1, -1)$, $(1, -1, 0, 0)$ and $(0, -1, 1, 0)$.

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- The Weber set is defined by $W(v) = \mathrm{conv}(m^\sigma, \sigma \in \mathfrak{S}(\mathcal{F}))$.

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$$m^\sigma_{\sigma(i)} = v(\{\sigma(1), \ldots, \sigma(i)\}) - v(\{\sigma(1), \ldots, \sigma(i-1)\})$$

- The Weber set is defined by $W(v) = \mathrm{conv}(m^\sigma, \sigma \in \mathfrak{S}(\mathcal{F}))$.

Theorem

(Derks & Gilles 95, Faigle & Kern 2000) Let v be a game on $\mathcal{F} = \mathcal{O}(N, \preceq)$. Then

$$\mathrm{core}(v) \subseteq W(v) + \mathrm{core}(0).$$

Extreme points

Generalization of the Shapley-Ichiishi Theorem:

Theorem

(Fujishige & Tomizawa 83, Derks & Gilles 95) Let v *be a game on* $\mathcal{F} = \mathcal{O}(N, \preceq)$. *The following propositions are equivalent.*

1. v *is convex;*
2. $m^\sigma \in \mathrm{core}(v)$ *for all* $\sigma \in \mathfrak{S}(\mathcal{F})$;
3. $\mathrm{core}(v) = W(v) + \mathrm{core}(0)$;
4. $\mathrm{ext}(\mathrm{core}(v)) = \{m^\sigma\}_{\sigma \in \mathfrak{S}(\mathcal{F})}$.

Part I: Basic Notions

Part II: Games with Restricted Cooperation

Part III: The Core of Games with Restricted Cooperation

Part IV: Bounded Faces of the Core

Bounded faces

- A *p-dim face* of a n-dim polyhedron P defined by $Ax \leqslant b$ is a set of points in P satisfying a subsystem of $n - p$ independent equalities $A'x = b'$.

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- Hence, searching *bounded faces* of the core amounts to turning some inequalities $x(S) \geqslant v(S)$ into equalities $x(S) = v(S)$, so that the new polyhedron is bounded (has no rays).

Bounded faces

- We call *normal collection* any collection $\mathcal{N} \subset \mathcal{F}$ of nonempty sets such that

$$\mathrm{core}_{\mathcal{N}}(v) = \{x \in \mathbb{R}^N \mid x(S) \geqslant v(S) \ \forall S \in \mathcal{F}, \\ x(S) = v(S) \ \forall S \in \mathcal{N}, \text{ and } x(N) = v(N)\}$$

is bounded. By convention, $N \notin \mathcal{N}$. We call $\mathrm{core}_{\mathcal{N}}(v)$ the *restricted core* w.r.t. $\mathcal{N}$ (G. 2011).

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- Problem: many normal collections exist.

Restricted cores

We say that an extremal ray r of $\mathrm{core}(0)$ is *deleted by equality* $x(S) = 0$ if $\mathrm{core}_{\{S\}}(0) = \{x \in \mathrm{core}(0) \mid x(S) = 0\}$ does not contain r any more.

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Lemma

(G. 2011) For $i, j \in N$ such that $j \prec\cdot i$, the extremal ray $1_{\{j\}} - 1_{\{i\}}$ is deleted by equality $x(S) = 0$ if and only if $S \ni j$ and $S \not\ni i$.

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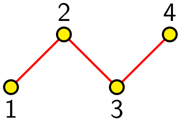

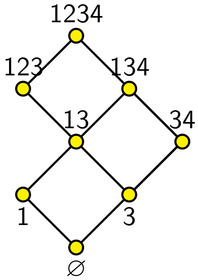

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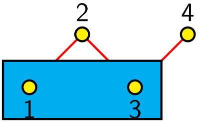
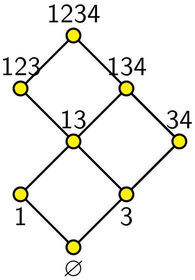




$\mathcal{N} = \{13\}$ is a normal collection, as well as $\mathcal{N} = \{1, 3\}$, $\{1, 3, 13\}$,...

Normal collections

Fact: a normal collection contains at least $h(N)$ subsets, where $h(N)$ is the height of $(N, \preceq)$.

Definition

Let $(N, \preceq)$ with $h(N) > 0$, and $\mathcal{N} = \{N_1, \dots, N_q\}$ be a normal collection.

1. $\mathcal{N}$ is a *minimal* collection if no proper subcollection is normal.
2. $\mathcal{N}$ is a *thin* collection if no $S \in \mathcal{N}$ may be replaced by a proper subset of S without losing normality.
3. $\mathcal{N}$ is a *short* collection if it contains exactly $h(N)$ subsets.
4. $\mathcal{N}$ is a *nested* collection if it is a chain in $\mathcal{F}$.

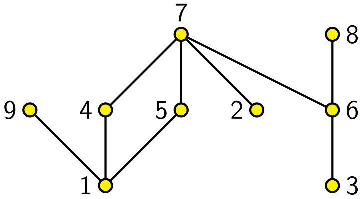
Note that any short normal collection is minimal, but the converse is not true

Normal collections

Examples of normal collections: the upwards collection (short, minimal and thin), the downwards collection (short, minimal and thin), the Grabisch-Xie collection (short, nested), etc.

Example

Consider the poset $(N, \preceq)$ of 9 elements depicted below.



The upwards collection is $\{123, 13456\}$, the downwards collection is $\{13, 123456\}$, and the Grabisch-Xie collection is $\{123, 1234569\}$.

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- We recall that to any maximal chain $C = \{\varnothing, S_1, S_2, \ldots, S_n = N\}$ with $S_i := \{\sigma(1), \ldots, \sigma(i)\}$, we associate the *marginal vector* $m^C \in \mathbb{R}^N$ (or m^σ)

$$m^C_{\sigma(i)} := v(S_i) - v(S_{i-1}), \quad i \in N,$$

and the *Weber set* is the convex hull of all marginal vectors: $\mathcal{W}(v) := \mathrm{conv}(m^C \mid C \in \mathcal{C})$.

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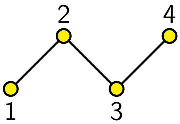
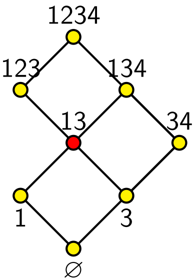
- A *restricted marginal vector* is a (classical) marginal vector whose underlying maximal chain is restricted.
- The *(restricted) Weber set* $\mathcal{W}_{\mathcal{N}}(v)$ is the convex hull of all restricted marginal vectors w.r.t. $\mathcal{N}$.

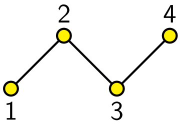
The Weber set

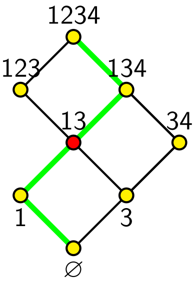




Normal collection $\{13\}$

The Weber set





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(G. 2011) Consider $\mathcal{N}$ a nested normal collection. If v is convex on $\mathcal{F} = \mathcal{O}(N)$, then $\mathrm{core}_{\mathcal{N}}(v) = \mathcal{W}_{\mathcal{N}}(v)$, i.e., the vertices of $\mathrm{core}_{\mathcal{N}}(v)$ are exactly the restricted marginal vectors w.r.t. $\mathcal{N}$.

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Theorem

(G. & Sudhölter 2013)

1. *For any convex game v and any nested normal collection $\mathcal{N}$ of $\mathcal{F}$, $\mathrm{core}_{\mathcal{N}}(v) \neq \varnothing$. Moreover, if v is strictly convex, then $\dim \mathrm{core}_{\mathcal{N}}(v) = n - |\mathcal{N}| - 1$.*
2. *For any convex game v,*
$$\mathrm{core}^b(v) = \bigcup_{\mathcal{N} \in \mathcal{MNNC}(\mathcal{F})} \mathrm{core}_{\mathcal{N}}(v)$$
Moreover, no term in the union is redundant if v is strictly convex.
3. *Let $\mathcal{N}$ be a normal collection of $\mathcal{F}$. If v is strictly convex, then $\mathrm{core}_{\mathcal{N}}(v) \neq \varnothing$ if and only if $\mathcal{N}$ is nested.*

Concluding remarks

- The core of games with restricted cooperation (on set systems), when nonempty, is a closed convex polyhedron, possibly unbounded, or without vertices. Its structure is elucidated.

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- The core of games with restricted cooperation (on set systems), when nonempty, is a closed convex polyhedron, possibly unbounded, or without vertices. Its structure is elucidated.
- Bounded faces of the core are induced by normal collections.
- When the game is strictly convex, bounded faces are in bijection with minimal nested normal collections.
- The union of all bounded faces, $\mathrm{core}^b(v)$, is called the *bounded core*. It can be defined as

$$\mathrm{core}^b(v) = \{x \in \mathrm{core}(v) \mid \forall j \prec\cdot\, i, \forall \epsilon > 0, \\ x + \epsilon(1_{\{i\}} - 1_{\{j\}}) \notin \mathrm{core}(v)\}$$

and has been axiomatized (G. & Sudhölter 2012).